Optimal investment with transient price impact

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A price impact model

Consider an investor whose trades affect market prices:

- investment strategy: \( X = (X^\uparrow_t, X^\downarrow_t)_{t \geq 0} \)
- \( X^\uparrow_t = \# \) shares bought up to time \( t \geq 0 \), \( X^\uparrow_0 \triangleq 0 \)
- \( X^\downarrow_t = \# \) shares sold up to time \( t \geq 0 \), \( X^\downarrow_0 \triangleq 0 \)
- \( X^\uparrow, X^\downarrow \) predictable, nondecreasing, right-continuous
A model with coupled bid-/ask-price dynamics

Bid and ask price dynamics $B^X, A^X$ for strategy $X = (X^\uparrow, X^\downarrow)$:

\[
\begin{align*}
\frac{dA^X_t}{dt} &= \frac{dP_t}{dt} + \eta dX^\uparrow_t - \frac{1}{2} \kappa (A^X_t - B^X_t) dt \\
\frac{dB^X_t}{dt} &= \frac{dP_t}{dt} - \eta dX^\downarrow_t + \frac{1}{2} \kappa (A^X_t - B^X_t) dt
\end{align*}
\]

- $dP_t$ exogenous fundamental price shock given by continuous semimartingale $P$ (unaffected mid price process)
- $\eta > 0$: price impact factor (finite market depth)
- $\kappa > 0$: resilience rate (recovery rate)

Autonomous spread dynamics: $\zeta^X_t = A^X_t - B^X_t$ satisfies

\[
\frac{d\zeta^X_t}{dt} = \eta (dX^\uparrow_t + dX^\downarrow_t) - \kappa \zeta^X_t dt, \quad \zeta^X_{0-} \geq 0
\]
Model discussion

Model similar to . . .

➤ . . . Obizhaeva & Wang (2013), but we allow for both buying and selling
➤ . . . Almgren & Chriss (2001), but our price impact depends linear on trading volume, not trading rate
➤ . . . Roch & Soner (2013), but our bid and ask prices revert to each other, not to a reference price

Model feature I:

➤ model captures Kyle’s (1985) three dimensions of illiquidity: finite market depth, finite resilience (transient price impact), market tightness (spread)
Wealth dynamics

Investor’s wealth at time $t$ described by:

$\varphi_t^X = \text{number of shares currently held:}$

$$d\varphi_t^X = dX_t^\uparrow - dX_t^\downarrow$$

$\xi_t^X = \text{amount of cash currently held:}$

$$d\xi_t^X = -(A_t^X + \frac{1}{2}\eta\Delta X_t^\uparrow)dX_t^\uparrow + (B_t^X - \frac{1}{2}\eta\Delta X_t^\downarrow)dX_t^\downarrow$$

$\text{Liquidation value of current position:}$

$$V_t(X) \triangleq \xi_t^X + \frac{1}{2}(A_t^X + B_t^X)\varphi_t^X - \left(\frac{1}{2}\zeta_t^X|\varphi_t^X| + \frac{1}{2}\eta(\varphi_t^X)^2\right)$$

book value \hspace{0.5cm} liquidation costs
Lemma

We have the decomposition

\[ V_t(X) = v_0 - \int_0^t \varphi_s^X dP_s - L_t(X) \]

with liquidity costs \( L_t(X) \) where

\[
L_t(X) \triangleq \frac{1}{4\eta} \left( \eta |\varphi_t^X| + (\zeta_t^X - e^{-\kappa t} \zeta_0^X) \right)^2 + \frac{1}{2} |\varphi_t^X| e^{-\kappa t} \zeta_0^X + \frac{\eta}{4} (\varphi_0^X)^2 \\
+ \frac{1}{2} \int_{[0,t]} e^{-\kappa s} \zeta_0^X (dX_s^\uparrow + dX_s^\downarrow) + \frac{\kappa}{2\eta} \int_0^t (\zeta_s^X - e^{-\kappa s} \zeta_0^X)^2 \, ds.
\]

In particular, \( L_t(X) \) is convex in \( X \) and satisfies

\[
L_t(X) \geq \frac{\eta}{4} e^{-2\kappa t} (X_t^\uparrow + X_t^\downarrow)^2 + \frac{\kappa \eta}{2} \int_0^t e^{-2\kappa s} (X_s^\uparrow + X_s^\downarrow)^2 \, ds \geq 0.
\]

Model feature II: Model dynamics induce convex costs
Optimal investment problem

Investor wants to maximize expected utility from terminal liquidation wealth at time \( T > 0 \), i.e.,

\[
\mathbb{E}u(V_T(X)) = \mathbb{E}u \left( v_0 - \int_0^T \varphi_t^X \, dP_t - L_T(X) \right) \to \max_{X=(X^+, X^-)}
\]

with utility function

\[
u : \mathbb{R} \to \mathbb{R} \text{ strictly concave and increasing with } u(\infty) < \infty.\]

Related literature:
Roch/Soner ('13), Gårleanu/Pedersen ('13, '16), Kallsen/Muhle-Karbe ('14), Guasoni/Weber ('15, '16, '17), Soner/Vukelja ('16), Forde/Weber/Zhang ('16), Moreau/Muhle-Karbe/Soner ('17), Cayé/Herdegen/Muhle-Karbe ('17), Ekren/Muhle-Karbe ('17), Chandra/Papanicolaou ('17) . . .
Existence and uniqueness of optimal strategies

**Theorem**

There exists a unique strategy $\hat{X} = (\hat{X}^\uparrow, \hat{X}^\downarrow)$ such that

$$E[u(V_T(\hat{X}))] \geq E[u(V_T(X))]$$

for all strategies $X = (X^\uparrow, X^\downarrow)$.

**Proof’s main tools:**

- convex compactness result for processes of finite variation (Guasoni ’02)
- convexity of liquidity costs $L_T(X)$ and continuity of the liquidation wealth $V_T(X)$ in $X$
Illiquid Bachelier model with exponential utility

Simplest setting:

- Bachelier model: \( dP_t = \mu dt + \sigma dW_t \) with \( \mu, \sigma > 0 \)
- exponential utility: \( u(x) = -e^{-\alpha x} \) with \( \alpha > 0 \)
Illiquid Bachelier model with exponential utility

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Frictionless case: \( (\eta = \zeta_{0-} = 0) \)

Optimal strategy \( \hat{X}^0 \) is a deterministic buy-and-hold strategy

\[
\begin{align*}
d\hat{X}^0_{t,\uparrow} &= \frac{\mu}{\alpha \sigma^2} \delta_0 (dt) \\
&\text{and} \\
d\hat{X}^0_{t,\downarrow} &= \frac{\mu}{\alpha \sigma^2} \delta_t (dt)
\end{align*}
\]

Optimal share holdings: Constant Merton portfolio

\[
\varphi_t \hat{X}^0 = \frac{\mu}{\alpha \sigma^2} \quad (0 \leq t \leq T)
\]
Proposition

The optimal investment strategy for the illiquid Bachelier model with exponential utility is deterministic. It suffices to minimize the convex cost functional

\[
J_T(X) \triangleq L_T(X) + \frac{\alpha \sigma^2}{2} \int_0^T \left( \varphi_t - \frac{\mu}{\alpha \sigma^2} \right)^2 dt \rightarrow \min_{X=(X^\uparrow, X^\downarrow)}
\]

Proof: Similar to Schied/Schöneborn/Tehranchi (2010).

\[\Rightarrow\] deterministic optimal tracking problem of frictionless optimal Merton portfolio \( \mu/(\alpha \sigma^2) \)

Model feature III: Problem can be solved explicitly
Singular control via convex analysis

Lemma

For any two deterministic strategies $X, Y$ with the same initial position $\varphi^Y_0 = \varphi^X_0$ and initial spread $\zeta_0 \geq 0$ we have

$$J_T(Y) - J_T(X) \geq \int_{[0, T]} \rho \nabla^\uparrow_t J_T(X)(dY_t^\uparrow - dX_t^\uparrow)$$

$$+ \int_{[0, T]} \rho \nabla^\downarrow_t J_T(X)(dY_t^\downarrow - dX_t^\downarrow)$$

with $\rho \nabla^\uparrow J_T(X)$ and $\rho \nabla^\downarrow J_T(X)$ given by

$$\rho \nabla^\uparrow_t, \downarrow J_T(X) = \int_t^T \left( \kappa e^{-\kappa(u-t)} \zeta_u^X \pm \alpha \sigma^2 \left( \varphi_u^X - \frac{\mu}{\alpha \sigma^2} \right) \right) du$$

$$\pm \frac{1}{2} \left( \eta |\varphi_T^X| + \zeta_T^X \right) \left( \text{sign}_\rho(\varphi_T^X) \pm e^{-\kappa(T-t)} \right).$$
Singular control via convex analysis

Lemma (First order conditions)

\( \hat{X} = (\hat{X}^+, \hat{X}^-) \) is optimal if the following conditions hold true:

- **buy-subgradient** \( \partial \nabla_t^+ J_T(\hat{X}) \geq 0 \), with \( ' = 0 \) on \( \{ d\hat{X}_t^+ > 0 \} \),
- **sell-subgradient** \( \partial \nabla_t^- J_T(\hat{X}) \geq 0 \), with \( ' = 0 \) on \( \{ d\hat{X}_t^- > 0 \} \).

**Obvious question:** How to construct \( \hat{X} \) solving first order conditions?
State space

- Three dimensional state space

\[ S \triangleq \{(\tau, \zeta, \varphi) : \tau \geq 0, \zeta \geq 0, \varphi \in \mathbb{R}\} \subset \mathbb{R}^3, \]

with time to maturity \( \tau \), initial spread \( \zeta \), initial number of shares \( \varphi \).

- Denote by \( \hat{X}^{\tau, \zeta, \varphi} \) the unique \textbf{optimal} strategy with problem data \( (\tau, \zeta, \varphi) \in S \), i.e.,

\[ \zeta_{0^{-}}^{\hat{X}^{\tau, \zeta, \varphi}} = \zeta, \quad \varphi_{0^{-}}^{\hat{X}^{\tau, \zeta, \varphi}} = \varphi. \]

- Describe evolution of optimally controlled state process

\[ (\tau - t, \zeta_{t}^{\hat{X}^{\tau, \zeta, \varphi}}, \varphi_{t}^{\hat{X}^{\tau, \zeta, \varphi}})_{0 \leq t \leq \tau} \subset S. \]
Buying-, selling-, waiting-region

Definition

We define

\[ \mathcal{R}_{\text{buy/sell}} \triangleq \left\{ (\tau, \zeta, \varphi) \in \mathcal{I} : \hat{X}^{\tau,\zeta,\varphi} \text{ satisfies } \varrho \nabla_0^{\uparrow,\downarrow} J_\tau(\hat{X}^{\tau,\zeta,\varphi}) = 0 \right\} , \]

for some \( \varrho > 0 \) and \( \hat{X}_0^{\tau,\zeta,\varphi,\uparrow,\downarrow} > 0 \),

\[ \partial \mathcal{R}_{\text{buy/sell}} \triangleq \left\{ (\tau, \zeta, \varphi) \in \mathcal{I} : \hat{X}^{\tau,\zeta,\varphi} \text{ satisfies } \varrho \nabla_0^{\uparrow,\downarrow} J_\tau(\hat{X}^{\tau,\zeta,\varphi}) = 0 \right\} , \]

for some \( \varrho > 0 \) and \( \hat{X}_0^{\tau,\zeta,\varphi,\uparrow,\downarrow} = 0 \),

\[ \mathcal{R}_{\text{wait}} \triangleq \mathcal{I} \setminus (\mathcal{R}_{\text{buy}} \cup \mathcal{R}_{\text{sell}}) , \]

where \( \mathcal{R}_{\text{buy/sell}} \triangleq \mathcal{R}_{\text{buy/sell}} \cup \partial \mathcal{R}_{\text{buy/sell}} \).

Goal: Characterize free boundaries \( \partial \mathcal{R}_{\text{buy}} \) and \( \partial \mathcal{R}_{\text{sell}} \) in \( \mathcal{I} \).
Main result

Theorem

There are two continuous free boundary functions

\[ \phi_{\text{buy}}(\tau, \zeta) < \phi_{\text{sell}}(\tau, \zeta) \]

explicitly available such that

\[ R_{\text{sell}} = \{(\tau, \zeta, \varphi) \in \mathcal{I} : \varphi > \phi_{\text{sell}}(\tau, \zeta)\} \]
\[ \partial R_{\text{sell}} = \{(\tau, \zeta, \varphi) \in \mathcal{I} : \varphi = \phi_{\text{sell}}(\tau, \zeta)\} \]

as well as

\[ R_{\text{buy}} = \{(\tau, \zeta, \varphi) \in \mathcal{I} : \varphi < \phi_{\text{buy}}(\tau, \zeta)\} \]
\[ \partial R_{\text{buy}} = \{(\tau, \zeta, \varphi) \in \mathcal{I} : \varphi = \phi_{\text{buy}}(\tau, \zeta)\} \]

In particular,

\[ R_{\text{wait}} = \{(\tau, \zeta, \varphi) \in \mathcal{I} : \phi_{\text{buy}}(\tau, \zeta) < \varphi < \phi_{\text{sell}}(\tau, \zeta)\} \]
\[ \partial R_{\text{wait}} = \partial R_{\text{buy}} \cup \partial R_{\text{sell}}. \]
Figure: State space $\mathcal{S}$ with time to maturity $\tau$, spread $\zeta$, number of shares $\varphi$; Merton plane at level $\mu/(\alpha\sigma^2) = 10$ (blue); boundary of buying-region $\partial R_{\text{buy}}$ (green); boundary of selling-region $\partial R_{\text{sell}}$ (red).
Illustration: State space with optimal state processes

Figure: Evolution of optimally controlled state processes embedded in the state space $S$. Dashed lines indicate waiting parts of the strategies; the big dots represent the corresponding initial and final triplets $(\tau, \zeta, \varphi)$ and $(0, \zeta', 0)$ for some final spread value $\zeta'$. 
Figure: Evolution of optimal share holdings for different initial problem data \((\tau, \zeta, \varphi) \in \mathcal{S}\) as functions in time to maturity \(\tau - t\) with \(0 \leq t \leq \tau\). The dots represent the initial position in the risky asset. The final position is always zero. The grey line represents the Merton position \(\mu/(\alpha \sigma^2) = 10\).
Conclusions

- price impact model which accounts for finite market depth, market tightness, finite resilience (transient price impact)
- coupled bid-/ask-price dynamics induce convex liquidity costs on singular controls
- general existence and uniqueness result of optimal investment strategies
- illiquid Bachelier model with exponential utility: resulting singular control problem reduces to a deterministic optimal tracking problem
- exploit convex analytic approach instead of more common dynamic programming methods
- state space and optimal strategies can be constructed explicitly
- surprisingly rich phenomenology of possible trajectories for the optimal share holdings
Reference

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Thank you very much!