A Full Asymptotic Series of European Call Option Prices in the SABR Model with $\beta = 1$

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November 17, 2018

SABR Model with $\beta = 1$

The SABR model is an extension of the Black Scholes model in which the volatility parameter follows a stochastic process:

$$dS_t = rS_t dt + \sigma_t S_t^{\beta}(\rho dW_t + \sqrt{1-\rho^2} dZ_t), \qquad (1)$$

$$d\sigma_t = \alpha \sigma_t dW_t. \qquad (2)$$

Approximation for Implied Volatilities of SABR Model

Hagan et al. derived, with perturbation techniques, an approximating direct formula for this implied volatility under the SABR model in [5]:

$$\sigma_{BS}(S_0,K) = \frac{\sigma_0}{(S_0K)^{\frac{1-\beta}{2}}[1+\frac{(1-\beta)^2}{24}\ln^2\frac{S_0}{K}+\frac{(1-\beta)^4}{1920}\ln^4\frac{S_0}{K}+\cdots]}\frac{z}{x(z)}\cdot \left[1+\left(\frac{(1-\beta)^2}{24}\frac{\sigma_0^2}{(S_0K)^{1-\beta}}+\frac{1}{4}\frac{\rho\beta\alpha\sigma_0}{(S_0K)^{(1-\beta)/2}}+\frac{2-3\rho^2}{24}\alpha^2\right)\tau+O(\tau^2)\right], \tag{3}$$

where $z := -\frac{\alpha}{\sigma_0}(S_0K)^{\frac{1-\beta}{2}}\ln(\frac{S_0}{K})$ and $x(z) = \ln\left(\frac{\sqrt{1-2\rho z+z^2}+z-\rho}{1-\rho}\right)$.

Approximation for Implied Volatilities of SABR Model

In the special case $\beta = 1$, the SABR implied volatility formula reduces to

$$\sigma_{BS}(S_0, K) = \sigma_0 \frac{y}{f(y)} \left[1 + \left(\frac{1}{4} \rho \alpha \sigma_0 + \frac{2 - 3\rho^2}{24} \alpha^2 \right) \tau + O(\tau^2) \right], \quad (4)$$

where $y := -\frac{\alpha}{\sigma_0} \ln(\frac{S_0}{K})$ and $f(y) = \ln\left(\frac{\sqrt{1-2\rho y + y^2} + y - \rho}{1-\rho} \right)$.

European call: $BS(t, x, \sigma_{BS}) = e^x N(d_+) - Ke^{-r(T-t)} N(d_-)$.

The Black-Scholes Theory

$$dS_t = rS_t dt + \sigma S_t dW_t. \tag{5}$$

Let $X_t = \ln S_t$ denote the logarithm of stock price. The price of an European call option with payoff $(X_T - K)_+$ at time t satisfy the Black-Scholes-Merton equation:

$$\mathcal{L}_{BS}(\sigma) BS(t, x, \sigma) = 0, \tag{6}$$

where $\mathcal{L}_{BS}(\sigma) = \mathcal{L}_{BS}(\sigma) = \frac{\partial}{\partial t} + \frac{1}{2}\sigma^2 \frac{\partial^2}{\partial x^2} + (r - \frac{1}{2}\sigma^2)\frac{\partial}{\partial x} - r\cdot$ is the Black-Scholes differential operator. And the closed-form solution of above PDE (6) is

$$BS(t, x, \sigma) = e^x N(d_+) - Ke^{-r(T-t)} N(d_-). \tag{7}$$

Generalization of Hull-White Formula

Consider the model under a risk-neutral probability:

$$dS_t = rS_t dt + \sigma_t S_t(\rho dW_t + \sqrt{1-\rho^2} dZ_t), t \in [0, T] \qquad (8)$$

Where W_t and Z_t are independent standard Brownian motions defined in a probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$, $\mathcal{F}_t = \mathcal{F}_t^W \cup \mathcal{F}_t^Z := \sigma\{W_s, Z_s, s \leq t\}$, and σ_t is a square integrable process adapted to $\{\mathcal{F}_t^W\}$.

Generalization of Hull-White Formula

Assume that hypotheses $(H1)$ to $(H4)$ in [1] by Alòs hold. Then, for all $t \in [0, T]$,

$$V_t = E[BS(t, X_t, v_t)|\mathcal{F}_t] + \frac{\rho}{2}\int_t^T e^{-r(s-t)} E\left[H(s, X_s, v_s)\Lambda_s \middle| \mathcal{F}_t\right] ds \tag{9}$$

where $v_s^2 = \frac{1}{T-s}\int_s^T \sigma_u^2 du$ is the future average volatility, and

$$H(s, X_s, v_s) := \left(\frac{\partial^3}{\partial x^3} - \frac{\partial^2}{\partial x^2}\right) BS(s, X_s, v_s),$$
$$\Lambda_s := \left(\int_s^T D_s^W \sigma_r^2 dr\right)\sigma_s.$$

We denote $V_{s,T} = v_s^2(T-s) = \int_s^T \sigma_u^2 du$ and $V_{t,s} = \int_t^s \sigma_u^2 du$.

Exponential Formula by Jin, Peng & Schellhorn

Theorem

Suppose $F \in \mathbb{D}_\infty([0,T])$ *satisfies the following condition:*

$$\frac{(T-t)^{2n}}{(2^n n!)^2} E\left[\left(\sup_{u_1,\dots u_n \in (t,T)} |(D^2_{u_n} \dots D^2_{u_1} F)(\omega^t)|\right)^2\right] \xrightarrow[n\to\infty]{} 0,$$

for fixed $t \in [0,T]$, *then*

$$E[F|\mathcal{F}_t] = \sum_{n=0}^{\infty} \frac{1}{2^n n!} \int_{[t,T]^n} (D^2_{s_n} \dots D^2_{s_1} F)(\omega^t) ds_n \dots ds_1. \tag{10}$$

Freezing Operator

Definition

Given $\omega \in \Omega$, a freezing operator ω^t is defined as:

$$W(s, \omega^t(\omega)) = \begin{cases} W(s, \omega), & \text{if } s \leq t; \\ W(t, \omega), & \text{if } t \leq s \leq T. \end{cases} \tag{11}$$

The freezing operator ω^t is a mapping from Ω to Ω.

Apply Exponential Formula to $F = H(s, X_s, v_s)\Lambda_s$

Goal: $E[H(s, X_s, v_s)\Lambda_s|\mathcal{F}_t]$, recall that

$$V_t = E[BS(t, X_t, v_t)|\mathcal{F}_t] + \frac{\rho}{2}\int_t^T e^{-r(s-t)} E\left[H(s, X_s, v_s)\Lambda_s \middle| \mathcal{F}_t\right] ds. \tag{12}$$

Let $F = H(s, X_s, v_s)\Lambda_s$, using iterated conditioning:

$$E[F|\mathcal{F}_t] = E\left[E\left[H(s, X_s, v_s)\Lambda_s|\mathcal{F}_T^W \cup \mathcal{F}_t^Z\right]\middle|\mathcal{F}_t\right] = E[\Lambda_s G_s|\mathcal{F}_t], \tag{13}$$

where $G_s = G(s, X_s, v_s) = E\left[H(s, X_s, v_s)|\mathcal{F}_T^W \cup \mathcal{F}_t^Z\right]$ depends only on Brownian motion $\{Z_t\}_{t\geq 0}$.

Apply Exponential Formula to $F = H(s, X_s, v_s)\Lambda_s$

Then the option price formula (9) becomes:

$$V_t = E[BS(t, X_t, v_t)|\mathcal{F}_t] + \frac{\rho}{2}\int_t^T e^{-r(s-t)}E[\Lambda_s G_s|\mathcal{F}_t]ds, \qquad (14)$$

where

$$\Lambda_s := \int_s^T D_s^W \sigma_r^2 dr\sigma_s = \int_s^T 2\alpha\sigma_r^2 dr\sigma_s = 2\alpha\sigma_s V_{s,T}, \qquad (15)$$

$$G_s = \sum_{n=0}^{\infty}\frac{1}{2^n n!}\omega_Z^t \circ \int_{[t,T]^n} D_{\tau^{\bigotimes n}}^{2n,Z} H(s, X_s, v_s)d\tau^{\bigotimes n}, \;\; t \leq s. \quad (16)$$

Goal: $E[G_s\Lambda_s|\mathcal{F}_t]$

Faà di Bruno's Formula for Malliavin derivative

The Faà di Bruno's formula can be generalized to Malliavin derivative in the following way:
If f and g are functions with a sufficient number of derivatives, then for a random variable $F \in \mathbb{D}^N([0,T])$ and $\forall n \leq N$, by chain rule and Faà di Bruno's formula we have

$$D_t^n f(g(F)) = \sum_{k=1}^{n} f^{(k)}(g(F)) \cdot B_{n,k}\Big(g'(F), \dots, g^{n-k+1}(F)\Big) D_t^n F, \tag{17}$$

where $B_{n,k}(x_1, \dots, x_{n-k+1})$ are the incomplete exponential Bell polynomials.

Malliavin derivative of H_s: $D^{2n,Z}_{T\bigotimes n} H(s, X_s, v_s)$

$$H_s = \Big(\frac{\partial^3}{\partial x^3} - \frac{\partial^2}{\partial x^2}\Big) BS(s, X_s, v_s) = \frac{-d_-}{\sqrt{2\pi} V_{s,T}} e^{X_s - \frac{d_+^2}{2}}.$$

Define two real-valued functions $p(\cdot)$ and $q(\cdot)$ such that$q\big(p(s, X_s, v_s)\big) = H_s$,

$$p(s, X_s, v_s) = X_s - \frac{d_+^2}{2} + \ln(-d_-), \tag{18}$$

$$q(x) = \frac{1}{\sqrt{2\pi} V_{s,T}} e^x. \tag{19}$$

Malliavin derivative of H_s: $D^{2n,Z}_{\tau \bigotimes n} H(s, X_s, v_s)$

$$D^Z_\tau X_s = D_\tau \int_t^s \sigma_u \sqrt{1-\rho^2} dZ_u = \sigma_\tau \sqrt{1-\rho^2} \mathbb{1}_{\{\tau \le s\}}, \qquad (20)$$

Then by Faà di Bruno's formula,

$$\begin{aligned} D^{2n,Z}_{\tau \bigotimes n} H_s &= D^{2n,Z}_{\tau \bigotimes n} q(p(s, X_s, v_s)) \\ &= \sum_{k=1}^{2n} q^{(k)}(p(s, X_s, v_s)) \cdot B_{2n,k}\left(b_1, \dots, b_{2n-k+1}\right) D^{2n,Z}_{\tau \bigotimes n} X_s \\ &= (1-\rho^2)^n H_s B_{2n}\left(b_1, \dots, b_{2n}\right) \Pi_{i=1}^n \sigma^2_{\tau_i} \mathbb{1}_{\{\tau_i \le s\}} \qquad (21) \end{aligned}$$

where $b_k = p^{(k)}(s, X_s, v_s), k = 1, \dots, 2n$

Expression of G_s

$$
\begin{aligned}
G_s &= \sum_{n=0}^{\infty} \frac{1}{2^n n!} \omega_Z^t \circ \int_{[t,T]^n} D^{2n,Z}_{\tau^{\bigotimes n}} H(s, X_s, v_s) d\tau^{\bigotimes n} \\
&= \sum_{n=0}^{\infty} \frac{(1-\rho^2)^n}{2^n n!} H_s^\omega B_{2n}\left(b_1^\omega, \ldots, b_{2n}^\omega\right) \int_t^s \Pi_{i=1}^n \sigma_{\tau_i}^2 d\tau^{\bigotimes n} \\
&= H_s^\omega \sum_{n=0}^{\infty} \frac{(1-\rho^2)^n}{2^n n!} V_{t,s}^n B_{2n}\left(b_1^\omega, \ldots, b_{2n}^\omega\right).
\end{aligned}
$$

$$
b_j = \frac{1}{\sqrt{V_{s,T}} d_-(s, X_s, v_s))^j} \begin{cases} (-1)^{j+1} - d_-^2(s, X_s, v_s), j = 1, 2; \\ (-1)^{j+1}(j-1)!, \quad j \geq 3. \end{cases}
$$

Interpretation of $\Lambda_s G_s$

Notice that

$$\begin{aligned} X_s^\omega &:= X_t + r(s-t) - \frac{1}{2}V_{t,s} + \frac{\rho}{\alpha}(\sigma_s - \sigma_t) + \omega_Z^t \circ \int_t^s \sigma_u \sqrt{1-\rho^2} dZ_u \\ &= X_t + r(s-t) - \frac{1}{2}V_{t,s} + \frac{\rho}{\alpha}(\sigma_s - \sigma_t) \quad (22) \end{aligned}$$

$$\begin{aligned} d_\pm^\omega(s, X_s, v_s) &:= \omega_Z^t \circ d_\pm((s, X_s, v_s)) = d_\pm(s, X_s^\omega, v_s) \\ &= \frac{X_s^\omega - \ln K + r(T-s) \pm \frac{1}{2}V_{s,T}}{\sqrt{V_{s,T}}} \quad (23) \end{aligned}$$

and recall that $\Lambda_s = 2\alpha V_{s,T}\sigma_s$, thus $\Lambda_s G_s$ is a function that depends on σ_s, $V_{t,s} = \int_t^s \sigma_u^2 du$ and $V_{s,T} = \int_s^T \sigma_u^2 du$.

Joint Density of $\left(\int_0^t e^{\sigma W_s} ds, W_t\right)$

Proposition 2 In [6] by Yor (1992): the joint density of $\left(\int_0^t e^{\sigma W_s} ds, W_t\right)$ has been derived for the case $\sigma = 2$,

$$\phi_{t,\sigma}(x,y) := \frac{1}{dxdy}\mathbb{P}\left(\int_0^t e^{\sigma W_s} ds \in dx, W_t \in dy\right)$$

$$= \frac{\sigma}{2x} e^{-\frac{2}{\sigma^2 x}(1+e^{\sigma y})} \cdot \theta\left(\frac{4e^{\sigma y/2}}{\sigma^2 x}, \frac{\sigma^2 t}{4}\right), \quad (24)$$

for $x > 0, y \in R, t > 0$, where

$$\theta(r,t) = \frac{r}{\sqrt{2\pi^3 t}} e^{\frac{\pi^2}{2t}} \int_0^\infty e^{-\frac{\xi^2}{2t}} \cdot e^{-r\cosh\xi} \sinh\xi \sin\frac{\pi\xi}{t} d\xi, r, t > 0. \quad (25)$$

Joint Density of $\left(\int_0^t e^{\sigma W_s-\mu s}ds, W_t\right)$

A straightforward application of the Cameron-Martin-Girsanov theorem implies that the joint density of $\left(\int_0^t e^{\sigma W_s-\mu s}ds, W_t\right)$, $\sigma > 0, \mu \in \mathbb{R}$, which we denote by $\phi_{t,\sigma,\mu}(x,y)$, $x > 0, y \in \mathbb{R}$, can be connected with the density $\phi_{t,\sigma}(x,y) = \phi_{t,\sigma,0}(x,y)$ through the formula

$$\phi_{t,\sigma,\mu}(x,y) = e^{-\frac{\mu}{\sigma}y+\frac{\mu^2 t}{2\sigma^2}}\phi_{t,\sigma,0}(x, y-\frac{\mu}{\sigma}t). \qquad (26)$$

Calculation of $E[\Lambda_s G_s|\mathcal{F}_t]$

Define $h(V_{t,s}, v_s, \sigma_s) = \Lambda_s G_s$, then $E[\Lambda_s G_s|\mathcal{F}_t]$ can be calculated as follows:

$$E[\Lambda_s G_s|\mathcal{F}_t] = E[h(V_{t,s}, v_s, \sigma_s)|\mathcal{F}_t] = E[E[h(V_{t,s}, v_s, \sigma_s)|\mathcal{F}_s]|\mathcal{F}_t]$$

$$= E\left[\int_0^\infty h(V_{t,s}, \frac{v}{\sqrt{T-s}}, \sigma_s) F'_{V_{s,T}}(v) dv \middle| \mathcal{F}_t \right]$$

$$= \int_0^\infty dx \int_{-\infty}^\infty dy \int_0^\infty dv\ h(\sigma_t^2 x, \frac{v}{\sqrt{T-s}}, \sigma_s(y)) F'_{V_{s,T}}(v) \phi_{s-t,2\alpha,\alpha^2}(x, y$$

where $\sigma_s(y) = \sigma_t exp(\alpha y - \frac{1}{2}\alpha^2(s-t))$.

Marginal Density of $\int_0^t e^{\sigma W_s - \mu s} ds$

The conditional density of $V_{s,T}$ is $F'_{V_{s,T}}(v) = \frac{1}{\sigma_s^2}\psi_{V_{s,T}}(\frac{v}{\sigma_s^2})$, where $\psi_{V_{s,T}}(v) = \int_{\mathbb{R}} \phi_{T-s,2\alpha,\alpha^2}(v,z)dz$, and

$$
\begin{aligned}
F_{V_{s,T}}(v) &= \mathbb{P}\Big(V_{s,T} \le v|\mathcal{F}_s\Big) = \mathbb{P}\Big(\int_s^T \sigma_u^2 du \le v|\sigma_s\Big) \\
&= \mathbb{P}\Big(\int_s^T \sigma_s^2 e^{2\alpha(W_u - W_s) - \alpha^2(u-s)} du \le v|\sigma_s\Big) \\
&= \mathbb{P}\Big(\int_0^{T-s} e^{2\alpha(W_u) - \alpha^2 u} du \le \frac{v}{\sigma_s^2}, W_{T-s} < \infty\Big) \\
&= \int_0^{\frac{v}{\sigma_s^2}} \int_{-\infty}^{\infty} \phi_{T-s,2\alpha,\alpha^2}(x,z)dzdx. \qquad (27)
\end{aligned}
$$

Marginal Density of $\int_0^t e^{\sigma W_s - \mu s} ds$

One straightforward application of (27) is using the conditional density of $V_{t,T}$ to obtain the first conditional expectation in (9):

$$
\begin{aligned}
E[BS(t,X_t,v_t)|\mathcal{F}_t] &= \int_0^\infty BS\left(t, X_t, \sqrt{\frac{v}{T-t}}\right) F'_{V_{t,T}}(v)dv \\
&= \int_0^\infty BS\left(t, X_t, \sqrt{\frac{v}{T-t}}\right) \frac{1}{\sigma_t^2}\psi_{V_{t,T}}(\frac{v}{\sigma_t^2})dv \\
&= \int_0^\infty \int_{-\infty}^\infty \frac{1}{\sigma_t^2} BS\left(t, X_t, \sqrt{\frac{v}{T-t}}\right)\phi_{T-t,2\alpha,\alpha^2}\left(\frac{v}{\sigma_t^2}, z\right) dzdv \qquad (28)
\end{aligned}
$$

A Formula for European Call Option Price

$$V_t = E[BS(t, X_t, v_t)|\mathcal{F}_t] + \frac{\rho}{2}\int_t^T e^{-r(s-t)} E[\Lambda_s G_s|\mathcal{F}_t]ds.$$

$$= \int_0^\infty \int_{-\infty}^\infty \frac{1}{\sigma_t^2} BS\left(t, X_t, \sqrt{\frac{v}{T-t}}\right)\phi_{T-t,2\alpha,\alpha^2}\left(\frac{v}{\sigma_t^2}, z\right) dzdv$$

$$+ \rho\alpha \int_t^T \int_0^\infty \int_{-\infty}^\infty \int_0^\infty \int_{-\infty}^\infty I(s,v,z,x,y)dydxdzdvds, \quad (29)$$

$$I(\cdot) = \frac{e^{-r(s-t)}}{\sigma_s(y)} \cdot f\left(s, X_s^{x,y}, \sqrt{\frac{v}{T-s}}\right) \cdot \phi_{T-s,2\alpha,\alpha^2}\left(\frac{v}{\sigma_s^2}, z\right) \cdot \phi_{s-t,2\alpha,\alpha^2}(x,y),$$

$$f(s, X_s, v_s) = V_{s,T} H_s^\omega \sum_{n=0}^\infty \frac{\left((1-\rho^2)V_{t,s}\right)^n}{2^n n!} B_{2n}(b_1^\omega, \ldots, b_{2n}^\omega).$$

Parameters of Approximation Results

In the following tables we compare the values of the approximate option prices. We have chosen $T - t = 1$, $\ln X_t = 100$, $r = 0.1$, $\sigma_t = 0.3$, $\alpha = 1$, $\rho = 0, \pm 0.5$ and varying values for the strike price K.

- Column 1: Strike price K;
- Column 2: Monte Carlo Simulation with number of simulation times $N = 10^6$;
- Column 3: Approximated prices obtained by Black-Scholes formula with volatility approximated by (4);
- Column 4: Approximated prices obtained by formula (29) with $f(\cdot)$ approximated by (40).

$\rho = 0$

K	Monte Carlo	Hagan	formula (29)
90	23.573138	23.415000	23.626726
95	20.440334	20.337570	20.457574
100	17.562962	17.624483	17.594033
105	15.066565	15.291032	15.063452
110	12.885739	13.322697	12.875527

$\rho = -0.5$

K	Monte Carlo	Hagan	1*st* order approx.
90	23.972526	22.025500	23.762565
95	20.640584	19.229952	20.539753
100	17.500136	16.889528	17.505670
105	14.688296	14.952772	14.836533
110	12.121686	13.353472	12.884976

$\rho = 0.5$

K	Monte Carlo	Hagan	$1st$ order approx.
90	22.352943	24.228522	22.979063
95	20.035690	20.836574	20.304502
100	17.186214	17.691469	17.555458
105	15.172375	14.842598	14.965057
110	13.080356	12.333288	12.802697

Second Approach to Calculate $E[H_s\Lambda_s|\mathcal{F}_t]$

Recall that $V_t = E[BS(t, X_t, v_t)|\mathcal{F}_t] + J$, where

$$
\begin{aligned}
J &:= \frac{\rho}{2}\int_t^T e^{-r(s-t)} E[H_s\Lambda_s|\mathcal{F}_t]ds \\
&= \frac{\rho}{2}\int_t^T e^{-r(s-t)} \frac{2\alpha}{\sqrt{2\pi}} E\Big[\sigma_s E\big[-d_- e^{X_s - \frac{d_+^2}{2}} |\mathcal{F}_T^W \cup \mathcal{F}_t^Z\big]\Big|\mathcal{F}_t\Big]ds \\
&= C_1\int_t^T E\Big[\sigma_s E\big[-d_- e^{-\frac{d_-^2}{2}} |\mathcal{F}_T^W \cup \mathcal{F}_t^Z\big]\Big|\mathcal{F}_t\Big]ds \qquad (30)
\end{aligned}
$$

for $d_\pm$ evaluated at (s, X_s, v_s), where $C_1 = \frac{1}{\sqrt{2\pi}}\rho\alpha K e^{-(T-t)}$.

Second Approach to Calculate $E[H_s\Lambda_s|\mathcal{F}_t]$

Denote $Q_s := E\big[-d_- e^{-\frac{d_-^2}{2}}|\mathcal{F}_T^W \cup \mathcal{F}_t^Z\big]$, then the correction term can be written as $J = C_1 \int_t^T E\Big[\sigma_s Q_s \Big| \mathcal{F}_t\Big] ds$.

$$d_-(s, X_s, v_s) \quad = \quad \lambda(V_{s,T})Z \quad + \quad \gamma(V_{t,s}, V_{s,T}, \sigma_s). \quad (31)$$

where $Z = \int_t^s \sigma_u dZ_u$ is conditional normal with variance $V_{t,s}$ i.e. $Z \sim \mathcal{N}(0, V_{t,s})$,

$$\gamma(V_{t,s}, V_{s,T}, \sigma_s) := \frac{\kappa + \frac{\rho}{\alpha}(\sigma_s - \sigma_t) - \frac{1}{2}(V_{t,s} + V_{s,T})}{\sqrt{V_{s,T}}},$$

$$\lambda(V_{s,T}) := \sqrt{\frac{1-\rho^2}{V_{s,T}}}.$$

Calculation of Q_s

Goal: $E[R(s, X_s, v_s)|\mathcal{F}_t]$

$$Q_s = \int_{\mathbb{R}} -(\lambda z + \gamma) e^{-\frac{(\lambda z+\gamma)^2}{2}} \frac{1}{\sqrt{2\pi V_{t,s}}} e^{-\frac{z^2}{2V_{t,s}}} dz = C_2 \gamma e^{C_3\gamma^2}$$

where $C_2 = -\frac{1}{(2-\rho^2)^{3/2}}$, $C_3 = -\frac{1}{2(2-\rho^2)}$. Thus we have

$$J = C_1 \int_t^T E\left[\sigma_s Q_s \middle| \mathcal{F}_t\right] ds = C_1 C_2 \int_t^T E\left[R_s \middle| \mathcal{F}_t\right] ds. \qquad (32)$$

where $R_s := R(s, X_s, v_s) = \sigma_s \gamma e^{C_3\gamma^2}$ is a random variable depends only on Brownian motion $\{W_t\}_{t\geq 0}$.

Calculation of $E[R_s|\mathcal{F}_t]$ by Exponential Formula

Goal: $E[R(s,X_s,v_s)|\mathcal{F}_t]$.
Now we can apply exponential formula (10) to $R(s,X_s,v_s)$ such that:

$$E\Big[R(s,X_s,v_s)\Big|\mathcal{F}_t\Big] = \sum_{n=0}^{\infty} \frac{1}{2^n n!} r_n(s,X_t,v_t), \ \ t \le s, \qquad (33)$$

where $r_n(s,X_t,v_t) = \omega_W^t \circ \int_{[t,T]^n} D^{2n,W}_{\tau^{\bigotimes n}} R(s,X_s,v_s) d\tau^{\bigotimes n}$.

First Order Approximation of $E\left[R(s, X_s, v_s)\middle|\mathcal{F}_t\right]$

Let $f(x,y) = yxe^{C_3x^2}$, then $R(s, X_s, v_s) = f(\gamma, \sigma_s)$, and

$$D_\tau^{2,W} R_s = f_x(\gamma,\sigma_s) D_\tau^{2,W}\gamma + f_{xx}(\gamma,\sigma_s)(D_\tau^W\gamma)^2 + f_y(\gamma,\sigma_s) D_\tau^{2,W}\sigma_s$$

By the structure of σ_t for $t \in [0, T]$, we have the following results:

$$\begin{aligned} D_\tau^W \sigma_s &= \alpha\sigma_s \mathbb{1}_{\{\tau\le s\}}, & D_\tau^{2,W}\sigma_s &= \alpha^2\sigma_s\mathbb{1}_{\{\tau\le s\}}, \\ D_\tau^W V_{s,T} &= 2\alpha V_{\tau\wedge s,T}, & D_\tau^{2,W} V_{s,T} &= 4\alpha^2 V_{\tau\wedge s,T}, \\ D_\tau^W V_{t,s} &= 2\alpha V_{\tau,s}\mathbb{1}_{\{\tau\le s\}}, & D_\tau^{2,W} V_{t,s} &= 4\alpha^2 V_{\tau,s}\mathbb{1}_{\{\tau\le s\}}. \end{aligned}$$

First Order Approximation for the Correction Term

$$
\begin{aligned}
Therefore, J &= \frac{\rho}{2}\int_t^T e^{-r(s-t)}E[H_s\Lambda_s|\mathcal{F}_t]ds = C_1C_2\int_t^T E[R_s|\mathcal{F}_t]ds \\
&\approx C_1C_2\int_t^T\sum_{n=0}^{1}\frac{1}{2^n n!}\omega_W^t\circ\int_{[t,T]^n} D_{\tau^{\otimes n}}^{2n,W}R(s,X_s,v_s)d\tau^{\otimes n}ds \\
&= C_1C_2\int_t^T 1+\frac{1}{2}\int_t^T D_\tau^{2,W}R_s^\omega d\tau ds \\
&= \frac{1}{2}C_1C_2\left[\int_t^T p_1(s)+p_2(s)ds+2(T-t)\right]. \quad (34)
\end{aligned}
$$

where $p_1(s) := \omega_W^t\circ\int_t^s D_\tau^{2,W}R_s d\tau, \quad p_2(s) := \omega_W^t\circ\int_s^T D_\tau^{2,W}R_s d\tau$

Conclusion

- Convergence Analysis
- Stochastic Volatility F.B.M

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Lemma:Faà di Bruno's formula

Faà di Bruno's formula. If f and g are functions with a sufficient number of derivatives, then

$$\frac{d^n}{dx^n} f(g(x)) = \sum \frac{n!}{\prod_{i=1}^n m_i!} f^{(\sum_{k=1}^n m_k)}(g(x)) \cdot \prod_{j=1}^n \left(\frac{g^{(j)}(x)}{j!} \right)^{m_j}, \tag{35}$$

subject that all nonnegetive integers $(m_1, \ldots, m_n)$ satisfying the constraint $\sum_{k=1}^n k m_k = n$. A simpler formula expressed in terms of Bell polynomials $B_{n,k}(x_1, \ldots, x_{n-k+1})$:

$$\frac{d^n}{dx^n} f(g(x)) = \sum_{k=1}^n f^{(k)}(g(x)) \cdot B_{n,k}\left(g'(x), \ldots, g^{n-k+1}(x)\right). \tag{36}$$

Exponential Bell polynomials

The partial or incomplete exponential Bell polynomials are a triangular array of polynomials given by

$$B_{n,k}(x_1, x_2, \ldots, x_{n-k+1}) = \sum \frac{n!}{\prod_{i=1}^{n-k+1} j_i!} \prod_{i=1}^{n-k+1} \left(\frac{x_i}{i!}\right)^{j_i}, \quad (37)$$

where the sum is taken over all sequences $j_1, j_2, \ldots, j_{n-k+1}$ non-negative integers such that these two conditions are satisfied: $\sum_{i=1}^{n-k+1} j_i = k$ and $\sum_{i=1}^{n-k+1} i \cdot j_i = n$.The sum

$$B_n(x_1, \ldots, x_n) = \sum_{k=1}^{n} B_{n,k}(x_1, x_2, \ldots, x_{n-k+1}) \quad (38)$$

is called the nth complete exponential Bell polynomials.

$1st$ order approximation of $f(\cdot)$ and option prices

Let $m > 0$, define $L_s^\omega = v_s^2(T-s)H_s^\omega$ and

$$f_m(s, X_s, v_s) := L_s^\omega \sum_{n=0}^{m} \frac{\left((1-\rho^2)V_{t,s}\right)^n}{2^n n!} B_{2n}\Big(p'(X_s^\omega), \ldots, p^{2n}(X_s^\omega)\Big) \quad (39)$$

then the first order approximation $f_1(s, v_s, X_s)$ is then calculated as following:

$$\begin{aligned} f_1(s, X_s, v_s) &= L_s^\omega \left(1 + \frac{(1-\rho^2)V_{t,s}}{2}\left[\left(p^{(1)}(X_s^\omega)\right)^2 + p^{(2)}(X_s^\omega)\right]\right) \\ &= \frac{-d_-^\omega}{\sqrt{2\pi}} e^{X_s^\omega - \frac{d_+^{\omega 2}}{2}} \left(1 + \frac{(1-\rho^2)V_{t,s}}{2} \frac{d_-^{\omega 2} - 3}{V_{s,T}}\right) \quad (40) \end{aligned}$$

for $d_\pm^\omega$ evaluated at (s, X_s, v_s).

Convergence Analysis

- Conditions on the convergence of the series

$$\frac{(T-t)^{2n}}{(2^n n!)^2} E\left[\left(\sup_{u_1,\dots u_n\in(t,T)} |(D^2_{u_n}\dots D^2_{u_1}F)(\omega^t)|\right)^2\right] \xrightarrow[n\to\infty]{} 0,$$

$$\frac{c^{2n}}{n!^2} E\left[\left(\sup_{\tau_i\in(t,T)} |H_s B_{2n}\left(b_1^\omega,\dots,b_{2n}^\omega\right)\Pi_{i=1}^n \sigma^2_{\tau_i}\mathbb{1}_{\{\tau_i\le s\}}\right)^2\right] \xrightarrow[n\to\infty]{} 0,$$

where $c = \frac{(T-t)\sqrt{1-\rho}}{\sqrt{2}}$, and $b_j = p^{(j)}(s, X_s, v_s)$ for $j = 1,\dots,2n$.

Full expression of $p_1(s)$

$$
\begin{aligned}
p_1(s) &:= \omega_W^t \circ \int_t^s D_\tau^{2,W} R_s^\omega \, d\tau \\
&= R_s^\omega \Bigg[\alpha^2 (s-t) + \left(\frac{1}{\gamma^\omega} + 2C_3 \gamma^\omega \right) \Bigg[\frac{\rho \alpha^2 (s-t)}{\sqrt{1 - e^{-\alpha^2 (T-s)}}} - \frac{2\alpha\sigma_t \left(\frac{1}{\alpha^2} (1 - e^{-\alpha^2 (s-t)}) - (s-t) e^{-\alpha^2 (s-t)} \right)}{\sqrt{e^{-\alpha^2 (s-t)} - e^{-\alpha^2 (T-t)}}} \\
&\quad - 2\alpha^2 \left(\sqrt{\frac{\sigma_t^2}{\alpha^2} (e^{-\alpha^2 (s-t)} - e^{-\alpha^2 (T-t)})} + \gamma^\omega \right) \frac{\frac{1}{\alpha^2} (1 - e^{-\alpha^2 (s-t)}) - (s-t) e^{-\alpha^2 (T-t)}}{e^{-\alpha^2 (s-t)} - e^{-\alpha^2 (T-t)}} \Bigg] + (6C_3 + 4C_3^2 \gamma^\omega) \cdot \\
&\quad \frac{(\rho \alpha e^{-\frac{1}{2}\alpha^2 (s-t)} + \sigma_t e^{-\alpha^2 (s-t)})^2 (s-t) - 2\frac{\rho}{\alpha} \sigma_t (e^{-\frac{1}{2}\alpha^2 (s-t)} - e^{-\frac{3}{2}\alpha^2 (s-t)}) + \frac{\sigma_t^2}{2\alpha} (1 - e^{-2\alpha^2 (s-t)})}{e^{-\alpha^2 (s-t)} - e^{-\alpha^2 (T-t)}} \\
&\quad - 2\sigma_t^3 \frac{\rho}{\alpha} e^{-\frac{1}{2}\alpha^2 (s-t)} \left(\frac{1}{\alpha^2} (1 - e^{-\alpha^2 (s-t)}) - (s-t) e^{-\alpha^2 (T-t)} \right) \mathcal{A}_1^\omega \\
&\quad + \frac{4\sigma_t^4}{\alpha^4} \Bigg[\left(\frac{1}{2} (1 + e^{-2\alpha^2 (s-t)}) + (e^{-\alpha^2 (T-t+s-t)} - e^{-\alpha^2 (s-t)} - e^{-\alpha^2 (T-t)}) + \alpha^2 e^{-\alpha^2 (T-t+s-t)} (s-t) \right) \mathcal{A}_2^\omega \\
&\quad + \left(\frac{1}{2} (1 - e^{-2\alpha^2 (s-t)}) + 2(e^{-\alpha^2 (T-t+s-t)} - e^{-\alpha^2 (T-t)}) + \alpha^2 e^{-2\alpha^2 (T-t)} (s-t) \right) \mathcal{A}_3^\omega \Bigg] \Bigg] \qquad (41)
\end{aligned}
$$

Full expression of $p_2(s)$

$$p_2(s) := \omega_W^t \circ \int_s^T D_\tau^{2,W} R_s^\omega \, d\tau = D_\tau^{2,W} R_s^\omega \int_s^T d\tau$$
$$= R_s^\omega \left[\left(\frac{1}{\gamma^\omega} + 2C_3\gamma^\omega \right) \left(-2\alpha^2 \left(\sqrt{V_{s,T}^\omega} + \gamma^\omega \right) \right) + \mathcal{A}_3^\omega (2\alpha V_{s,T}^\omega)^2 \right] (T-s)$$
$$= R_s^\omega \left[-2\alpha^2 \left(2C_3 (\gamma^{\omega 2} + \sqrt{V_{s,T}^\omega}\gamma^\omega) + 1 + \frac{\sqrt{V_{s,T}^\omega}}{\gamma^\omega} \right) + \alpha^2 \mathcal{B}_3^\omega \right] (T-s) \quad (42)$$

where

$$\mathcal{B}_3^\omega = 4 {V_{s,T}^\omega}^2 \mathcal{A}_3^\omega = 4 \Big(C_3^2 \gamma^{\omega 3} + (2C_3^2 \sqrt{V_{s,T}^\omega} + 3C_3)\gamma^{\omega 2} + (C_3^2 V_{s,T}^\omega + 4C_3 \sqrt{V_{s,T}^\omega})\gamma^\omega \Big)$$
$$+ 6C_3 V_{s,T}^\omega + 3 + \frac{2\sqrt{V_{s,T}^\omega}}{\gamma^\omega}. \quad (43)$$

$\mathcal{A}_1^\omega, \mathcal{A}_2^\omega, \mathcal{A}_3^\omega$

$$\mathcal{A}_1^\omega := \omega_W^t \circ \mathcal{A}_1 = \frac{4C_3^2\gamma^{\omega 2} + (4C_3^2\sqrt{V_{s,T}^\omega} + 8C_3)\gamma^\omega + 6C_3\sqrt{V_{s,T}^\omega}}{\sqrt{{V_{s,T}^\omega}^3}} + \frac{1}{\sqrt{{V_{s,T}^\omega}^3}\gamma^\omega},$$

$$\mathcal{A}_2^\omega := \omega_W^t \circ \mathcal{A}_2 = \frac{C_3(2C_3\gamma^{\omega 2} + (V_{s,T}^\omega + 2C_3\sqrt{V_{s,T}^\omega} + 3)\gamma^\omega + 3\sqrt{V_{s,T}^\omega})}{\sqrt{{V_{s,T}^\omega}^3}} + \frac{1}{2\sqrt{V_{s,T}^\omega}\gamma^\omega},$$

$$\mathcal{A}_3^\omega := \omega_W^t \circ \mathcal{A}_3 = \frac{C_3^2\gamma^{\omega 3} + (2C_3^2\sqrt{V_{s,T}^\omega} + 3C_3)\gamma^{\omega 2} + (C_3^2 V_{s,T}^\omega + 4C_3\sqrt{V_{s,T}^\omega})\gamma^\omega}{{V_{s,T}^\omega}^2}$$
$$+ \frac{6C_3 V_{s,T}^\omega + 3}{4{V_{s,T}^\omega}^2} + \frac{1}{2\sqrt{{V_{s,T}^\omega}^3}\gamma^\omega}.$$