Optimal investment with transient price impact

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A price impact model

Consider an investor whose trades affect market prices:

- investment strategy: $X = (X_t^{\uparrow}, X_t^{\downarrow})_{t \geq 0}$
- $X_t^{\uparrow} = \#$ shares bought up to time $t \geq 0$, $X_{0-}^{\uparrow} \triangleq 0$
- $X_t^{\downarrow} = \#$ shares sold up to time $t \geq 0$, $X_{0-}^{\downarrow} \triangleq 0$
- $X^{\uparrow}$, $X^{\downarrow}$ predictable, nondecreasing, right-continuous

A model with coupled bid-/ask-price dynamics

Bid and ask price dynamics B^X, A^X for strategy $X = (X^{\uparrow}, X^{\downarrow})$:

$$dA_t^X = dP_t + \eta dX_t^{\uparrow} - \frac{1}{2}\kappa(A_t^X - B_t^X)dt$$
$$dB_t^X = dP_t - \eta dX_t^{\downarrow} + \frac{1}{2}\kappa(A_t^X - B_t^X)dt$$

- dP_t exogenous fundamental price shock given by continuous semimartingale P (unaffected mid price process)
- $\eta > 0$: price impact factor (finite market depth)
- $\kappa > 0$: resilience rate (recovery rate)

Autonomous spread dynamics: $\zeta_t^X = A_t^X - B_t^X$ satisfies

$$d\zeta_t^X = \eta(dX_t^{\uparrow} + dX_t^{\downarrow}) - \kappa\zeta_t^X dt, \quad \zeta_{0-}^X \geq 0$$

Model discussion

Model similar to ...

- ... Obizhaeva & Wang (2013), but we allow for both buying and selling
- ... Almgren & Chriss (2001), but our price impact depends linear on trading volume, not trading rate
- ... Roch & Soner (2013), but our bid and ask prices revert to each other, not to a reference price

Model feature I:

- model captures Kyle's (1985) three dimensions of illiquidity: finite market depth, finite resilience (transient price impact), market tightness (spread)

Wealth dynamics

Investor's wealth at time t described by:

- $\varphi_t^X =$ number of shares currently held:

$$d\varphi_t^X = dX_t^{\uparrow} - dX_t^{\downarrow}$$

- $\xi_t^X =$ amount of cash currently held:

$$d\xi_t^X = -(A_{t-}^X + \frac{1}{2}\eta\Delta X_t^{\uparrow})dX_t^{\uparrow} + (B_{t-}^X - \frac{1}{2}\eta\Delta X_t^{\downarrow})dX_t^{\downarrow}$$

- Liquidation value of current position:

$$V_t(X) \triangleq \underbrace{\xi_t^X + \frac{1}{2}(A_t^X + B_t^X)\varphi_t^X}_{\text{book value}} - \underbrace{\left(\frac{1}{2}\zeta_t^X|\varphi_t^X| + \frac{1}{2}\eta(\varphi_t^X)^2\right)}_{\text{liquidation costs}}$$

Liquidation wealth process

Lemma

We have the decomposition

$$V_t(X) = v_{0-} + \int_0^t \varphi_s^X \, dP_s - L_t(X)$$

with liquidity costs $L_t(X)$ *where*

$$L_t(X) \triangleq \frac{1}{4\eta}\left(\eta|\varphi_t^X| + (\zeta_t^X - e^{-\kappa t}\zeta_{0-}^X)\right)^2 + \frac{1}{2}|\varphi_t^X| e^{-\kappa t}\zeta_{0-}^X + \frac{\eta}{4}(\varphi_{0-}^X)^2$$
$$+ \frac{1}{2}\int_{[0,t]} e^{-\kappa s}\zeta_{0-}^X (dX_s^{\uparrow} + dX_s^{\downarrow}) + \frac{\kappa}{2\eta}\int_0^t (\zeta_{s-}^X - e^{-\kappa s}\zeta_{0-}^X)^2 \, ds.$$

In particular, $L_t(X)$ *is convex in* X *and satisfies*

$$L_t(X) \geq \frac{\eta}{4} e^{-2\kappa t}(X_t^{\uparrow} + X_t^{\downarrow})^2 + \frac{\kappa\eta}{2}\int_0^t e^{-2\kappa s}(X_s^{\uparrow} + X_s^{\downarrow})^2 \, ds \geq 0.$$

Model feature II: Model dynamics induce convex costs

Optimal investment problem

Investor wants to maximize expected utility from terminal liquidation wealth at time $T > 0$, i.e.,

$$\mathbb{E}u(V_T(X)) = \mathbb{E}u\left(v_{0-} + \int_0^T \varphi_t^X \, dP_t - L_T(X)\right) \to \max_{X=(X^\uparrow, X^\downarrow)}$$

with utility function

$u : \mathbb{R} \to \mathbb{R}$ strictly concave and increasing with $u(\infty) < \infty$.

Related literature:

Roch/Soner ('13), Gârleanu/Pedersen ('13, '16), Kallsen/Muhle-Karbe ('14), Guasoni/Weber ('15, '16, '17), Soner/Vukelja ('16), Forde/Weber/Zhang ('16), Moreau/Muhle-Karbe/Soner ('17), Cayé/Herdegen/Muhle-Karbe ('17), Ekren/Muhle-Karbe ('17), Chandra/Papanicolaou ('17) ...

Existence and uniqueness of optimal strategies

Theorem

There exists a unique strategy $\hat{X} = (\hat{X}^{\uparrow}, \hat{X}^{\downarrow})$ *such that* $\mathbb{E}u(V_T(\hat{X})) \geq \mathbb{E}u(V_T(X))$ *for all strategies* $X = (X^{\uparrow}, X^{\downarrow})$.

Proof's main tools:

- convex compactness result for processes of finite variation (Guasoni '02)
- convexity of liquidity costs $L_T(X)$ and continuity of the liquidation wealth $V_T(X)$ in X

□

Illiquid Bachelier model with exponential utility

Simplest setting:

- Bachelier model: $dP_t = \mu dt + \sigma dW_t$ with $\mu, \sigma > 0$
- exponential utility: $u(x) = -e^{-\alpha x}$ with $\alpha > 0$

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Frictionless case: $(\eta = \zeta^X_{0-} = 0)$
Optimal strategy $\hat{X}^0$ is a deterministic buy-and-hold strategy

$$d\hat{X}_t^{0,\uparrow} = \frac{\mu}{\alpha\sigma^2}\delta_0(dt) \quad \text{and} \quad d\hat{X}_t^{0,\downarrow} = \frac{\mu}{\alpha\sigma^2}\delta_T(dt)$$

Optimal share holdings: Constant Merton portfolio

$$\varphi_t^{\hat{X}^0} \equiv \frac{\mu}{\alpha\sigma^2} \qquad (0 \leq t \leq T)$$

Optimal investment strategy

Proposition

The optimal investment strategy for the illiquid Bachelier model with exponential utility is **deterministic**. *It suffices to minimize the convex cost functional*

$$J_T(X) \triangleq L_T(X) + \frac{\alpha\sigma^2}{2} \int_0^T \left(\varphi_t^X - \frac{\mu}{\alpha\sigma^2}\right)^2 dt \to \min_{X=(X^\uparrow, X^\downarrow)}$$

Proof: Similar to Schied/Schöneborn/Tehranchi (2010). □

⇝ deterministic **optimal tracking problem** of frictionless optimal Merton portfolio $\mu/(\alpha\sigma^2)$

Model feature III: Problem can be solved explicitly

Singular control via convex analysis

Lemma

For any two deterministic strategies X, Y *with the same initial position* $\varphi^Y_{0-} = \varphi^X_{0-}$ *and initial spread* $\zeta_0 \geq 0$ *we have*

$$J_T(Y) - J_T(X) \geq \int_{[0,T]} {}^{\varrho}\nabla^{\uparrow}_t J_T(X)(dY^{\uparrow}_t - dX^{\uparrow}_t) + \int_{[0,T]} {}^{\varrho}\nabla^{\downarrow}_t J_T(X)(dY^{\downarrow}_t - dX^{\downarrow}_t)$$

with ${}^{\varrho}\nabla^{\uparrow} J_T(X)$ *and* ${}^{\varrho}\nabla^{\downarrow} J_T(X)$ *given by*

$$ {}^{\varrho}\nabla^{\uparrow,\downarrow}_t J_T(X) = \int_t^T \left(\kappa e^{-\kappa(u-t)} \zeta^X_u \pm \alpha\sigma^2 \left(\varphi^X_u - \frac{\mu}{\alpha\sigma^2} \right) \right) du \pm \frac{1}{2} \left(\eta |\varphi^X_T| + \zeta^X_T \right) \left(\mathrm{sign}_{\varrho}(\varphi^X_T) \pm e^{-\kappa(T-t)} \right). $$

Singular control via convex analysis

Lemma (First order conditions)

$\hat{X} = (\hat{X}^{\uparrow}, \hat{X}^{\downarrow})$ *is optimal if the following conditions hold true:*

- *buy-subgradient* ${}^{\varrho}\nabla_t^{\uparrow} J_T(\hat{X}) \geq 0$, *with* '$= 0$' *on* $\{d\hat{X}_t^{\uparrow} > 0\}$,
- *sell-subgradient* ${}^{\varrho}\nabla_t^{\downarrow} J_T(\hat{X}) \geq 0$, *with* '$= 0$' *on* $\{d\hat{X}_t^{\downarrow} > 0\}$.

Obvious question: How to construct $\hat{X}$ solving first order conditions?

State space

- Three dimensional state space

$$\mathscr{S} \triangleq \{(\tau, \zeta, \varphi) : \tau \geq 0, \zeta \geq 0, \varphi \in \mathbb{R}\} \subset \mathbb{R}^3,$$

with time to maturity τ, initial spread ζ, initial number of shares φ.

- Denote by $\hat{X}^{\tau,\zeta,\varphi}$ the unique **optimal** strategy with **problem data** $(\tau, \zeta, \varphi) \in \mathscr{S}$, i.e.,

$$\zeta_{0-}^{\hat{X}^{\tau,\zeta,\varphi}} = \zeta, \quad \varphi_{0-}^{\hat{X}^{\tau,\zeta,\varphi}} = \varphi.$$

- Describe evolution of optimally controlled state process

$$(\tau - t, \zeta_t^{\hat{X}^{\tau,\zeta,\varphi}}, \varphi_t^{\hat{X}^{\tau,\zeta,\varphi}})_{0 \leq t \leq \tau} \subset \mathscr{S}.$$

Buying-, selling-, waiting-region

Definition

We define

$$\mathscr{R}_{\mathrm{buy/sell}} \triangleq \Big\{(\tau,\zeta,\varphi) \in \mathscr{S} : \hat{X}^{\tau,\zeta,\varphi} \text{ satisfies } {}^{\varrho}\nabla_0^{\uparrow,\downarrow} J_\tau(\hat{X}^{\tau,\zeta,\varphi}) = 0 \text{ for some } \varrho > 0 \text{ and } \hat{X}_0^{\tau,\zeta,\varphi,\uparrow,\downarrow} > 0\Big\},$$

$$\partial\mathscr{R}_{\mathrm{buy/sell}} \triangleq \Big\{(\tau,\zeta,\varphi) \in \mathscr{S} : \hat{X}^{\tau,\zeta,\varphi} \text{ satisfies } {}^{\varrho}\nabla_0^{\uparrow,\downarrow} J_\tau(\hat{X}^{\tau,\zeta,\varphi}) = 0 \text{ for some } \varrho > 0 \text{ and } \hat{X}_0^{\tau,\zeta,\varphi,\uparrow,\downarrow} = 0\Big\},$$

$$\mathscr{R}_{\mathrm{wait}} \triangleq \mathscr{S} \setminus (\bar{\mathscr{R}}_{\mathrm{buy}} \cup \bar{\mathscr{R}}_{\mathrm{sell}}),$$

where $\bar{\mathscr{R}}_{\mathrm{buy/sell}} \triangleq \mathscr{R}_{\mathrm{buy/sell}} \cup \partial\mathscr{R}_{\mathrm{buy/sell}}$.

Goal: Characterize free boundaries $\partial\mathscr{R}_{\mathrm{buy}}$ and $\partial\mathscr{R}_{\mathrm{sell}}$ in $\mathscr{S}$.

Main result

Theorem

There are two continuous **free boundary functions**

$$\phi_{\text{buy}}(\tau,\zeta) < \phi_{\text{sell}}(\tau,\zeta)$$

explicitly available such that

$$\begin{aligned}\mathscr{R}_{\text{sell}} &= \{(\tau,\zeta,\varphi)\in\mathscr{S} : \varphi > \phi_{\text{sell}}(\tau,\zeta)\},\\ \partial\mathscr{R}_{\text{sell}} &= \{(\tau,\zeta,\varphi)\in\mathscr{S} : \varphi = \phi_{\text{sell}}(\tau,\zeta)\}\end{aligned}$$

as well as

$$\begin{aligned}\mathscr{R}_{\text{buy}} &= \{(\tau,\zeta,\varphi)\in\mathscr{S} : \varphi < \phi_{\text{buy}}(\tau,\zeta)\},\\ \partial\mathscr{R}_{\text{buy}} &= \{(\tau,\zeta,\varphi)\in\mathscr{S} : \varphi = \phi_{\text{buy}}(\tau,\zeta)\}.\end{aligned}$$

In particular,

$$\begin{aligned}\mathscr{R}_{\text{wait}} &= \{(\tau,\zeta,\varphi)\in\mathscr{S} : \phi_{\text{buy}}(\tau,\zeta) < \varphi < \phi_{\text{sell}}(\tau,\zeta)\},\\ \partial\mathscr{R}_{\text{wait}} &= \partial\mathscr{R}_{\text{buy}}\cup\partial\mathscr{R}_{\text{sell}}.\end{aligned}$$

Illustration: State space

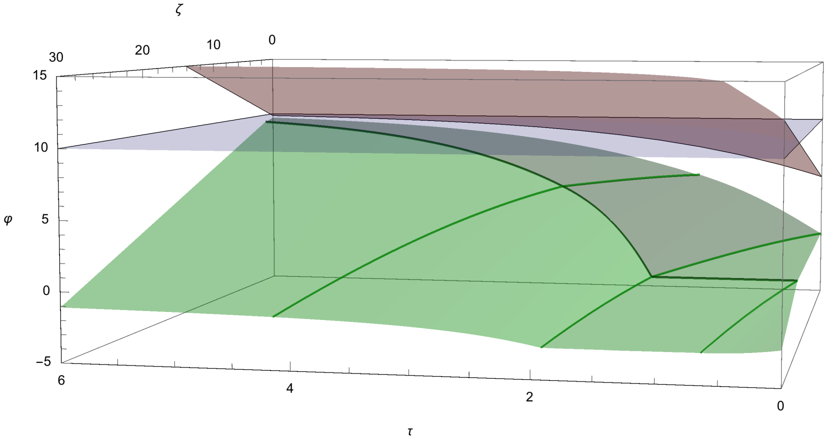

Figure: State space $\mathscr{S}$ with time to maturity τ, spread ζ, number of shares φ; Merton plane at level $\mu/(\alpha\sigma^2) = 10$ (blue); boundary of buying-region $\partial\mathscr{R}_{\text{buy}}$ (green); boundary of selling-region $\partial\mathscr{R}_{\text{sell}}$ (red).

Illustration: State space with optimal state processes

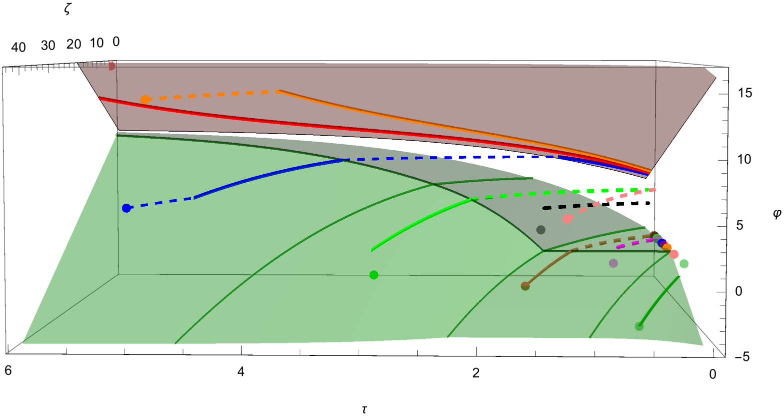

Figure: Evolution of optimally controlled state processes embedded in the state space $\mathscr{S}$. Dashed lines indicate waiting parts of the strategies; the big dots represent the corresponding initial and final triplets (τ, ζ, φ) and $(0, \zeta', 0)$ for some final spread value ζ'.

Illustration: Phenomenology of optimal trading trajectories

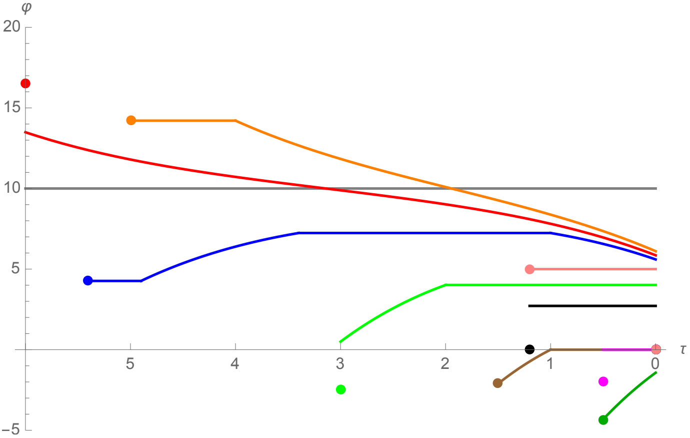

Figure: Evolution of optimal share holdings for different initial problem data $(\tau, \zeta, \varphi) \in \mathscr{S}$ as functions in time to maturity $\tau - t$ with $0 \leq t \leq \tau$. The dots represent the initial position in the risky asset. The final position is always zero. The grey line represents the Merton position $\mu/(\alpha\sigma^2) = 10$.

Conclusions

- price impact model which accounts for finite market depth, market tightness, finite resilience (transient price impact)
- coupled bid-/ask-price dynamics induce convex liquidity costs on singular controls
- general existence and uniqueness result of optimal investment strategies
- illiquid Bachelier model with exponential utility: resulting singular control problem reduces to a deterministic optimal tracking problem
- exploit convex analytic approach instead of more common dynamic programming methods
- state space and optimal strategies can be constructed explicitly
- surprisingly rich phenomenology of possible trajectories for the optimal share holdings

Reference

Optimal investment with transient price impact
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Preprint on arXiv:1804.07392.

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Thank you very much!