PARTIAL DIFFERENTIAL EQUATIONS QUALIFYING EXAM–Fall 2017

Choose THREE out of four problems: only three will be graded. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Start each problem on a fresh sheet of paper and write on only one side of the paper.

1. Consider the Burgers' equation

$$u_t + uu_x = 0, \qquad x \in \mathbb{R},\ t > 0,$$
$$u(x,0) = g(x), \qquad x \in \mathbb{R}.$$

where $g(x)$ is a given piecewise continuous function.

   (a) (5pts) Show that the characteristics are straight lines.

   (b) (2.5pts) Give an example of $g(x)$ so that the characteristics do not cover the entire (x,t) space.

   (c) (2.5pts) Give an example of $g(x)$ so that the characteristics intersect.

2. Let $U \subset \mathbb{R}^n$ be an open set.

   (a) (5pts) Let $u \in C^2(U)$. Show that for any Ball $\overline{B}(x_0, r) \subset U$ it holds

$$\frac{d}{dr} \fint_{\partial B(0,1)} u(x_0 + rz) d\mathcal{S}(z) = \frac{r}{n} \fint_{B(0,1)} (\Delta u)(x_0 + rz) dz.$$

   Here we have used the notation $\fint_A f = \frac{1}{|A|} \int_A f$.

   (b) (3pts) Let $u \in C^2(U)$ be such that for any ball $\overline{B}(x_0, r) \subset U$ it holds

$$u(x_0) = \fint_{\partial B(x_0,r)} u d\mathcal{S}.$$

   Show that then $\Delta u = 0$ in U.

   (c) (2pts) Does the implication of part (b) still hold if you just assume $u \in C(U)$? Briefly explain your answer.

3. Let U be the unit ball in $\mathbb{R}^n$.

   (a) (5pts) For $u(x) = |x|^{-a}$ for $x \in U$, determine the values of $a,\ n,\ p$ for which u belongs to the Sobolev space $W^{1,p}(U)$.

   (b) (5pts) Let $n \geq 2$. If $u(x) = \ln\ln(1 + \frac{1}{|x|})$ for $x \in U$, show that $u \in W^{1,n}(U)$ but not in $L^\infty(U)$.

4. (a) (5pts) Let $U \subset \mathbb{R}^n$ be a bounded open set, let $T > 0$ be fixed and define $U_T = U \times (0, T)$. Assume $u \in C_1^2(\overline{U_T})$ solves the following initial boundary value problem

$$\begin{aligned} u_t - \Delta u = f \quad & \text{in} \quad U_T, \\ \frac{\partial u}{\partial \nu} + u = h \quad & \text{on} \quad \partial U \times (0, T), \\ u = g \quad & \text{on} \quad U \times \{t = 0\}, \end{aligned}$$

where f, g, and h are given smooth functions, and ν is the outward pointing unit normal field of ∂U. Prove that there exists at most one such solution.

(b) (5pts) Let $U \subset \mathbb{R}^2$ be a bounded open set and let $a > 0, b, c \in \mathbb{R}$ be given constants. Show that any solution $u \in C^2(\overline{U})$ of

$$\Delta u - au + b\partial_x u + c\partial_y u = 0 \quad \text{in} \quad U,$$

cannot attain a positive maximum or negative minimum inside U.

PARTIAL DIFFERENTIAL EQUATIONS QUALIFYING EXAM–Spring 2018

Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Start each problem on a fresh sheet of paper and write on only one side of the paper.

1. Let $\Omega \subset \mathbb{R}^n$ be open and bounded with normal vector field ν and let $u_0 \in C_b(\Omega)$ with $u_0 \geq 0$ be non-trivial. Show that the problem

$$\begin{aligned} \partial_t u - \Delta u &= u^2 \text{ in } \Omega \times (0,T), \\ \partial_\nu u &= 0 \text{ on } \partial\Omega \times (0,T), \\ u(x,0) &= u_0(x) \text{ for } x \in \Omega, \end{aligned}$$

exists for at most a finite time T.

Hint: Show that the mean $m(t) = \frac{1}{|\Omega|}\int_\Omega u(t,x)dx$ satisfies $\partial_t m(t) \geq m(t)^2$.

2. Let $\Omega \subset \mathbb{R}^n$ be open, connected and bounded and let $R > 0$ such that $\Omega \subset B_R(0)$.

(a) Let $v \in C^2(\Omega) \cap C^0(\overline{\Omega})$ with $\Delta v = 0$ in Ω. Show that

$$\max_{x \in \overline{\Omega}} v(x) = \max_{x \in \partial\Omega} v(x).$$

(b) Let $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ be a solution of

$$\begin{aligned} -\Delta u &= 1 \text{ in } \Omega, \\ u &= 0 \text{ on } \partial\Omega. \end{aligned}$$

Show that

$$0 \leq u(x) \leq \frac{R^2 - |x|^2}{2n}$$

for all $x \in \overline{\Omega}$.

3. Let u be a classical solution of the following initial boundary value problem:

$$\begin{aligned} &u_t = u_{xx}, \qquad \text{in } (a,b) \times (0,T) \\ &u(a,t) = u(b,t) = 0 \\ &u(x,0) = u_0(x) \end{aligned}$$

where u_0 is a continuous function.

(a) Show that the solutions are unique.

(b) Show that there exists a constant $\alpha > 0$ such that

$$\|u(\cdot,t)\|_{L^2}^2 \leq e^{-\alpha t}\|u_0\|_{L^2}^2.$$

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1. Let $\Omega \subset \mathbb{R}^n$ be open and bounded and let $g_j \in C(\partial\Omega)$ converge uniformly to $g \in C(\partial\Omega)$ (recall that this means that $\lim_{j\to\infty} \sup_{x\in\partial\Omega} |g_j(x) - g(x)|$). Let $u_j \in C^2(\Omega) \cap C(\overline{\Omega})$ be the solutions of

$$\begin{aligned} \Delta u_j &= 0 \text{ in } \Omega, \\ u_j &= g_j \text{ on } \partial\Omega. \end{aligned}$$

Show that u_j converges uniformly to a function $u \in C^2(\Omega) \cap C(\overline{\Omega})$ and that u solves

$$\begin{aligned} \Delta u &= 0 \text{ in } \Omega, \\ u &= g \text{ on } \partial\Omega. \end{aligned}$$

2. Let $u_0 \in C^2(B_1(0)), u_1 \in C^1(B_1(0)), f \in C((0,T) \times B^1(0))$. Show that the problem

$$\begin{aligned} \partial_{tt}u - \Delta u + u &= f \text{ in } (0,T) \times B^1(0), \\ u(0,x) &= u_0(x), \\ \partial_t u(0,x) &= u_1(x), \\ u(t,x) &= 0 \text{ on } (0,T) \times \partial B_1(0), \end{aligned}$$

has at most one solution $u \in C^2([0,T] \times \overline{B_1(0)})$.

3. Consider the equation

$$\partial_t u + u\partial_x u = 0$$

on $(0,T) \times \mathbb{R}$. Show that a classical solution with initial data $u(0,x) = \frac{\pi}{2} - \arctan(x)$ can exist at most for a finite time.