Minimum Drawdown Portfolios

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What is Drawdown?

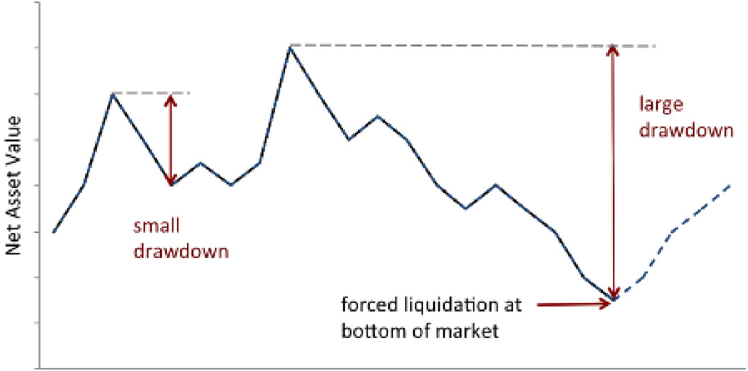


Figure: Drawdown example[1]

[1] Goldberg and Mahmoud 2016

Why drawdown matters

- Widely used indicator of risk in the fund management industry
- Levered investor can get caught in a liquidity trap: forced to sell valuable positions if unable to secure funding after an abrupt market decline
- Pathwise: end horizon risk diagnostics like volatility, Value-at-Risk, and Expected Shortfall are less significant conditioned on a large drawdown

What this work is about?

- Risk based asset allocation using Conditional Expected Drawdown (CED)[2]
- The challenge of computing the CED
- Optimal CED portfolios and risk attribution
- Properties of optimal CED portfolios and a new balance of risk and return

[2] Goldberg and Mahmoud 2016

Setup & Definitions
Risk Measures

Risk measure $\delta : \mathcal{L} \mapsto \mathbb{R}$

- Normalization: for all constant deterministic $C \in \mathcal{L}$, $\delta(C) = 0$.
- Positivity: for all $X \in \mathcal{L}$, $\delta(X) \geqslant 0$.
- Shift invariance: for all $X \in \mathcal{L}$ and all constant deterministic $C \in \mathcal{L}$, $\delta(X + C) = \delta(X)$.
- Convexity: for all $X, Y \in \mathcal{L}$ and $\lambda \in [0, 1]$, $\delta(\lambda + (1 - \lambda)Y) \leqslant \lambda\delta(X) + (1 - \lambda)\delta(Y)$.
- Positive degree-one homogeneity: for all $X \in \mathcal{L}$ and $\lambda > 0$, $\delta(\lambda X) = \lambda\delta(X)$.

Tail Risk

General VaR and CVaR Measures

- Realizations of a value process: $V_t, t \in [0, T]$
- Loss function $\rho(V)$
- Value-at-Risk for ρ:

$$\beta_\alpha = \inf\{\beta : \mathbb{P}(\rho(V) \leqslant \beta) \geqslant \alpha\}$$

- Conditional-Value-at-Risk for ρ:

$$\mathrm{CVaR}_\alpha = (1-\alpha)^{-1}\mathbb{E}[\rho(V) \mid \rho(V) \geqslant \beta_\alpha]$$

Setup & Definitions

Return & Max Drawdown Loss

- Return Loss: $\rho(V) = -V_T$
- Return Loss yields traditional VaR and CVaR (aka Expected Shortfall)
- Drawdown:

$$D_t = M_t - V_t$$

where $M_t = \sup_{s\in[0,T]} V_s$.

- Drawdown-on-Drawdown process

$$\mathrm{DD}_t = \mathrm{MD}_t - D_t$$

- Maximum Drawdown Loss:

$$\rho(V) = \sup_{s\in[0,T]} D_s = \sup_{s\in[0,T]} \left\{ \sup_{r\in[0,s]} V_r - V_s \right\}$$

Setup & Definitions

Max Drawdown-at-Risk and Conditional Expected Drawdown

Conditional Expected Drawdown as a path-dependent risk measure [3]

- $\rho(V) = \sup_{s\in[0,T]} D_s,\ \ D_t = M_t - V_t,\ \ M_t = \sup_{s\in[0,T]} V_s.$
- Max Drawdown-at-Risk:

$$\mathrm{DT}_\alpha(V) = \inf\{\gamma \mid \mathbb{P}(\rho(V) > \gamma) \leqslant 1-\alpha\}$$

- Conditional Expected Drawdown (CED):

$$\mathrm{CED}_\alpha(V) = \mathbb{E}[\rho(V) \mid \rho(V) > \mathrm{DT}_\alpha(\rho(V))]$$

[3] Goldberg and Mahmoud 2016

Setup & Definitions

Other Drawdown Risk Measures

- Average drawdown

$$\mathrm{ADD} = \frac{1}{T}\int_0^t D_s ds$$

- Conditional-Drawdown (CDD)[4]: mean of the worst $(1-\alpha)\times 100\%$ drawdowns
- Portfolio problems routinely impose these measures in a linear programming approach
- ADD or CDD smooths out drawdown while CED is purely about extreme risk

[4] Chekhlov, Uryasev, and Zabarankin 2005

Optimizing CVaR Tail Risk

- From Rockafellar and Uryasev 2000, consider,

$$\Phi_\alpha(\pi, y) = y + (1-\alpha)^{-1}\mathbb{E}[(\rho(R^\pi) - y)^+]$$

- Equivalent results:

$$\min_{\pi\in\Pi} \mathrm{CVaR}_\alpha(\pi) = \min_{(\pi,y)\in(\Pi,\mathbb{R}^+)} \Phi_\alpha(\pi, y) \quad (1)$$

$$(\pi^*, \beta_\alpha) = \arg\min_{(\pi,y)\in(\Pi,\mathbb{R}^+)} \Phi_\alpha(\pi, y) \quad (2)$$

where Π are portfolio constraints

Optimizing CVaR Tail Risk

- Loss function ρ is quite general, ie. it can be horizon-wise, path-wise, non-convex
- Convexity is obtained through the tail expectation[5]
- For $\rho(R^\pi) = -R^\pi_T$, we get traditional CVaR aka Expected Shortfall
- For $\rho(R^\pi) = \sup_{s\in[0,T]} \left\{ \sup_{r\in[0,s]} R^\pi_r - R^\pi_s \right\}$, we get CED

[5] Rockafellar and Uryasev 2000

Risk-Based Asset Allocation

Optimizing CED

Optimization problem

$$\begin{aligned} \underset{\pi, y}{\text{minimize}} \quad & \Phi_\alpha(\pi, y) \\ \text{subject to} \quad & \pi \geqslant -\gamma, \pi^T \mathbb{1} = 1, \pi^T \mu \geqslant R \end{aligned}$$

where γ is short sell limit, R is target return, μ is expected return, and

$$\Phi_\alpha(\pi, y) = y + (1-\alpha)^{-1} \mathbb{E}\left[(\rho(R^\pi) - y)^+ \right]$$

$\Rightarrow$ We need to compute $\Phi_\alpha(\pi, y)$

Computing $\Phi_\alpha(\pi, \beta)$

Reflection Process

- Drawdown process D_t is the Skorohod reflection process for return process
- For now restrict to Normal returns:

$$R_t^\pi = \pi^T \mu t + \pi^T C B_t$$

where μ is expected return and covariance $\Sigma = CC^T$ and B_t is a d-dimensional Brownian Motion.

- Brownian return process $\Rightarrow$ reflected Brownian drawdown process

$$dR_t^\pi = dM_t - \pi^T \mu dt - \pi^T C dB_t$$

where the maximum process M_t is a local time at 0 creating the reflection.

Computing $\Phi_\alpha(\pi, \beta)$
PDE Formulation

$u(t,x,y) = \mathbb{E}\left[(\mathrm{MD}_T - \mathrm{MD}_t)^+ \mid D_t = x, \mathrm{DD}_t = y\right]$ solves a mixed BVP:

$$\begin{aligned} u_t + \mathcal{L}_\pi u &= 0, \quad t < T, \\ u(T,x,y) &= 0, \quad (x,y) \in \mathbb{R}_2^+ \\ u_y(t,0,y) - u_x(t,0,y) &= 0, \quad t \in [0,T), \quad y > 0 \\ -u_y(t,x,0) &= 1, \quad t \in [0,T), \quad x > 0, \end{aligned} \tag{3}$$

Infinitesimal generator:

$$\mathcal{L}_\pi u = \mu_\pi[u_y - u_x] + \frac{1}{2}{\sigma_\pi}^2[u_{xx} - 2u_{xy} + u_{yy}]$$
$$\mu_\pi = \pi^T\mu, \;\; {\sigma_\pi}^2 = \pi^T\Sigma\pi$$

Computing $\Phi_\alpha(\pi, \beta)$

Change of Variables

Define,

$$w(t,x,y) = \frac{\rho}{2} u(T(1-t/\lambda), x/\rho, y/\rho),$$

$$\lambda = \frac{\mu^2 T}{2\sigma^2}, \quad \sqrt{\lambda} = \frac{|\mu|\sqrt{T}}{\sqrt{2}\sigma}, \quad \rho = \frac{|\mu|}{\sigma^2}.$$

$w(t,x,y)$ satisfies the new parameterless PDE,

$$\begin{aligned} w_t - \mathcal{L}w &= 0, \quad && t > 0 \\ w(0,x,y) &= 0, \quad && (x,y) \in \mathbb{R}_2^+, \\ w_y(t,0,y) - w_x(t,0,y) &= 0, \quad && y > 0, \\ -w_y(t,x,0) &= \tfrac{1}{2}, \quad && x > 0. \end{aligned} \tag{4}$$

where the new PDE operator $\mathcal{L}$ is given by,

$$\mathcal{L}w = 2 \cdot \mathrm{sign}(\mu)[w_y - w_x] + [w_{xx} - 2w_{xy} + w_{yy}].$$

Computing $\Phi_\alpha(\pi, \beta)$

- The key quantity of interest:

$$\mathbb{E}[(\mathrm{MD}_T - y)^+ \mid D_0 = 0, \mathrm{MD}_0 = y]$$

$$= F(y, \mu, \sigma^2, T) = \begin{cases} \frac{2}{\rho} Q(\lambda, \rho y) & \mu < 0 \\ \frac{2}{\rho} Q(\lambda, \rho y) & \mu > 0 \\ 2\sigma^2 Q\left(\frac{T}{2\sigma^2}, \frac{y}{\sigma^2}\right) & \mu = 0 \end{cases}$$

where the function $Q(r, s)$ is defined as

$$Q(r, s) = \begin{cases} w^-(r, 0, s) & \mu < 0 \\ w^+(r, 0, s) & \mu > 0 \\ w^0(r, 0, s) & \mu = 0. \end{cases}$$

Computing $\Phi_\alpha(\pi, \beta)$

Numerical PDE

- Can solve for w^-, w^+, and w^0 and store the results for later use.
- Use FEniCS finite element library to produce solution

So what can we say about drawdown and optimal drawdown portfolios?

- How do optimal CED portfolios compare to mean-variance or mean-shortfall portfolios?
- How do optimal CED portfolios depend on horizon T?

Properties of Max Drawdown Portfolios

- Optimal CED portfolio is on the mean-variance frontier
- Optimal CED converges to Minimum Variance as $T \to 0$

Properties of Max Drawdown Portfolios

- Assume a 1-Factor model:

$$R_j^i = a^i + \beta^i \Gamma_j + e_j^i,$$
$$\mu = a + \mu_m \beta, \ \Sigma = \sigma_m^2 \beta \beta^T + \Delta$$

- Market factor:

$$\mu_m = 0.05, \sigma_m = 0.16$$

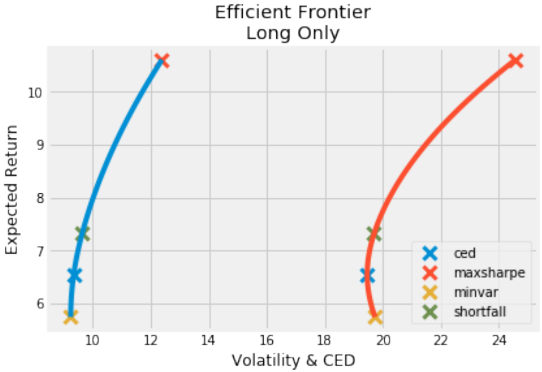
- Excess Return, loadings, and idiosyncratic risk:

$$a_i \sim \mathcal{N}(0, 0.05), \quad \beta_i \sim \mathcal{N}(1, 0.1),$$
$$\sqrt{\Delta_{ii}} \sim \mathrm{Unif}[0.32, 0.64]$$

- Parameters are reasonable approximations to empirical observations

Properties of Max Drawdown Portfolios

- Optimal CED portfolio is a balance of Risk and Return



Properties of Max Drawdown Portfolios

Other Features

Compared to minimum shortfall portfolios...

- Optimal CED portfolio are relatively indifferent to α (this is likely due to Normal returns)
- Optimal CED portfolio are relatively indifferent to expected return (shortfall portfolios tilt more on security return)

Drawdown Risk Attribution

- Marginal Risk Contribution (MRC) of component i[6]:

$$\mathrm{MRC}_i^{\mathrm{CED}_\alpha}(R^\pi) = \frac{\partial \mathrm{CED}_\alpha(R^\pi)}{\partial \pi_i}$$

- Exact probabilistic expression for MRC [7]:

$$\mathrm{MRC}_i^{\mathrm{CED}_\alpha}(R^\pi) = \mathbb{E}[F^i_{s^*} - F^i_{t^*} | \rho(P) > \mathrm{DT}_\alpha(\rho(P))]$$

  where $0 \leqslant s^* < t^* \leqslant T$ such that $\rho(P) = P_{s^*} - P_{t^*}$

- From this work:

$$\mathrm{MRC}_i^{\mathrm{CED}_\alpha}(R^\pi) = \nabla_\pi \Phi(\pi, \beta_\alpha)$$

  for some portfolio π and $\beta_\alpha = \arg\min_y \Phi(\pi, y)$.

[6] Goldberg, Hayes, et al. 2009

[7] Goldberg and Mahmoud 2016

Beyond Normal or Elliptical Returns

Monte Carlo and Linear Programming

Optimal CED Linear Program

$$
\begin{aligned}
\min_{\pi,y,z,u} \quad & y + \frac{1}{K}\sum_{i=1}^{M} z_i \\
\text{s.t.} \quad & z_i + y \geqslant u_{i,j} - \pi^T R_{i,j}, \\
& z_i \geqslant 0, \quad u_{i,0} = 0, \quad u_{i,j} \geqslant 0,
\end{aligned}
$$

- Vector of returns at time t_j in sim i: $R_{i,j}$
- M total simulations and $K = \lfloor M\alpha \rfloor$
- Need Monte Carlo generation of return vectors

Beyond Normal or Elliptical Returns

Leverage Effect

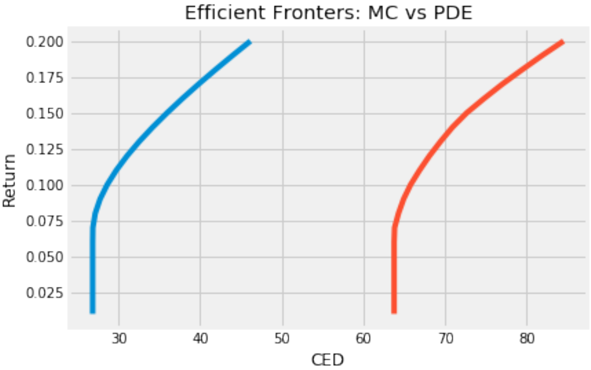
Example: One-factor Heston market model:

$$R_t = \alpha + \beta \Gamma_t + \epsilon_t,$$
$$R_t, \alpha, \beta, \epsilon_t \in \mathbb{R}^n, \quad \Gamma_t \in \mathbb{R},$$
$$d\Gamma_t = \mu dt + \sqrt{\nu_t} dB_t^\Gamma, \;\; d\nu_t = \kappa(\theta - \nu_t)dt + \xi\sqrt{\nu_t} dB_t^\nu,$$
$$\langle dB^\Gamma, dB^\nu \rangle_t = \rho dt$$

- Stochastic volatility and ρ should create asymmetry in returns
- Leverage effect for $\rho < 0$
- $E[(\rho(R^\pi) - y)^+]$ and related quantities from numerical PDE solutions
- Differentiate PDE with respect to π for gradients/MRC

Beyond Normal or Elliptical Returns

One-factor Heston market model



Beyond Normal or Elliptical Returns

- MC approach doesn't scale well unless an efficient scheme can be developed: for 95% CED level, only 5% of simulations are used
- PDE approach not prone to MC error but still computationally intensive
- Monte Carlo more flexible if a PDE solution cannot be formulated; would still require calibration of any parameters

Concluding Remarks

- Computation of optimal CED portfolios as well as risk attribution
- Optimal CED portfolios overlap with mean-variance portfolios; tend to be near the Minimum Variance portfolio; much less sensitive to expected returns
- To be addressed...
  - Non-elliptical returns likely key to more novel portfolio allocations
  - Estimation uncertainty and whether more optimal portfolios can even be found

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