Minimum Drawdown Portfolios

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What is Drawdown?

Figure: Drawdown example\(^1\)

\(^1\)Goldberg and Mahmoud 2016
Why drawdown matters

- Widely used indicator of risk in the fund management industry
- Levered investor can get caught in a liquidity trap: forced to sell valuable positions if unable to secure funding after an abrupt market decline
- Pathwise: end horizon risk diagnostics like volatility, Value-at-Risk, and Expected Shortfall are less significant conditioned on a large drawdown
What this work is about?

- Risk based asset allocation using Conditional Expected Drawdown (CED)\(^2\)
- The challenge of computing the CED
- Optimal CED portfolios and risk attribution
- Properties of optimal CED portfolios and a new balance of risk and return

\(^2\)Goldberg and Mahmoud 2016
Risk measure $\delta : \mathcal{L} \mapsto \mathbb{R}$

- **Normalization**: for all constant deterministic $C \in \mathcal{L}$, $\delta(C) = 0$.
- **Positivity**: for all $X \in \mathcal{L}$, $\delta(X) \geq 0$.
- **Shift invariance**: for all $X \in \mathcal{L}$ and all constant deterministic $C \in \mathcal{L}$, $\delta(X + C) = \delta(X)$.
- **Convexity**: for all $X, Y \in \mathcal{L}$ and $\lambda \in [0, 1]$, $\delta(\lambda + (1 - \lambda)Y) \leq \lambda \delta(X) + (1 - \lambda) \delta(Y)$.
- **Positive degree-one homogeneity**: for all $X \in \mathcal{L}$ and $\lambda > 0$, $\delta(\lambda X) = \lambda \delta(X)$. 
• Realizations of a value process: $V_t, t \in [0, T]$

• Loss function $\rho(V)$

• Value-at-Risk for $\rho$:

$$\beta_\alpha = \inf\{\beta : \mathbb{P}(\rho(V) \leq \beta) \geq \alpha\}$$

• Conditional-Value-at-Risk for $\rho$:

$$\text{CVaR}_\alpha = (1 - \alpha)^{-1} \mathbb{E}[\rho(V) \mid \rho(V) \geq \beta_\alpha]$$
Setup & Definitions

Return & Max Drawdown Loss

- Return Loss: $\rho(V) = -V_T$
- Return Loss yields traditional VaR and CVaR (aka Expected Shortfall)
- Drawdown:
  \[ D_t = M_t - V_t \]
  where $M_t = \sup_{s \in [0,T]} V_s$.
- Drawdown-on-Drawdown process
  \[ DD_t = MD_t - D_t \]
- Maximum Drawdown Loss:
  \[
  \rho(V) = \sup_{s \in [0,T]} D_s = \sup_{s \in [0,T]} \left\{ \sup_{r \in [0,s]} V_r - V_s \right\}
  \]
Conditional Expected Drawdown as a path-dependent risk measure\(^3\)

- \(\rho(V) = \sup_{s \in [0, T]} D_s\), \(D_t = M_t - V_t\), \(M_t = \sup_{s \in [0, T]} V_s\).
- Max Drawdown-at-Risk:
  \[
  DT_\alpha(V) = \inf\{\gamma \mid \mathbb{P}(\rho(V) > \gamma) \leq 1 - \alpha\}
  \]
- Conditional Expected Drawdown (CED):
  \[
  CED_\alpha(V) = \mathbb{E}[\rho(V) \mid \rho(V) > DT_\alpha(\rho(V))]
  \]

\(^3\)Goldberg and Mahmoud 2016
Average drawdown

\[ \text{ADD} = \frac{1}{T} \int_{0}^{t} D_s ds \]

Conditional-Drawdown (CDD)\(^4\): mean of the worst \((1 - \alpha) \times 100\%\) drawdowns

Portfolio problems routinely impose these measures in a linear programming approach

ADD or CDD smooths out drawdown while CED is purely about extreme risk

\(^4\)Chekhlov, Uryasev, and Zabarankin 2005
Optimizing CVaR Tail Risk

From Rockafellar and Uryasev 2000, consider,

\[ \Phi_\alpha(\pi, y) = y + (1 - \alpha)^{-1} \mathbb{E}[(\rho(R^\pi) - y)^+] \]

Equivalent results:

\[
\min_{\pi \in \Pi} \text{CVaR}_\alpha(\pi) = \min_{(\pi, y) \in (\Pi, \mathbb{R}^+)} \Phi_\alpha(\pi, y) \quad (1)
\]

\[
(\pi^*, \beta_\alpha) = \arg \min_{(\pi, y) \in (\Pi, \mathbb{R}^+)} \Phi_\alpha(\pi, y) \quad (2)
\]

where \( \Pi \) are portfolio constraints
Loss function $\rho$ is quite general, i.e. it can be horizon-wise, path-wise, non-convex.

Convexity is obtained through the tail expectation\(^5\).

For $\rho(R^\pi) = -R^\pi_T$, we get traditional CVaR aka Expected Shortfall.

For $\rho(R^\pi) = \sup_{s \in [0, \tau]} \left\{ \sup_{r \in [0, s]} R^\pi_r - R^\pi_s \right\}$, we get CED.

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\(^5\)Rockafellar and Uryasev 2000
Optimization problem

\[
\begin{align*}
\text{minimize} & \quad \Phi_\alpha(\pi, y) \\
\text{subject to} & \quad \pi \geq -\gamma, \pi^T \mathbb{1} = 1, \pi^T \mu \geq R
\end{align*}
\]

where \(\gamma\) is short sell limit, \(R\) is target return, \(\mu\) is expected return, and

\[
\Phi_\alpha(\pi, y) = y + (1 - \alpha)^{-1} \mathbb{E} \left[ (\rho(R^\pi) - y)^+ \right]
\]

\(\Rightarrow\) We need to compute \(\Phi_\alpha(\pi, y)\)
Computing $\Phi_\alpha(\pi, \beta)$

Reflection Process

- Drawdown process $D_t$ is the Skorohod reflection process for return process.
- For now restrict to Normal returns:
  \[
  R_t^\pi = \pi^T \mu t + \pi^T CB_t
  \]
  where $\mu$ is expected return and covariance $\Sigma = CC^T$ and
  $B_t$ is a $d$-dimensional Brownian Motion.
- Brownian return process $\Rightarrow$ reflected Brownian drawdown process
  \[
  dR_t^\pi = dM_t - \pi^T \mu dt - \pi^T CdB_t
  \]
  where the maximum process $M_t$ is a local time at 0 creating the reflection.
Computing  $\Phi_{\alpha}(\pi, \beta)$

**PDE Formulation**

$u(t, x, y) = \mathbb{E}[(MD_T - MD_t)^+ | D_t = x, DD_t = y]$ solves a mixed BVP:

$$
\begin{align*}
  u_t + \mathcal{L}_\pi u &= 0, \quad t < T, \\
  u(T, x, y) &= 0, \quad (x, y) \in \mathbb{R}_2^+ \\
  u_y(t, 0, y) - u_x(t, 0, y) &= 0, \quad t \in [0, T), \quad y > 0 \\
  -u_y(t, x, 0) &= 1, \quad t \in [0, T), \quad x > 0,
\end{align*}
$$

(3)

Infinitesimal generator:

$$
\mathcal{L}_\pi u = \mu_\pi [u_y - u_x] + \frac{1}{2} \sigma_\pi^2 [u_{xx} - 2u_{xy} + u_{yy}]
$$

$$
\mu_\pi = \pi^T \mu, \quad \sigma_\pi^2 = \pi^T \Sigma \pi
$$
**Computing $\Phi_\alpha(\pi, \beta)$**

**Change of Variables**

Define,

$$w(t, x, y) = \frac{\rho}{2} u \left( T(1 - t/\lambda), x/\rho, y/\rho \right),$$

$$\lambda = \frac{\mu^2 T}{2\sigma^2}, \quad \sqrt{\lambda} = \frac{|\mu| \sqrt{T}}{\sqrt{2}\sigma}, \quad \rho = \frac{|\mu|}{\sigma^2}.$$

$w(t, x, y)$ satisfies the new parameterless PDE,

$$w_t - \mathcal{L}w = 0, \quad t > 0$$

$$w(0, x, y) = 0, \quad (x, y) \in \mathbb{R}^+_2,$$

$$w_y(t, 0, y) - w_x(t, 0, y) = 0, \quad y > 0,$$

$$-w_y(t, x, 0) = \frac{1}{2}, \quad x > 0.$$

where the new PDE operator $\mathcal{L}$ is given by,

$$\mathcal{L}w = 2 \cdot \text{sign}(\mu) [w_y - w_x] + [w_{xx} - 2w_{xy} + w_{yy}].$$
The key quantity of interest:

\[ E[(MD_T - y)^+ | D_0 = 0, MD_0 = y] \]

\[ = F(y, \mu, \sigma^2, T) = \begin{cases} 
\frac{2}{\rho} Q(\lambda, \rho y) & \mu < 0 \\
\frac{2}{\rho} Q(\lambda, \rho y) & \mu > 0 \\
2\sigma^2 Q\left(\frac{T}{2\sigma^2}, \frac{y}{\sigma^2}\right) & \mu = 0
\end{cases} \]

where the function \( Q(r, s) \) is defined as

\[ Q(r, s) = \begin{cases} 
-w^-(r, 0, s) & \mu < 0 \\
w^+(r, 0, s) & \mu > 0 \\
w^0(r, 0, s) & \mu = 0
\end{cases} \]
Computing $\Phi_\alpha(\pi, \beta)$

Numerical PDE

- Can solve for $w^-, w^+, \text{ and } w^0$ and store the results for later use.
- Use FEniCS finite element library to produce solution
So what can we say about drawdown and optimal drawdown portfolios?

- How do optimal CED portfolios compare to mean-variance or mean-shortfall portfolios?
- How do optimal CED portfolios depend on horizon $T$?
Properties of Max Drawdown Portfolios

- Optimal CED portfolio is on the mean-variance frontier
- Optimal CED converges to Minimum Variance as $T \rightarrow 0$
Properties of Max Drawdown Portfolios

- Assume a 1-Factor model:
  \[ R^i_j = a^i + \beta^i \Gamma^j + e^i_j, \]
  \[ \mu = a + \mu_m \beta, \quad \Sigma = \sigma^2_{m \beta \beta^T} + \Delta \]

- Market factor:
  \[ \mu_m = 0.05, \sigma_m = 0.16 \]

- Excess Return, loadings, and idiosyncratic risk:
  \[ a_i \sim N(0, 0.05), \quad \beta_i \sim N(1, 0.1), \]
  \[ \sqrt{\Delta_{ii}} \sim \text{Unif}[0.32, 0.64] \]

- Parameters are reasonable approximations to empirical observations
Properties of Max Drawdown Portfolios

- Optimal CED portfolio is a balance of Risk and Return

![Efficient Frontier](image.png)
Properties of Max Drawdown Portfolios

Other Features

Compared to minimum shortfall portfolios...

- Optimal CED portfolio are relatively indifferent to $\alpha$ (this is likely due to Normal returns)
- Optimal CED portfolio are relatively indifferent to expected return (shortfall portfolios tilt more on security return)
Marginal Risk Contribution (MRC) of component $i^6$:

$$\text{MRC}_{i}^{\text{CED} \alpha}(R^{\pi}) = \frac{\partial \text{CED} \alpha(R^{\pi})}{\partial \pi_i}$$

Exact probabilistic expression for MRC $^7$:

$$\text{MRC}_{i}^{\text{CED} \alpha}(R^{\pi}) = \mathbb{E}[F_{s^*}^{i} - F_{t^*}^{i} | \rho(P) > DT_\alpha(\rho(P))]$$

where $0 \leq s^* < t^* \leq T$ such that $\rho(P) = P_{s^*} - P_{t^*}$

From this work:

$$\text{MRC}_{i}^{\text{CED} \alpha}(R^{\pi}) = \nabla_\pi \Phi(\pi, \beta_\alpha)$$

for some portfolio $\pi$ and $\beta_\alpha = \arg \min_y \Phi(\pi, y)$.

$^7$Goldberg and Mahmoud 2016
Beyond Normal or Elliptical Returns
Monte Carlo and Linear Programming

Optimal CED Linear Program

\[
\min_{\pi, y, z, u} \quad y + \frac{1}{K} \sum_{i=1}^{M} z_i \\
\text{s.t.} \quad z_i + y \geq u_{i,j} - \pi^T R_{i,j}, \\
\quad z_i \geq 0, \quad u_{i,0} = 0, \quad u_{i,j} \geq 0,
\]

- Vector of returns at time \( t_j \) in sim \( i \): \( R_{i,j} \)
- \( M \) total simulations and \( K = \lfloor M\alpha \rfloor \)
- Need Monte Carlo generation of return vectors
Beyond Normal or Elliptical Returns

Leverage Effect

Example: One-factor Heston market model:

\[ R_t = \alpha + \beta \Gamma_t + \epsilon_t, \]
\[ R_t, \alpha, \beta, \epsilon_t \in \mathbb{R}^n, \quad \Gamma_t \in \mathbb{R}, \]
\[ d\Gamma_t = \mu dt + \sqrt{\nu_t} dB^\Gamma_t, \quad d\nu_t = \kappa(\theta - \nu_t) dt + \xi \sqrt{\nu_t} dB^\nu_t, \]
\[ \langle dB^\Gamma, dB^\nu \rangle_t = \rho dt \]

- Stochastic volatility and \( \rho \) should create asymmetry in returns
- Leverage effect for \( \rho < 0 \)
- \( E[(\rho(R^\pi) - y)^+] \) and related quantities from numerical PDE solutions
- Differentiate PDE with respect to \( \pi \) for gradients/MRC
Beyond Normal or Elliptical Returns

One-factor Heston market model
Beyond Normal or Elliptical Returns

- MC approach doesn’t scale well unless an efficient scheme can be developed: for 95% CED level, only 5% of simulations are used
- PDE approach not prone to MC error but still computationally intensive
- Monte Carlo more flexible if a PDE solution cannot be formulated; would still require calibration of any parameters
Concluding Remarks

- Computation of optimal CED portfolios as well as risk attribution
- Optimal CED portfolios overlap with mean-variance portfolios; tend to be near the Minimum Variance portfolio; much less sensitive to expected returns
- To be addressed...
  - Non-elliptical returns likely key to more novel portfolio allocations
  - Estimation uncertainty and whether more optimal portfolios can even be found

