Unbiased Monte Carlo Estimation for Multivariate Jump-Diffusions

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Background

- Jump-Diffusion processes are widely used in finance as models for asset prices, interest and exchange rates, and the timing of defaults.
- Jump-Diffusion processes are rarely tractable, so **Monte Carlo** methods are often used to treat the pricing, risk management, and statistical estimation problems arising in applications.
- We develop an **unbiased estimator** for jump-diffusions with *general and state-dependent drift, volatility and jump coefficients*.

Jump-Diffusion

Let X be a Markov process on $\mathcal{D} = \mathbb{R}^d$ solving the SDE.

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t + dJ_t \tag{1}$$

defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$.

The $\mu : \mathcal{D} \to \mathbb{R}^d$, $\sigma : \mathcal{D} \to \mathbb{R}^{d\times m}$ are the drift and volatility coefficients and W is a m-dimensional standard Brownian motion. The jump term,

$$J_t = \sum_{n=1}^{N_t} h(X_{T_n^-}, Z_n) \tag{2}$$

- N_t is a non-explosive counting process with stopping times $(T_n)_{n\geq 1}$ and state-dependent intensity $\Lambda(X)$ for $\Lambda : \mathcal{D} \to (0, \infty)$.
- $h : \mathcal{D} \times \mathcal{M} \to \mathcal{D}$ governs the jump sizes, and $(Z_n)_{n\geq 1}$ is a sequence of i.i.d $\mathcal{F}_{T_n}$-measurable random variables with law ν on $\mathcal{M} \subseteq \mathcal{D}$.

Objective

We wish to compute the expectation $\mathbb{E}[f(X_T)]$ for a fixed $T > 0$ and a suitable class of "payoff functions" $f : \mathcal{D} \to \mathbb{R}$.

- Numerical Approaches
  - Direct analytical computation (limited scope)
  - Semi-analytic Fourier techniques (curse of dimensionality)
  - Numerical solution of the PIDE associated with $\mathbb{E}[f(X_T)]$

- Monte Carlo (widest scope, appropriate for higher dimension)
  - Discretization method (discretization bias + statistical error)
  - Exact method (only statistical error)
    - Exact sampling: sample X_T according to its true law
    - **Unbiased estimation**: estimator Y s.t. $\mathbb{E}[Y] = \mathbb{E}[f(X_T)]$

Literature Review

- Discretization method
  - Diffusions: extensive, see Kloeden & Platen (1999)
  - Jump-Diffusions: Glasserman & Merener (2004), Giesecke et al (2015)
- 1-dim exact method for diffusion and jump-diffusion
  - Beskos & Roberts (2005), Chen & Huang (2013)
  - Casella & Roberts (2013), Giesecke & Smelov (2013), and Goncalves & Roberts (2014)
- d-dim exact method for diffusion
  - Exact sampling: Blanchet & Zhang (2017)
    - infinite expected running time
  - Unbiased estimation: Wagner (1989), Bally & Higa (2015), Labordere et al (2017), and Rhee & Glynn (2015)
    - demanding conditions for coefficients

No results on (general state-dependent coefficient) multivariate jump diffusions yet, to our knowledge.

Our Work

We combine three ingredients to develop our unbiased estimator for jump diffusion of general coefficients:

- Parametrix method
- Change of measure
- Iterative Monte-Carlo scheme

Parametrix Method

Consider estimating $\mathbb{E}[f(Y_T)]$, where Y_t is a diffusion following

$$dY_t = \mu(Y_t)dt + \sigma(Y_t)dW_t$$

Key idea : approximate Y by its Euler process Y^π, and debias by using some "weight function" $\theta_t(\cdot,\cdot)$. The **parametrix formula** states that

$$\mathbb{E}[f(Y_T)] = e^T \mathbb{E}\left[f\left(Y^\pi_T\right) \prod_{j=0}^{K_T-1} \theta_{\tau_{j+1}-\tau_j}(Y^\pi_{\tau_j}, Y^\pi_{\tau_{j+1}}) \right] \tag{3}$$

- K_T is a standard Poisson process and arrival times $\tau_1, \cdots, \tau_{K_T}$.
- Y^π is the Euler approximation to Y on $0 < \tau_1 < \cdots \tau_{K_T} < T$.

Cannot apply the parametrix method to a jump diffusion directly because its generator in the presence of jumps leads to technical challenges.

Change of measure

Let $T_1, \cdots, T_{N_T}$ denote the jump times by horizon T. Let,

$$L_T = \exp\left(\int_0^T -\left(\Lambda(X_s) - \Lambda(X_{T_{N_s}})\right) ds\right) \prod_{n=1}^{N_t} \frac{\Lambda(X_{T_n^-})}{\Lambda(X_{T_{n-1}})}$$

Theorem 3.1 of Giesecke & Shkolink (2018) combined with Kolmogorov's extension theorem yields that $\mathbb{E}\left[1/L_T\right] = 1$. Therefore, we can define

$$\mathbb{Q}(\mathcal{A}) = \mathbb{E}[1_{\mathcal{A}}/L_T] \quad \mathcal{A} \in \mathcal{F}_T$$

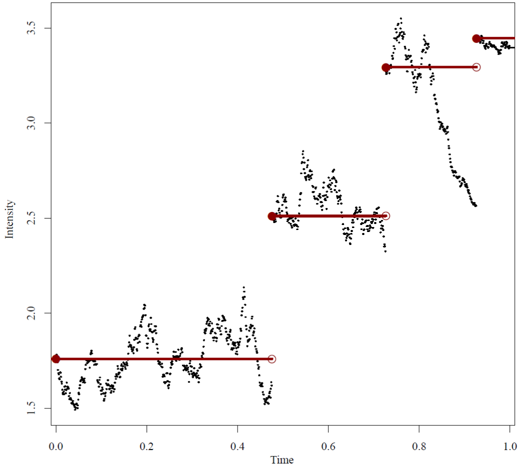
a probability measure on $\mathcal{F}_T$. Furthermore, we have

$$\mathbb{E}[f(X_T)] = \mathbb{E}_{\mathbb{Q}}[f(X_T)L_T]$$

.

Change of intensity

Under $\mathbb{Q}$ the paths of the intensity of the jump counting process N are step function $s \to \Lambda(X_{T_{N_s}})$. This facilitates exact sampling of the jump times!



Iterative Monte-Carlo Scheme

Now go back to our problem of estimating $\mathbb{E}_x[f(X_T)]$
Denote T_i the i-th jump time of X_t, assuming we start from $t = 0$, it is clear that

$$\mathbb{E}_x[f(X_T)] = \mathbb{E}_x[f(X_T)1_{\{T_1 \leq T\}}] + \mathbb{E}_x[f(X_T)1_{\{T_1 > T\}}] \tag{4}$$

Applying change of measure and taking $g(X_{T_1}) = \mathbb{E}_{\mathcal{F}_{T_1}}[f(X_T)]$ will give us

$$\mathbb{E}_x[f(X_T)1_{\{T_1 \leq T\}}] = \mathbb{E}_x^{\mathbb{Q}}\left[1_{\{T_1 \leq T\}} L_{T_1} g(X_{T_1})\right] \tag{5}$$

Main idea: sample T_1 under measure $\mathbb{Q}$, then estimate $\mathbb{E}_x[f(X_T)1_{\{T_1 \leq T\}}]$ and $\mathbb{E}_x[f(X_T)1_{\{T_1 > T\}}]$ separately conditioned on T_1. Estimate $g(X_{T_1}) = \mathbb{E}_{\mathcal{F}_{T_1}}[f(X_T)]$ the same way as (4), hence moving forward iteratively till $T_n > T$.

Estimation of $\mathbb{E}_x[f(X_T)1_{\{T_1 \le T\}}]$

Recall

$$\mathbb{E}_x[f(X_T)1_{\{T_1 \le T\}}] = \mathbb{E}_x^{\mathbb{Q}}\left[1_{\{T_1 \le T\}} L_{T_1} g(X_{T_1})\right] \tag{5}$$

This could be estimated unbiasedly if we can apply parametrix method to

$$L_{T_1} g(X_{T_1}) = \exp\left(-\int_0^{T_1} \Lambda(X_s)ds + \Lambda(x)T_1\right) \frac{\Lambda(X_{T_1^-})}{\Lambda(x)} g(X_{T_1}) \tag{6}$$

above is a function of $\int_0^{T_1} \Lambda(X_s)ds$ and $X_{T_1^-}$, but the law of $\int_0^{T_1} \Lambda(X_s)ds$ is not trivial.

Apply Parametrix Method

Recall the key observation of parametrix method: for any process that we don't know its law, we can approximate the law by Euler's method.
So we can look at a different process: $\bar{Y}_t = (\bar{X}_t, \bar{Z}_t)$, such that

$$\begin{aligned} d\bar{X}_t &= \mu(\bar{X}_t)dt + \sigma(\bar{X}_t)dW_t \\ d\bar{Z}_t &= \Lambda(\bar{X}_t)dt \\ \bar{X}_0 &= x, \ \bar{Z}_0 = z \end{aligned} \tag{7}$$

After sampling T_1 under measure Q, estimation of (6) is equivalent to estimating

$$\tilde{f}(\bar{Y}_T) = \exp\left(-\left(\bar{Z}_{T_1} - \Lambda(x)T_1\right)\right)\frac{\Lambda(\bar{X}_{T_1})}{\Lambda(x)}g(\bar{X}_{T_1} + \Delta(\bar{X}_{T_1})) \tag{8}$$

Estimation for the second term

Recall the equation

$$\mathbb{E}_x[f(X_T)] = \mathbb{E}_x[f(X_T)1_{\{T_1 \leq T\}}] + \mathbb{E}_x[f(X_T)1_{\{T_1 > T\}}] \tag{5}$$

We already get an estimator for $\mathbb{E}_x[f(X_T)1_{\{T_1 \leq T\}}]$, the estimation of $\mathbb{E}_x[f(X_T)1_{\{T_1 > T\}}]$ will be simpler and in a similar spirit. Reuse the framwork $\bar{Y}_t = (\bar{X}_t, \bar{Z}_t)$. Notice we can apply parametrix method to

$$\mathbb{E}_x[f(X_T)1_{\{T_1 > T\}}] = \mathbb{E}\left[\exp\left(-\int_0^T \Lambda(X_s)ds\right) f(X_T)\right]$$

Main Result

Theorem 1

For $f \in C(\mathbb{R}^d)$ with sub-Gaussian growth, $\mu(x) \in C_b^2$, $\sigma(x) \in C_b^2$, and $\lambda(x)$ being sub-Gaussian. Let X be a jump-diffusion process following

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t + dJ_t, \quad J_t = \sum_{n=1}^{N_t} h(X_{T_n^-}, Z_n)$$

For $T > 0$, we have

$$\mathbb{E}[\Xi] = \mathbb{E}[f(X_T)]$$

where Ξ has the expression

$$\sum_{i=1}^{N_t} \left(\prod_{j=1}^{i-1} \left(\exp\left(\bar{Z}^{\pi\ (j,1)}_{T_j - T_{j-1}} - \Lambda(x_j)(T_j - T_{j-1}) \right) \frac{\Lambda(x_j)}{\Lambda(\bar{X}^{\pi\ (j,1)}_{T_j - T_{j-1}})} \prod_{l=0}^{K^{(j,1)}_{T_j - T_{j-1}} - 1} \theta_{\tau^{(j,1)}_{l+1} - \tau^{(j,1)}_l}(\bar{Y}^{\pi\ (j,1)}_{\tau^{(j,1)}_l}, \bar{Y}^{\pi\ (j,1)}_{\tau^{(j,1)}_{l+1}}) \right) \right.$$
$$\left. \times \exp\left(-\bar{Z}^{\pi\ (i,2)}_{T - T_{i-1}} \right) f(\bar{X}^{\pi\ (i,2)}_{T - T_{i-1}}) \prod_{l=0}^{K^{(i,2)}_{T - T_{i-1}} - 1} \theta_{\tau^{(i,2)}_{l+1} - \tau^{(i,2)}_l}(\bar{Y}^{\pi\ (i,2)}_{\tau^{(i,2)}_l}, \bar{Y}^{\pi\ (i,2)}_{\tau^{(i,2)}_{l+1}}) \right)$$

Numerical experiment

In Bally's paper, the assumptions are that $\mu(x), \sigma(x)$ should be bounded smooth, but it is hard to find the exact law for $f(X_T)$ given that assumption.

One thing we can do is to let $f(x) = 1$ and our estimator should have expectation 1. This is to test if the mechanism of our algorithm is correct.

If we pass that test, then we can try to simulate some process where we have an analytical solution for $\mathbb{E}[f(X_T)]$. Here we let $f(x) = x$ and X_t follows an affine jump-diffusion process.

Numerical experiment: $f(x) = 1$

Our first experiment use the setting

$$\mu(x) = 2 + \sin(x)$$

$$\sigma(x) = \sqrt{2 + \sin(x)}$$

$$\lambda(x) = 2 + \sin(x)$$

ntrials	Error	Variance	99 CI
2^{20}	-0.0133	799	0.07
2^{24}	-0.0118	2157	0.029
2^{26}	0.0018	1458	0.012
2^{28}	-0.00139	2195	0.0073
2^{30}	-0.00097	2195	0.0035
2^{32}	-0.00047	3243	0.0022

Numerical experiment: $f(x) = 1$

Our second experiment use the affine model

$$\mu(x) = 1 - x$$

$$\sigma(x) = \sqrt{2 + x}$$

$$\lambda(x) = \lambda(x) = 1 + x$$

ntrials	Error	Variance	99 CI
2^{20}	-0.005672	941	0.077
2^{24}	-0.010449	1004	0.01996
2^{26}	0.001205	2146	0.01458
2^{28}	-0.00066	3048	0.00869
2^{30}	-0.00016	3671	0.00477
2^{32}	-0.00042	3099	0.00219
2^{34}	0.000384	4369	0.00130

Numerical experiment: $f(x) = x$

In the affine model, we are able to calculate $\mathbb{E}[X_T]$ explicitly. Let the jump size be uniform on $\{0.5, 0.1\}$, $\mu(x) = 1 - x$, $\sigma(x) = \sqrt{2 + x}$ and $\lambda(x) = 1 + x$. We can solve analytically that $\mathbb{E}[X_1] = 3.105996$.

Numerical experiment: $f(x) = x$

Let the jump size be uniform on $\{0.5, 0.1\}$

$$\mu(x) = 1 - x$$

$$\sigma(x) = \sqrt{2 + x}$$

$$\lambda(x) = 1 + x$$

ntrials	Error	Variance	99 CI
2^{20}	0.05704	31818	0.4494
2^{24}	−0.07943	38575	0.1237
2^{26}	0.031501	439663	0.2088
2^{28}	−0.0254	217940	0.0735
2^{30}	−0.01132	127982	0.02817
2^{32}	−0.01102	175582	0.01650
2^{34}	−0.00203	350492	0.01165

Conclusion

By adopting Parametrix method and change of measure in an iterative Monte Carlo scheme, We develop an unbiased Monte Carlo estimator for multivariate Jump Diffusions and it works for general processes. We think the future work could be

- Analyze the best choice of parameters that will minimize variance.
- This change of measure setting could be applied to other unbiased algorithm for diffusions.