Hedging Non-tradable Risks with Transaction Costs and Price Impact

9^{th} Western Conference on Mathematical Finance

Los Angeles, California, USA

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November 17, 2018

Motivation

- Consider a framework of trading which incorporates price impact (both temporary and permanent)
- An agent has exposure to a factor which is not traded or in which the agent's trading is prohibited
- Such exposure can occur if an agent is endowed (or expected to be endowed in the future) with shares of a particular asset
- The agent is still allowed to trade in an asset which is correlated with the exposure

Related Literature

- (Almgren and Chriss 2001) - optimal execution with temporary and permanent price impact
- (Henderson 2002) - considers valuation of claims on non-tradable assets
- (Leung and Sircar 2009b), (Leung and Sircar 2009a), and (Grasselli and Henderson 2009) - study valuation of employee stock options by trading partially correlated assets
- (Leung and Lorig 2016) - consider the problem of statically hedging a contingent claim written on a correlated asset

Model - Inventory and Fundamental Price

- The agent's inventory, Q_t, is controlled through the speed of trading, ν_t:

$$dQ_t = \nu_t dt$$

- A large volume of trades in a short time will tend to impact the price of the asset
- This effect is captured through a permanent impact on the fundamental price of the traded asset:

$$dS_t = (\mu + b\nu_t)dt + \sigma dW_t$$

- If the agent refrains from trading altogether then the fundamental price follows an arithmetic Brownian motion

Model - Wealth

- Large trade volumes will also tend to “walk the book” executing at prices beyond the best bid and ask
- This effect is captured through a temporary price impact
- The agent’s cash process changes according to:

$$
\begin{aligned}
dX_t &= -\hat{S}_t \nu_t dt \\
\hat{S}_t &= S_t + k\nu_t
\end{aligned}
$$

Model - Non-traded Factor

- If the agent refrains from trading then the non-traded factor also follows an arithmetic Brownian motion correlated with W
- There is evidence that trading in one asset is accompanied by an increase in activity in closely related assets (Tristan Buchs 2017 EPFL Master Thesis)
- This effect is captured through a permanent cross impact:

$$\begin{aligned} dU_t &= (\beta + c\nu_t)dt + \eta dB_t \\ d[W, B]_t &= \rho dt \end{aligned}$$

Terminal Wealth

- We fix a terminal horizon for the trading period denoted T
- At time T the agent liquidates any position in the traded asset and incurs a penalty
- Restrictions on the non-traded factor are lifted and exposure to U is liquidated
- Terminal wealth is equal to

$$X_T + Q_T(S_T - \alpha Q_T) + \psi(U_T)$$

where ψ represents the explicit dependence of the exposure to the non-traded factor U

Optimal Trading Program

- The agent desires to maximize expected utility of terminal wealth by controlling the inventory trade process
- At time t the agent's value function is

$$H(t, x, q, S, U) = \sup_{\nu} \mathbb{E}_t \left[-e^{-\gamma(X_T + Q_T(S_T - \alpha Q_T) + \psi(U_T))} \right]$$

- This stochastic optimal control problem has an associated HJB equation

HJB simplification

- The following ansatz gives an appropriate form of the value function:

$$H(t,x,q,S,U) = -e^{-\gamma(x+qS+h(t,q,U))}$$

- This simplifies the HJB equation to one for h:

$$\begin{aligned}\partial_t h + \mu q - \frac{1}{2}\gamma\sigma^2 q^2 + (\beta - \gamma\rho\sigma\eta q)\partial_U h \\ + \frac{1}{2}\eta^2\partial_{UU} h - \frac{1}{2}\gamma\eta^2(\partial_U h)^2 \\ + \sup_\nu\left\{\nu\partial_q h + c\nu\partial_U h + bq\nu - k\nu^2\right\} = 0\,, \\ h(T,q,U) = \psi(U) - \alpha q^2\,.\end{aligned}$$

Linear Exposure

Linear Exposure

- The simplest case is when the exposure depends linearly on U:

$$\psi(U) = \mathfrak{N}U$$

- The agent has been endowed with $\mathfrak{N}$ shares of an asset but is restricted from trading in that asset until time T
- Analysis of the previous PDE suggests that we may take $h(t, q, U)$ to also have linear dependence on U
- Further analysis suggests quadratic dependence on q (common for models with this structure)

Linear Exposure - Value Function

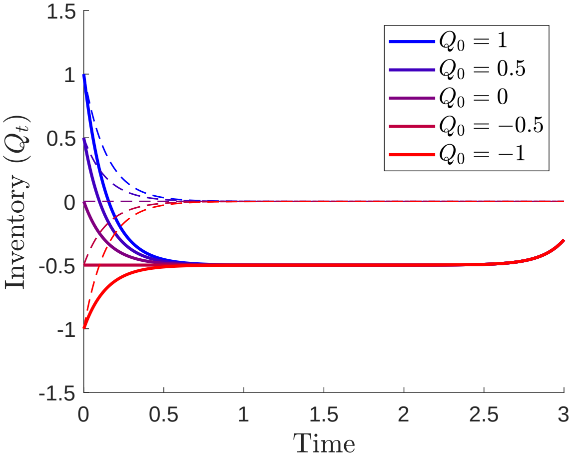
- The value function can be expressed in closed form
- Of more interest is the optimal (feedback) trading strategy, also in closed form:

$$\nu^*(t,q) = \frac{c\mathfrak{N} + h_1(t) + (2h_2(t) + b)q}{2k}$$

  where the functions h_1 and h_2 are deterministic (explicit formula available)
- The trading speed has no dependence on U, therefore the inventory process will be deterministic

Linear Exposure - Inventory Path

- Exposed versus unexposed trading strategies



Linear Exposure - Long Term Position

- The inventory appears to approach a particular level and stay there for most of the trading period
- If the trading horizon is sufficiently long, the desired inventory position of the agent is

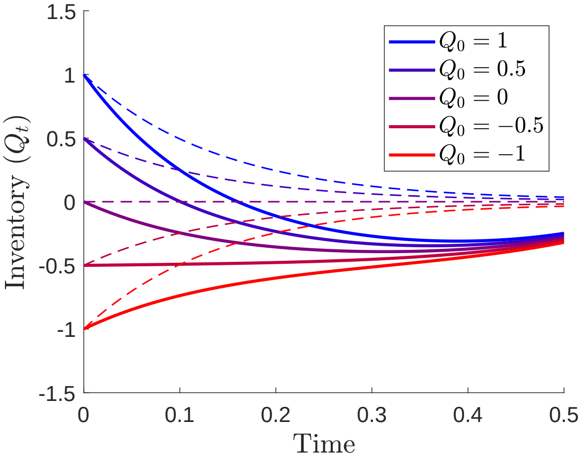
$$\frac{\mu - \gamma\rho\sigma\eta\mathfrak{N}}{\gamma\sigma^2}$$

- The agent trades as if the traded asset has modified drift:

$$\mu \mapsto \mu - \gamma\rho\sigma\eta\mathfrak{N}$$

Linear Exposure - Inventory Path

- Shorter trading horizons don't offer enough time to enter the risk-adjusted optimal position efficiently



Non-Linear Exposure

Non-Linear Exposure

- For general ψ the non-linear terms in the PDE do not allow for easy treatment
  - The value function is no longer quadratic with respect to q
- For small values of some parameters, an asymptotic expansion allows for quadratic dependence on q
- Let θ be a small expansion parameter and replace risk-aversion and cross impact with

$$c \mapsto \theta c$$
$$\gamma \mapsto \theta \gamma$$

- We suppose the function h can be expanded in θ

$$h(t,q,U) = h_0(t,q,U) + \theta\Big(ch_1(t,q,U) + \gamma h_2(t,q,U)\Big) + o(\theta)$$

Value Function - Order Zero

- The order zero component can be decomposed into dependence on q and dependence on U separately:

$$h_0(t,q,U) = f_0(t) + f_1(t)q + f_2(t)q^2 + g(t,U) \tag{1}$$

$$g(t,U) = \mathbb{E}[\psi(\tilde{U}_T)|\tilde{U}_t = U] \tag{2}$$

$$d\tilde{U}_t = \beta dt + \eta dB_t \tag{3}$$

$$f_0(t) = \int_t^T \frac{f_1^2(s)}{4k} ds \tag{4}$$

$$f_1(t) = \frac{\mu(T-t)(4k + m(T-t))}{4k + 2m(T-t)} \tag{5}$$

$$f_2(t) = \frac{-km}{2k + m(T-t)} - \frac{b}{2} \tag{6}$$

$$m = 2\alpha - b \tag{7}$$

Value Function - Order Zero

- Dependence on U comes through the expected payoff of a European option on the non-traded factor with Bachelier dynamics
  - This option value assumes unaffected prices of the non-traded factor (no cross impact)
- Dependence on q comes through the value of a modified optimal trading program:
  - Risk-neutral agent
  - Exposure only to traded asset S

Trading Strategy - Approximation

- From the HJB equation, the feedback form of the optimal trading speed is

$$\nu^*(t, q, U) = \frac{\partial_q h + c\partial_U h + bq}{2k} \tag{8}$$

- The optimal trading strategy can be approximated by

$$\nu^*(t, q, U) = \nu_0(t, q) + \theta\Big(c\nu_1(t, U) + \gamma\nu_2(t, q, U)\Big) + o(\theta) \tag{9}$$

- The zero order term is given by

$$\nu_0(t, q) = \frac{1}{2k}\Big(f_1(t) + (2f_2(t) + b)q\Big) \tag{10}$$

- This is the optimal trading strategy of a risk-neutral execution program as in (Almgren and Chriss 2001)

Trading Strategy - First Order Approximation

$$\nu^*(t,q,U) = \nu_0(t,q) + \theta\Big(c\nu_1(t,U) + \gamma\nu_2(t,q,U)\Big) + o(\theta)$$

- The first order terms are given by a stochastic representation:

$$\nu_1(t,U) = \frac{1}{2k}\Big(\partial_U g(t,U) + \lambda_1(t,U)\Big) \tag{11}$$

$$\nu_2(t,q) = \frac{1}{2k}\Big(\Lambda_1(t,U) + 2\Lambda_2(t)q\Big) \tag{12}$$

$$\lambda_1(t,U) = \frac{-m}{2k+m(T-t)}\mathbb{E}\left[\int_t^T \partial_U g(s,\tilde{U}_s)ds\middle|\tilde{U}_t = U\right] \tag{13}$$

$$\Lambda_1(t,U) = \mathbb{E}\left[\int_t^T \frac{2k+m(T-s)}{2k+m(T-t)}\frac{f_1(s)\Lambda_2(s) - k\rho\sigma\eta\partial_U g(s,\tilde{U}_s)}{k}ds\middle|\tilde{U}_t = U\right] \tag{14}$$

$$\Lambda_2(t) = \frac{-\sigma^2(T-t)(12k^2 + 6km(T-t) + m^2(T-t)^2)}{6(2k+m(T-t))^2} \tag{15}$$

- In particular, dependence on U means inventory is no longer deterministic

Trading Strategy - First Order Approximation

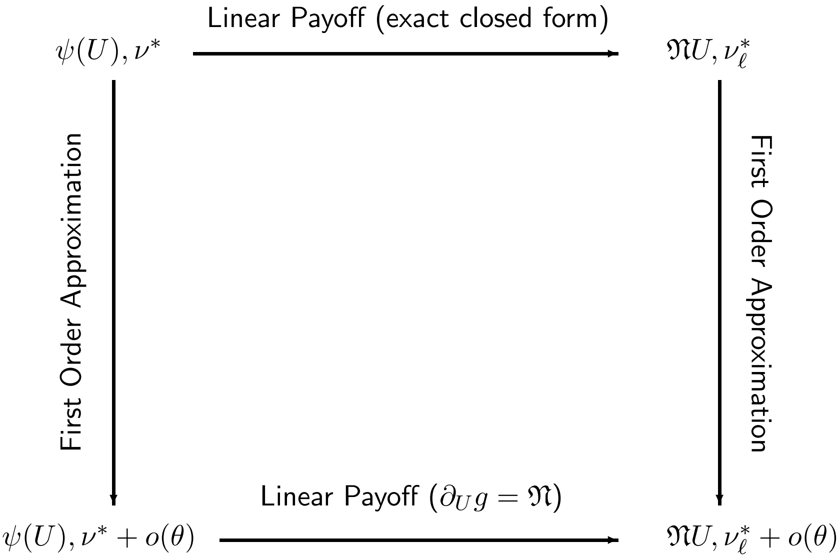
- Inspection of the stochastic representations of ν_1 and ν_2 show that dependence on U comes in a specific form
  - Any dependence on U comes from the "Delta" of the Bachelier option: $\partial_U g(t, U)$
- If the payoff is linear $(\psi(U) = \mathfrak{N}U)$ then the Delta is $\mathfrak{N}$

Theorem (Closed Form Approximation)

Denote the optimal strategy for the linear payoff by $\nu_\ell^*(t, q; \mathfrak{N})$*.Then the optimal strategy with exposure* $\psi(U_T)$ *is approximated by*

$$\nu^*(t, q, U) = \nu_\ell^*(t, q; \partial_U g(t, U)) + o(\theta)$$

Trading Strategy - First Order Approximation



Simulation of Optimal Trading Strategy

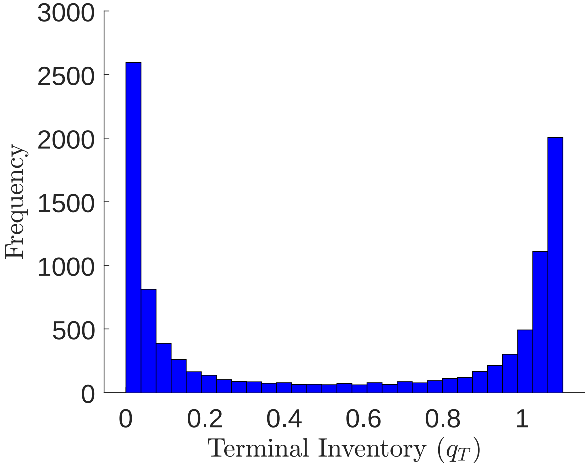
- We would like to know how the optimal trading strategy behaves relative to the order zero strategy
- The specific payoff considered is that of 100 at-the-money call options

$$\psi(U_T) = 100(U_T - U_0)_+$$

- We consider an agent beginning with no inventory in the traded asset: $Q_0 = 0$
- Also specify unimpacted prices to be martingales: $\mu = \beta = 0$
- In this setting the order zero strategy is to make no trades

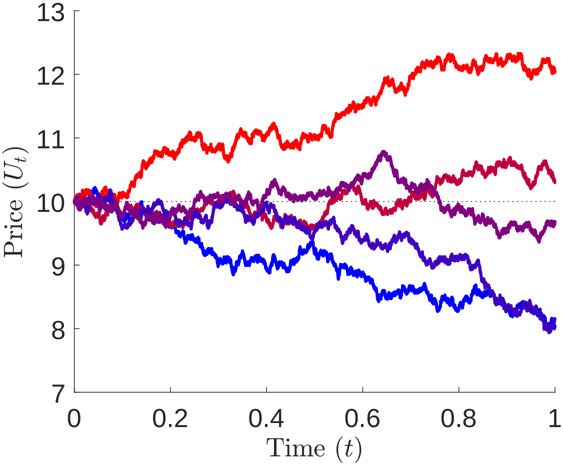
Effect of Cross Impact on Inventory Distribution

- Inventory at time T appears to be drawn towards two possible values ($c = 10^{-3}$)



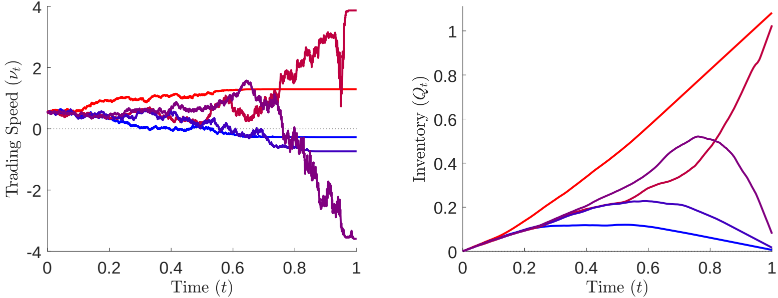
Non-traded Factor Paths

- Individual paths influence agent's optimal decisions in very different ways



Effect of Cross Impact on Inventory Path

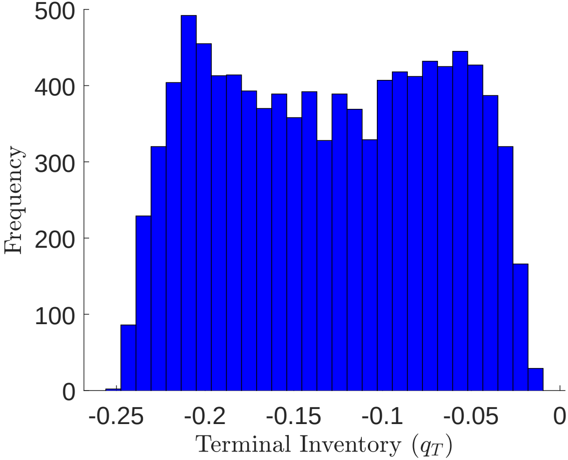
- Agent's action depends on probability of expiring in-the-money



- With high confidence of expiring in-the-money, agent targets terminal inventory of $\frac{100c}{2\alpha - b}$, otherwise targets zero

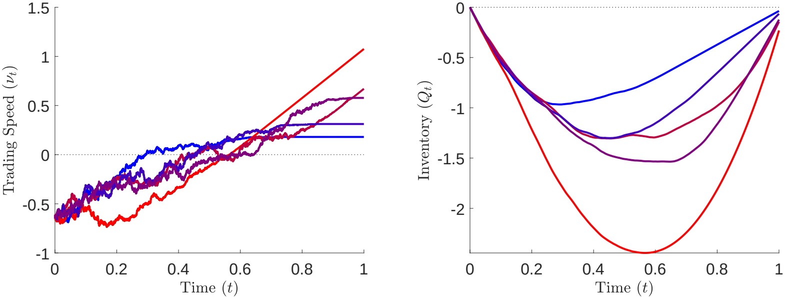
Effect of Risk-Aversion on Inventory Distribution

- Distribution still appears somewhat bimodal ($\gamma = 10^{-3}$)



Effect of Risk-Aversion on Inventory Path

- The agent's action depends on the future integrated Delta of the option



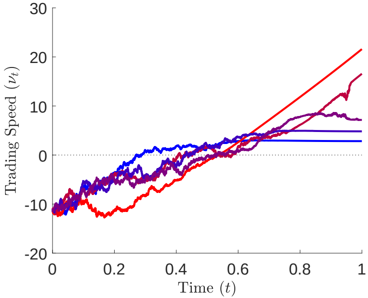
- Higher future Delta requires a larger short position to hedge the option

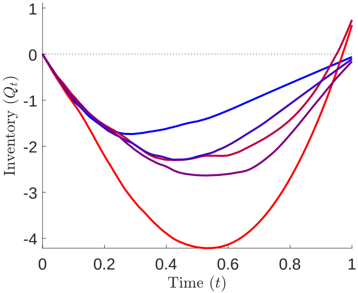
Counteracting Effects

- The two parameters which induce behaviour that differs from the order zero strategy have significantly different effects
- Risk-aversion gives incentive for the agent to hold a short position with large variance in the middle of the trading period, low variance at the end
- Cross impact gives incentive for a long position with variance increasing monotonically

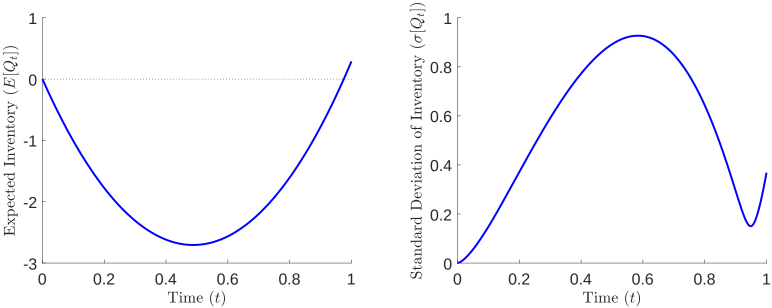
Counteracting Effects on Inventory Path

- Paths which are expected to expire in-the-money induces a short position over most of the horizon, but a long position at maturity





Counteracting Effects - Distribution Through Time



Thanks for your attention!

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Linear Exposure Value Function

$$
\begin{aligned}
h(t,q,U) &= h_0(t) + h_1(t)q + h_2(t)q^2 + \mathfrak{N}U \\
h_0(t) &= (\beta\mathfrak{N} - \frac{1}{2}\gamma\eta^2\mathfrak{N}^2)(T-t) + \frac{1}{4k}\int_t^T (h_1(s) + c\mathfrak{N})^2 ds \\
h_1(t) &= \quad \frac{\zeta k}{\omega(\phi^- e^{-\frac{\omega}{k}(T-t)} + \phi^+ e^{\frac{\omega}{k}(T-t)})}\Big(\phi^-(1 - e^{-\frac{\omega}{k}(T-t)}) - \phi^+(1 - e^{\frac{\omega}{k}(T-t)})\Big) \\
&\qquad + \frac{2\omega c\mathfrak{N}}{\phi^- e^{-\frac{\omega}{k}(T-t)} + \phi^+ e^{\frac{\omega}{k}(T-t)}} - c\mathfrak{N} \\
h_2(t) &= \quad \omega \frac{\phi^- e^{-\frac{\omega}{k}(T-t)} - \phi^+ e^{\frac{\omega}{k}(T-t)}}{\phi^- e^{-\frac{\omega}{k}(T-t)} + \phi^+ e^{\frac{\omega}{k}(T-t)}} - \frac{b}{2} \\
\omega &= \sqrt{\frac{k\gamma\sigma^2}{2}} \\
\phi^{\pm} &= \omega \pm \alpha \mp \frac{b}{2} \\
\zeta &= \mu - \gamma\rho\sigma\eta\mathfrak{N}
\end{aligned}
$$

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