A Class of Stochastic Games and Moving Free Boundary Problems

Renyuan Xu

University of California, Berkeley

Joint work with Xin Guo (UC Berkeley) and Wenpin Tang (UCLA)

9th WCMF

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Stochastic Game: Set-up

- N players in the system

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- **Dynamics of each player i:** $(\in \mathbb{R})$

$$dX_t^i = b^i(\boldsymbol{X}_t)dt + \boldsymbol{\sigma_i}(\boldsymbol{X}_t)d\boldsymbol{B}_t, \;\; X_{0-}^i = x^i$$

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- **Admissibility:**
  - $(\xi^{i,+}, \xi^{i,-}) \in \mathcal{U}$: measurability, adaptiveness, non-decreasing càdlàg, $\int_0^\infty e^{-\alpha_i t} d\check{\xi}_t^i < \infty$, where $\check{\xi}_t^i = \xi_t^{i,+} + \xi_t^{i,-}$

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  $$J^i(\boldsymbol{x}, \boldsymbol{\xi}) = \mathbb{E} \int_0^\infty e^{-\alpha^i t} \left[h^i(X_t^1, \cdots .X_t^N)dt + \lambda^i d\check{\xi}_t^i \right]$$

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- **Admissibility:**
  - $(\xi^+, \xi^-) \in \mathcal{U}_N^i$: non-decreasing càdlàg, measurability, adaptiveness, $\int_0^\infty e^{-\alpha_i t} d\check{\xi}_t^i < \infty$, where $\check{\xi}^i = \xi^{i,+} + \xi^{i,-}$
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Stochastic Game: Choice of RAC

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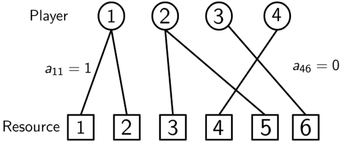
  - **Adjacent matrix:** $\boldsymbol{A} = (a_{ij})_{1 \leq i \leq N, 1 \leq j \leq M}$, $a_{ij} = 0$ or 1

$$A = \begin{array}{|c|c|c|c|c|c|} \hline 1 & 1 & & & & \\ \hline & & 1 & & 1 & \\ \hline & & & & & 1 \\ \hline & & & 1 & & \\ \hline \end{array}$$



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- **Sharing game:** N players M resources:
  - **Adjacent matrix:** special cases

$$A = \begin{array}{|c|c|c|c|}\hline 1 & 1 & 1 & 1 \\ \hline \end{array}$$

(a) Pooling

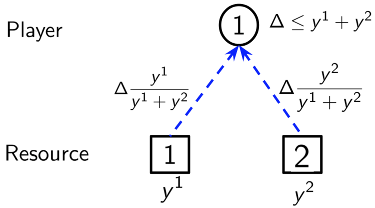
$$A = \begin{array}{|c|c|c|c|}\hline 1 & & & \\ \hline & 1 & & \\ \hline & & 1 & \\ \hline & & & 1 \\ \hline \end{array}$$

(b) Dividing

Stochastic Game: Choice of RAC

- **Pooling game:** $\sum_{i=1}^{N} \int_0^\infty d\check{\xi}_t^i \le y$, $\Delta \le y^1 + y^2$
- **Dividing game:** $\int_0^\infty d\check{\xi}_t^i \le y^i$ $(i = 1, 2, \cdots, N)$
- **Sharing game:** N players M resources:
  - **Adjacent matrix:** $\boldsymbol{A} = (a_{ij})_{1 \le i \le N, 1 \le j \le M}$, $a_{ij} = 0$ or 1
  - **Resource allocation policy:** $(Y_t^1, \ldots, Y_t^M;\ t \ge 0)$

$$Y_t^j = y^j - \sum_{i=1}^{N} \int_0^t \frac{a_{ij} Y_{s-}^j}{\sum_{j=1}^{M} a_{ij} Y_{s-}^j} d\check{\xi}_s^i \ge 0, \quad \text{and} \quad Y_{0-}^j = y^j$$



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- **Sharing game:** N players M resources:
  - **Adjacent matrix:** $\boldsymbol{A} = (a_{ij})_{1 \leq i \leq N, 1 \leq j \leq M}$, $a_{ij} = 0$ or 1
  - **Resource constraint:** $(Y_t^1, \ldots, Y_t^M;\ t \geq 0)$

$$Y_t^j = y^j - \sum_{i=1}^{N} \int_0^t \frac{a_{ij} Y_{s-}^j}{\sum_{j=1}^{M} a_{ij} Y_{s-}^j} d\check{\xi}_s^i \geq 0, \quad \text{and} \quad Y_{0-}^j = y^j$$

  - **Well-definedness:** $\sum_{j=1}^{M} a_{ij} \geq 1$, $\sum_{i=1}^{N} a_{ij} \geq 1$

Stochastic Game: Set-up

Sharing game

$$
\begin{aligned}
J^i(\boldsymbol{x}, \boldsymbol{y}; \boldsymbol{\xi}) &:= \mathbb{E}\int_0^\infty e^{-\alpha^i t}(h^i(X_t^1, \cdots, X_t^N)dt + \lambda_i d\check{\xi}_t^i) \\
dX_t^i &= b^i(\boldsymbol{X}_t)dt + \boldsymbol{\sigma}^i(\boldsymbol{X}_t)d\boldsymbol{B}_t + d\xi_t^{i,+} - d\xi_t^{i-}, \qquad X_{0-}^i = x^i \\
dY_t^j &= -\sum_{i=1}^N \frac{a_{ij} Y_{t-}^j}{\sum_{k=1}^M a_{ik} Y_{t-}^k} d\check{\xi}_t^i, \qquad Y_{0-}^j = y^j
\end{aligned}
$$

- $\boldsymbol{\xi} \in \mathcal{S}(\boldsymbol{y}) := \left\{\boldsymbol{\xi} \,:\, \xi^i \in \mathcal{U},\ Y_t^j \geq 0,\ \ \forall i, j\right\}$,
- $$\mathcal{U} := \{(\xi^+, \xi^-) : \xi^+ \text{ and } \xi^- \text{ are } \mathcal{F}^{\boldsymbol{X},\boldsymbol{Y}}\text{-progressively measurable, càdlàg, and non-decreasing, with } \xi_{0-}^+ = \xi_{0-}^- = 0\},$$
- h^i: convex, symmetric, $0 < k \leq h'' < K$,
- $\alpha^i > 0$: discount factor.

Stochastic Game: Special Case ($N = 1$)

- **Two-dimensional control problem:**

$$
\begin{aligned}
v(x,y) &= \inf_{\xi \in \mathcal{S}(y)} \int_0^\infty e^{-\alpha t} \left[h(X_t) dt + \lambda d\check{\xi}_t \right] \\
dX_t &= \mu(X_t) dt + \sigma(X_t) dB_t + d\xi_t^+ - d\xi_t^-, \quad X_{0-} = x \\
dY_t &= -d\xi_t^+ - d\xi_t^-, \quad Y_{0-} = y
\end{aligned}
$$

- **Partial references:**
  - **Finite fuel problem:** Beneš, Shepp & Witsenhausen (1980), Karatzas (1983), Ma (1993)
  - **Transaction cost analysis:** Davis & Norman (1990), Soner & Shreve (1994), Dai & Yi (2009), Kallsen & Muhle-Karbe (2010)
  - **Optimal execution/price impact:** Guo & Zervos (2015), Motairi & Zervos (2017)

Stochastic Game: Nash equilibrium

Define the N-player game with the cost functions $(J^1(\boldsymbol{x},\boldsymbol{y};\boldsymbol{\xi}),\dots,J^N(\boldsymbol{x},\boldsymbol{y};\boldsymbol{\xi}))$

Definition (Nash equilibrium)

A tuple of admissible controls $\boldsymbol{\xi}^* := (\xi^{1*},\cdots,\xi^{N*})$ is a Markovian Nash equilibrium strategy (NES) if for each admissible $(\boldsymbol{\xi}^{-i*},\xi^i)$,

$$J^i\left(\boldsymbol{x},\boldsymbol{y};\boldsymbol{\xi}^*\right) \le J^i\left(\boldsymbol{x},\boldsymbol{y};\left(\boldsymbol{\xi}^{-i*},\xi^i\right)\right).$$

Here the strategies ξ^{i*} and ξ^i are deterministic functions of time t and $\boldsymbol{X}_t = (X_t^1,\dots,X_t^N)$ with $\boldsymbol{X}_{0-} = \boldsymbol{x}$, and $\boldsymbol{Y}_t = (Y_t^1,\dots,Y_t^M)$ with $\boldsymbol{Y}_{0-} = \boldsymbol{y}$.

Stochastic Game: Different Regions

Definition (Action and waiting regions)

The i^{th} player's action region is

$$\mathcal{A}_i := \{(\boldsymbol{x}, \boldsymbol{y}) \in \mathbb{R}^N \times \mathbb{R}_+^M : d\xi^i(\boldsymbol{x}, \boldsymbol{y}) \neq 0\},$$

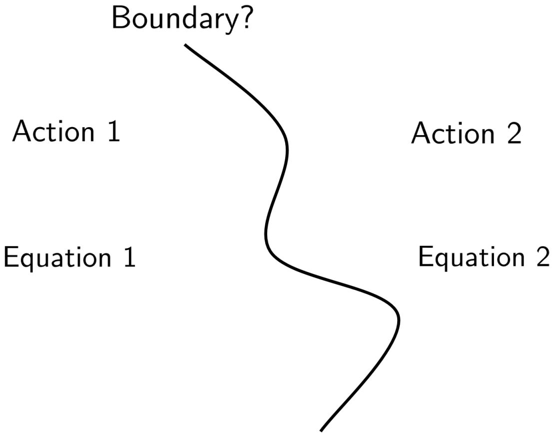
and the waiting region is $\mathcal{W}_i := (\mathbb{R}^N \times \mathbb{R}_+^M) \setminus \mathcal{A}_i$.

- $\mathcal{W}_{-i} := \cap_{j \neq i} \mathcal{W}_j$: common waiting region other than player i
- $\mathcal{W}_{NE}$: common waiting region of all players
- $\mathcal{W}_{NE}(\boldsymbol{y}) := \{\boldsymbol{x} \in \mathbb{R}^N : (\boldsymbol{x}, \boldsymbol{y}) \in \mathcal{W}_{NE}\}$

Assumption

$\mathcal{A}_i \cap \mathcal{A}_j = \emptyset, i \neq j$

Stochastic Game: Free Boundary Problem



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Stochastic Game: Different Scenarios

We focus on:

1		1	1
	1		1
		1	1
1		1	1

(a) Sharing game $N = M$

1	1	1	1
1	1	1	1
1	1	1	1
1	1	1	1

(b) Special case: pooling game

1			
	1		
		1	
			1

(c) Special case: dividing game

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Pooling: Set-up

Pooling game (on $\mathbb{R}^N \times \mathbb{R}_+$)

$$
\begin{aligned}
J^i(\boldsymbol{x}, y; \boldsymbol{\xi}) &:= \mathbb{E}\int_0^\infty e^{-\alpha t} h(X_t^i - \overline{X}_t)dt \\
dX_t^i &= dB_t^i + d\xi_t^{i,+} - d\xi_t^{i-}, \quad X_{0-}^i = x^i \\
dY_t &= -\sum_{i=1}^N d\check{\xi}_t^i, \qquad\qquad Y_{0-} = y
\end{aligned}
$$

- $\overline{X}_t = \frac{\sum_{i=1}^N X_t^i}{N}$: mean position
- h: convex, symmetric, $0 < k \leq h'' < K$
- $\alpha > 0$: discount factor
- **Constraints:**
  - **Zero-borrowing:** $Y_t \geq 0$ for all t a.s.
  - **No simultaneous jump:** $\mathbb{P}(d\xi_t^i d\xi_t^j \neq 0, \forall t \geq 0) = 1$, $i \neq j$

Pooling: Solution Derivation

Step 1: Hamilton-Jacobi-Bellman system for N players

Step 2: Candidate solution of game value

Step 3: NE strategies via

- Skorokhod problem
- Sequential jumps at time 0

Step 1: HJB System

HJB system (on $\mathbb{R}^N \times \mathbb{R}_+$)

$$\min_{(x^i,y)\in\mathbb{R}\times\mathbb{R}_+} \left\{ -\alpha v^i + h + \frac{1}{2}\sum_{j=1}^{N} v^i_{x^j x^j}, -v^i_y + v^i_{x^i}, -v^i_y - v^i_{x^i} \right\} = 0$$

in $\mathcal{W}_{-i}$

- First equation. Player i solves a usual control problem with three choices:
  - Choice 1: do nothing
  - Choice 2: increase the state variable
  - Choice 3: decrease the state variable

Step 1: HJB System

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$$\text{in } \mathcal{A}_j, j \neq i$$

- Second equation. If player j intervenes, by the definition of Nash equilibrium, we expect that player i has no incentive to move

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HJB system (on $\mathbb{R}^N \times \mathbb{R}_+$)

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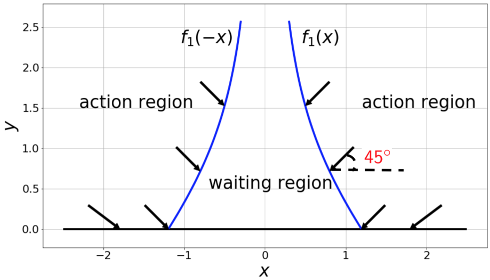
  - Choice 3: decrease the state variable
- Second equation. If player j intervenes, by the definition of Nash equilibrium, we expect that player i has no incentive to move

Step 2: Game Value (Special Case $N = 1$)[1]

$$v(x,y) = \begin{cases} \frac{x^2}{\alpha} + \frac{1}{\alpha^2} + A_1(y)\cosh(x\sqrt{2\alpha}) & \text{if} \quad |x| \le f_1^{-1}(y) \\ v(x_+, f_1(x_+)) & \text{if} \quad x > f_1^{-1}(y) \\ v(x_-, f_1(x_-)) & \text{if} \quad x < -f_1^{-1}(y) \end{cases}$$



[1] Beneš, Shepp and Witsenhausen (1980)

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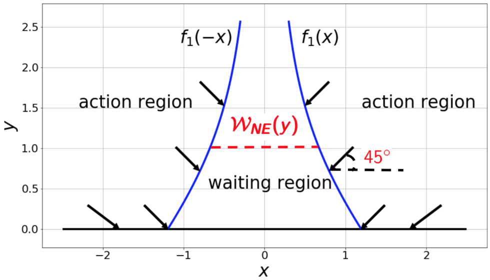

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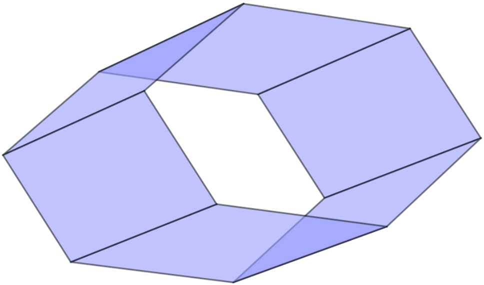
Step 2: Boundary of Free Boundary Problem

$\mathcal{W}_{NE}(y)$ when $N = 3$:



Step 2: Candidate Game Value

Candidate game value (Guo, Tang & X. (2018))

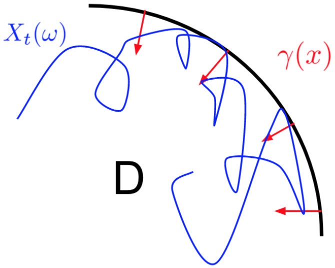
$$v^i(\boldsymbol{x}, y) = \begin{cases} p_N(\widetilde{x}^i) + A_N(y)\cosh(\widetilde{x}^i\sqrt{\frac{2(N-1)\alpha}{N}}) & \text{in } \mathcal{W}_i \\ v^i\left(\boldsymbol{x}^{-i}, x^i_+ + \frac{\sum_{k\neq i} x^k}{N-1}, f_N(x^i_+)\right) & \text{in } \mathcal{A}_i^+ \\ v^i\left(\boldsymbol{x}^{-i}, \frac{\sum_{k\neq i} x^k}{N-1} - x^i_-, f_N(x^i_-)\right) & \text{in } \mathcal{A}_i^- \\ v^i\left(\boldsymbol{x}^{-j}, x^j_+ + \frac{\sum_{k\neq j} x^k}{N-1}, f_N(x^j_+)\right) & \text{in } \mathcal{A}_j^+, j \neq i \\ v^i\left(\boldsymbol{x}^{-j}, \frac{\sum_{k\neq j} x^k}{N-1} - x^j_-, f_N(x^j_-)\right) & \text{in } \mathcal{A}_j^-, j \neq i \end{cases}$$

- $\widetilde{x}^i = x^i - \frac{\sum_{j\neq i} x^j}{N-1}$, $x^i_\pm$: unique positive root of $z \mp f_N(z) = \widetilde{x}^i \mp y$
- $f_N(\cdot)$: threshold function

Step 3: NE via Skorokhod

A heuristic description of Skorokhod: A Skorokhod problem is to find a stochastic differential equation with some reflection directions on the boundary.

- Domain with boundary
- Reflection direction
- Dynamics



Step 3: NE via Skorokhod

Ingredient 1: common waiting region (unbounded)

$$\mathcal{W}_{NE}(y) = \{\boldsymbol{x} \in \mathbb{R}^N : |\widetilde{x}^i| < f_N^{-1}(y) \text{ for } 1 \leq i \leq N\}$$

$$F_i(y) = \{\boldsymbol{x} \in \mathbb{R}^N : \frac{1}{N-1}(-\mathbf{1} + N\boldsymbol{e}_i) \cdot \boldsymbol{x} = f_N^{-1}(y)\} \cap \overline{\mathcal{W}_{NE}(y)}$$

$$F_{N+i}(y) = \{\boldsymbol{x} \in \mathbb{R}^N : \frac{1}{N-1}(-\mathbf{1} + N\boldsymbol{e}_i) \cdot \boldsymbol{x} = -f_N^{-1}(y)\} \cap \overline{\mathcal{W}_{NE}(y)}$$

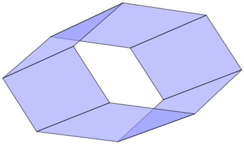


Figure 3: $\mathcal{W}_{NE}(y)$ when $N = 3$

Step 3: NE via Skorokhod

Ingredient 2: reflection direction

$\boldsymbol{\gamma}(\boldsymbol{x}) = -\boldsymbol{e}_i$ on $F_i(y)$
$\boldsymbol{\gamma}(\boldsymbol{x}) = \boldsymbol{e}_i$ on $F_{i+N}(y)$, $i = 1, 2, \cdots, N$

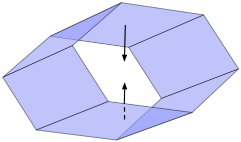


Figure 4: Reflection direction when $N = 3$

Step 3: NE via Skorokhod

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Ingredient 3: dynamic without control

$$\boldsymbol{X}_t = \boldsymbol{B}_t$$

Step 3: NE via Skorokhod

Lemma (Skorokhod solution given y)

For fixed $y > 0$, there exists a reflected process $\boldsymbol{R}_y(t) = (R_y^1(t), \ldots, R_y^N(t))$ with $\boldsymbol{R}_y(0) = \boldsymbol{x} \in \overline{\mathcal{W}_{NE}(y)}$ such that

$$R_y^i(t) = x^i + B^i(t) + \eta_y^i(t) - \eta_y^{i+N}(t) \in \overline{\mathcal{W}_{NE}(y)} \quad \text{for } 1 \le i \le N,$$

where $(j = 1, 2, \cdots, 2N)$

- $(\eta_y^j(t);\ t \ge 0)$ *is the local time process on the boundary*
- η_y^j *increases only at times t such that $R_y^j(t) \in F_j(y)$*

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Key idea:

- Skew symmetry condition for bounded polyhedron in Ruth Williams (1987)
- Localization argument

Step 3: NE via Skorokhod

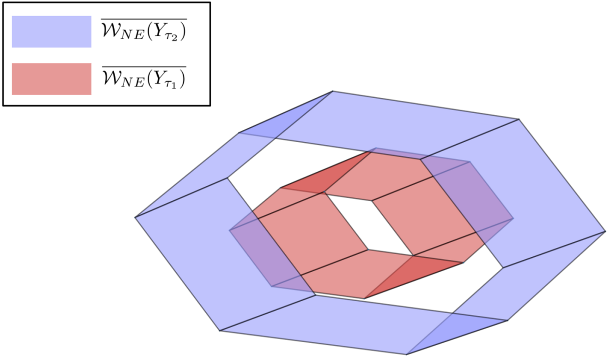

Figure 5: Pooling: evolving domain when $N = 3$

Step 3: NE via Skorokhod

Theorem (Skorokhod solution (Guo, Tang & X. 2018))

Inductively, for $k \geq 2$, let

$$\tau_k := \inf\left\{t > \tau_{k-1} : \boldsymbol{R}_{Y_{\tau_{k-1}}}(t-\tau_{k-1}) \in \partial\mathcal{W}_{NE}(Y_{\tau_{k-1}})\right\},$$

where $\boldsymbol{R}_{Y_{\tau_{k-1}}}$ is a copy of the reflected process in $\mathcal{W}_{NE}(Y_{\tau_{k-1}})$, starting at $\boldsymbol{X}_{\tau_{k-1}}$ and driven by $\boldsymbol{B}_k = (B_k^1, \dots, B_k^N)$. Then we have for $\tau_{k-1} \leq t \leq \tau_k$,

$$X_t^i = X_{\tau_{k-1}}^i + B_k^i(t-\tau_{k-1}) + \eta^i_{Y_{\tau_{k-1}}}(t-\tau_{k-1}) - \eta^{i+N}_{Y_{\tau_{k-1}}}(t-\tau_{k-1}),$$

and

$$Y_t = Y_{\tau_{k-1}} - \eta^i_{Y_{\tau_{k-1}}}(t-\tau_{k-1}) - \eta^{i+N}_{Y_{\tau_{k-1}}}(t-\tau_{k-1})$$

are the NE strategies.

Step 3: Sequential Jumps at Time 0

$\mathcal{A}_i$ is defined in the way

- Player who is *furtherest away* controls
- Player with the *largest index* will control if ties occur

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Sharing Game: Set-up

Sharing game

$$
\begin{aligned}
J^i(\boldsymbol{x},\boldsymbol{y};\boldsymbol{\xi}) &:= \mathbb{E}\int_0^\infty e^{-\alpha t} h(X_t^i - \overline{X}_t)dt \\
dX_t^i &= dB_t^i + d\xi_t^{i,+} - d\xi_t^{i-}, & X_{0-}^i = x^i \\
dY_t^i &= -\sum_{j=1}^N \frac{a_{ji} Y_{t-}^i}{\sum_{k=1}^N a_{jk} Y_{t-}^k} d\check{\xi}_t^j, & Y_{0-}^i = y^i
\end{aligned}
$$

- $\overline{X}_t = \frac{\sum_{i=1}^N X_t^i}{N}$: mean position
- h: convex, symmetric, $0 < k \le h'' < K$
- $\alpha > 0$: discount factor
- **Constraints:**
  - **Zero-borrowing:** $Y_t^i \ge 0$ for all t a.s. and i
  - **No simultaneous jump:** $\mathbb{P}(d\xi_t^i d\xi_t^j \ne 0, \forall t \ge 0) = 1$, $i \ne j$

Sharing Game: HJB

HJB system ($\mathbb{R}^{M+N}$)

$$\begin{cases} \min\limits_{(x^i,\boldsymbol{y})\in\mathbb{R}\times\mathbb{R}_+^N} \Big\{ -\alpha v^i + h + \dfrac{1}{2}\displaystyle\sum_{j=1}^N v^i_{x^jx^j}, -\sum_{j=1}^N \frac{a_{ij}y^j}{\sum_{j=1}^N a_{ij}y^j} v^i_{y^j} + v^i_{x^i}, \\ \qquad\qquad -\sum_{j=1}^N \frac{a_{ij}y^j}{\sum_{j=1}^N a_{ij}y^j} v^i_{y} - v^i_{x^i} \Big\} = 0, \\ \qquad\qquad \text{for } (\boldsymbol{x},\boldsymbol{y}) \in \mathcal{W}_{-i}, \\ \min_{(x^j,\boldsymbol{y})\in\mathbb{R}\times\mathbb{R}_+^N} \Big\{ -\sum_{k=1}^N \frac{a_{jk}y^k}{\sum_{s=1}^N a_{js}y^s} v^i_{y^k} + v^i_{x^j}, \\ \qquad\qquad -\sum_{k=1}^N \frac{a_{jk}y^k}{\sum_{s=1}^N a_{js}y^s} v^i_{y^k} - v^i_{x^j} \Big\} = 0 \\ \qquad\qquad \text{for } (\boldsymbol{x},\boldsymbol{y}) \in \mathcal{A}_j, j \neq i. \end{cases}$$

Sharing Game: Game Value

Game value of sharing(Guo, Tang & X. (2018))

$$v^i(\boldsymbol{x}, y) = \begin{cases} p_N(\widetilde{x}^i) + A_N(y)\cosh(\widetilde{x}^i\sqrt{\frac{2(N-1)\alpha}{N}}) & \text{in } \mathcal{W}_i \\ v^i\left(\boldsymbol{x}^{-i}, x^i_+ + \frac{\sum_{k\neq i} x^k}{N-1}, f_N(x^i_+)\right) & \text{in } \mathcal{A}_i^+ \\ v^i\left(\boldsymbol{x}^{-i}, \frac{\sum_{k\neq i} x^k}{N-1} - x^i_-, f_N(x^i_-)\right) & \text{in } \mathcal{A}_i^- \\ v^i\left(\boldsymbol{x}^{-j}, x^j_+ + \frac{\sum_{k\neq j} x^k}{N-1}, f_N(x^j_+)\right) & \text{in } \mathcal{A}_j^+, j \neq i \\ v^i\left(\boldsymbol{x}^{-j}, \frac{\sum_{k\neq j} x^k}{N-1} - x^j_-, f_N(x^j_-)\right) & \text{in } \mathcal{A}_j^-, j \neq i \end{cases}$$

- $\widetilde{x}^i = x^i - \frac{\sum_{j\neq i} x^j}{N-1}$, $x^i_\pm$: unique positive root of $z \mp f_N(z) = \widetilde{x}^i \mp \sum_{j=1}^N a_{ij} y^j$
- $f_N(\cdot)$: threshold function

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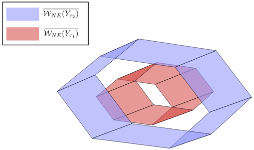
Game Comparison: NE Strategies

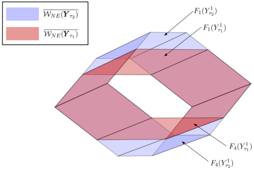
1	1	1
1	1	1
1	1	1

1		
	1	
		1

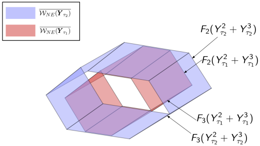
1		
	1	1
	1	1



(a) Pooling



(b) dividing



(c) Sharing

Comparison: Game Values

Proposition (Game value comparison (Guo & Tang and X. 2018))

$\forall(\boldsymbol{x},\boldsymbol{y}) \in \mathbb{R}^N \times \mathbb{R}^N_+$, *when* $y = \sum_{j=1}^N y^j$, $(\boldsymbol{x}, y) \in \mathcal{W}_i^{pool}$, *and* $(\boldsymbol{x},\boldsymbol{y}) \in \mathcal{W}_i^{share} \cap \mathcal{W}_i^{divide}$, *for each* $i = 1, 2, \cdots, N$,

$$v^i_{pool}(\boldsymbol{x}, y) \leq v^i_{share}(\boldsymbol{x}, y) \leq v^i_{divide}(\boldsymbol{x}, \boldsymbol{y}).$$

- Sharing has lower cost than playing selfishly
- Among all sharing strategies, pooling provides the lowest cost

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Controlled Rank-dependent SDEs

Controlled rank-dependent SDE (Guo, Tang & X. (2018))

$$
\begin{aligned}
dX_t^i &= \sum_{j=1}^{N} 1_{F^i(\boldsymbol{X}_t,\boldsymbol{Y}_t)=F^{(j)}(\boldsymbol{X}_t,\boldsymbol{Y}_t)} \left(b_j dt + \sigma_j dB_t^j + \lambda^{j,+} d\xi_t^{j,+} - \lambda^{j,-} d\xi_t^{j,-} \right) \\
Y_t^i &= y^i - \xi_t^{i,+} - \xi_t^{i,-} \quad \text{for } 1 \le i \le N
\end{aligned}
$$

- Pooling game: $F^i(\boldsymbol{x},\boldsymbol{y}) = |x_i - \frac{\sum_{j\neq i} x_j}{N-1}|$
- Dividing game: $F^i(\boldsymbol{x},\boldsymbol{y}) = |x_i - \frac{\sum_{j\neq i} x_j}{N-1} - f_N^{-1}(y^i)|$
- Sharing game: $F^i(\boldsymbol{x},\boldsymbol{y}) = |x_i - \frac{\sum_{j\neq i} x_j}{N-1} - f_N^{-1}(\sum_{j=1}^N a_{ij} y^j)|$
- $F^{(1)} \le \ldots \le F^{(N)}$: the order statistics of $(F^i)_{1\le i\le N}$

Controlled Rank-dependent SDEs

Controlled rank-dependent SDEs

$$
\begin{aligned}
dX_t^i &= \sum_{j=1}^{N} 1_{F^i(\boldsymbol{X}_t,\boldsymbol{Y}_t)=F^{(j)}(\boldsymbol{X}_t,\boldsymbol{Y}_t)} \left(b_j dt + \sigma_j dB_t^j + \lambda^{j,+} d\xi_t^{j,+} - \lambda^{j,+} d\xi_t^{j,-} \right) \\
Y_t^i &= y^i - \xi_t^{i,+} - \xi_t^{i,-} \quad \text{for } 1 \le i \le N
\end{aligned}
$$

- F^i : rank function depends on both $\boldsymbol{X}$ and $\boldsymbol{Y}$
- $F^{(1)} \le \ldots \le F^{(N)}$: the order statistics of $(F^i)_{1 \le i \le N}$
- $b_i \in \mathbb{R}$, $\sigma_i \ge 0$
- $(\xi^{i,+}, \xi^{i,-})$: the controls

Controlled Rank-dependent SDEs

Controlled rank-dependent SDEs

$$
\begin{aligned}
dX_t^i &= \sum_{j=1}^{N} 1_{F^i(\boldsymbol{X}_t,\boldsymbol{Y}_t)=F^{(j)}(\boldsymbol{X}_t,\boldsymbol{Y}_t)} \left(\delta_j dt + \sigma_j dB_t^j + \lambda^{j,+} d\xi_t^{j,+} - \lambda^{j,-} d\xi_t^{j,-}\right) \\
Y_t^i &= y^i - \xi_t^{i,+} - \xi_t^{i,-} \quad \text{for } 1 \le i \le N
\end{aligned}
$$

- $F^i(\boldsymbol{X}_t, \boldsymbol{Y}_t) = x^i$ and $\lambda^{i,+} = \lambda^{i,-} = 0$: rank-dependent SDE
  - **"Up the River problem":** Aldous (2002)
  - **Stochastic portfolio:** Fernholz (2002)
  - **Atlas model ($\delta_1 = 1$, $\delta_2 = \cdots = \delta_N = 0$):** Banner, Fernholz and Karatzas (2005), Ichiba, Karatzas and Shkolnikov (2013), Pal and Pitman (2008), Cabezas, Dembo and Sarantsev (2017), Tang and Tsai (2018)

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