Functional portfolio generation

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My collaborators

▶ Soumik Pal (UW) - discrete time formulation of SPT and functionally generated portfolio
▶ Christa Cuchiero and Walter Schachermayer (Vienna) - Cover’s universal portfolio and SPT
▶ Peter Baxendale (USC) - random concave functions
Outline

Two parts:
1. Concave functions on the simplex $\Delta_n$ have financial applications:
   - Measure of market diversity and volatility
   - Trading strategy
2. Models for generating random concave functions
Motivations

- **Stochastic portfolio theory** (Fernholz, Karatzas, ...)
- Macroscopic behaviors of stock markets, and trading strategies that exploit them.
- Mathematical framework: market weight $\mu(t) = (\mu_1(t), \ldots, \mu_n(t))$ as a process in the simplex $\Delta_n$. 

![SPY weights graph](image_url)
Diversity and volatility

\[ \varphi(\mu(t)) \]

\[ \sum_i D[\mu(t_{i+1})|\mu(t_i)] \]

▶ Can we have portfolios such that

portfolio value relative to market = diversity + volatility?
Functional portfolio generation

Definition (W. 2018)
A trading strategy is functionally generated, if there exist $g, \varphi$ and $D[\cdot|\cdot] \geq 0$ such that the following pathwise decomposition holds:

$$g(V(t)) - g(V(0)) = \varphi(\mu(t)) - \varphi(\mu(0)) + \sum_{s<t} D[\mu(s + 1)|\mu(s)].$$

Here, the relative value of a self-financing strategy $\eta$ is defined by

$$V(t) - V(0) := \sum_{s<t} \eta(s) \cdot \Delta \mu(s).$$

- Multiplicative generation (Fernholz 1999): $g(x) = \log x$
- Additively generation (Karatzas and Ruf 2017): $g(x) = x$
Characterization

Theorem (W. 2018)

Under suitable regularity conditions:

(i) Up to a linear transformation $g$ is of the form
$$g(x) = \frac{1}{\alpha} \log(C + x), \text{ where } \alpha > 0, \ C \geq 0, \ \text{or } g(x) = x.$$

(ii) $e^{\alpha \varphi} > 0$ is concave.


$$D[q|p] = \frac{1}{\alpha} \log(1 + \alpha \nabla \varphi(p) \cdot (q - p)) - (\varphi(q) - \varphi(p)).$$

The trading strategy is given by

$$\eta_i(t) = \alpha(C + V(t))D_{e^{\mu(t)}} \varphi(\mu(t)) + V(t).$$
Random concave functions

- **Problem:** Construct probability measures $\nu$ on

$$\mathcal{C} := \{\psi : \Delta_n \to [0, \infty) \text{ concave}\}.$$

- **Motivation:** $\nu$ can serve as the *prior distribution* in Cover’s universal portfolio and nonparametric statistics.

- **Main idea:** Let $\Psi = \min_\alpha \ell_\alpha$, the minimum of a random family of affine functions.
More generally, we consider the soft minimum

\[ m_\lambda(x_1, \ldots, x_K) := \frac{-1}{\lambda} \log \left( \frac{1}{K} \sum_{k=1}^{K} e^{-\lambda x_k} \right) \]

which is smooth and concave in \( x \) for any \( \lambda > 0 \).
Probabilistic model

- Fix random vector $C \in (0, \infty)^n$ as coefficients of $\ell(p) = C \cdot p$.
- Let $\ell_1, \ldots, \ell_K, \ldots$ be i.i.d. copies of $\ell$.
- Let $\Psi_K = c_K m_{\lambda_K}(\ell_1, \ldots, \ell_K)$, $c_K > 0$ suitable scaling constant.

This gives a distribution $\nu = \nu_{K,\lambda_K,\text{Law}(C),c_K}$ on $\mathcal{C}$.

We are interested in the limiting properties as $K \to \infty$. 
Case 1: $\lambda_K \equiv \lambda < \infty$

Theorem (Baxendale and W. (2018))

Let $C$ be a random vector with values in $(0, \infty)^n$ which generates $\ell(p) = C \cdot p$. Let $\psi(t) = \log \mathbb{E} e^{t \cdot C}$ be its cgf. For $\lambda \in (0, \infty)$ fixed, let $\Psi_K = m_\lambda(\ell_1, \ldots, \ell_K)$. Then, almost surely,

$$
\Psi_K \rightarrow \Psi_\infty, \quad \Psi_\infty(p) := \frac{-1}{\lambda} \psi(-\lambda p).
$$
Case 2: $\lambda_K \equiv \infty$

Conditions on the random vector $C = (C_1, \ldots, C_n)$:

- $C$ has a joint density $f(x)$ on $(0, \infty)^n$.
- $f$ is asymptotically homogeneous of order $\alpha \geq 0$ near the origin, i.e.,
  \[ f(\lambda x) \sim \lambda^\alpha h(x), \quad \lambda \to 0^+. \]
  (And some technical conditions.)

Theorem (Baxendale and W. (2018))

As $K \to \infty$ (with $\lambda_K \equiv \infty$) the law of the random concave function $\Psi_K = K^{1/(n+\alpha)} \min(\ell_1, \ldots, \ell_K)$ converges weakly to a measure $\nu$ supported on

\[ \mathcal{C}_+ = \{ \psi : \Delta_n \to (0, \infty) \text{ concave} \}. \]
Realization by Poisson point process

Consider a point process \( N = N(\omega) \) which is a random set in \((0, \infty)^n\). It induces

\[
\Psi(p) = \inf_{x \in N(\omega)} x \cdot p.
\]

Theorem (Baxendale and W. (2018))

Let \( N \) be a Poisson point process on \((0, \infty)^n\) with rate measure \( m(A) = \int_A h(x)dx \). Then the law of \( \Psi(p) = \inf_{x \in N} x \cdot p \) is the limiting measure \( \nu \) in the previous theorem.
Convex duality

This gives a map $\psi \in \mathcal{C} \mapsto \hat{R}(\psi)$ (lower left part).
In fact, if $\Psi \sim \nu$, then

$$
\mathbb{P}(\Psi \geq \psi) = \exp(-m(\hat{R}(\psi))) = \exp \left( - \int_{\hat{R}(\psi)} h(x) dx \right).
$$
When intensity measure $= \gamma \times$ Lebesgue

- Let $\Psi \sim \nu$, when $h = \text{const} = \gamma$. Then $\mathbb{E}\Psi(p) \propto (p_1 \cdots p_n)^{1/n}$. It generates multiplicatively the equal-weighted portfolio.

Let $\pi$ be portfolio generated multiplicatively by $\Phi \sim \nu$. Then for $p \in \Delta_n$, $\pi(p)$ is uniformly distributed in $\Delta_n$.

- Let $\Psi \sim \nu$. For $\psi \in \mathcal{C} \cap C^2$, $\psi|_{\partial \Delta_n} \equiv 0$, we have

$$\mathbb{P}(\Psi \geq \psi) = \exp\left(\frac{-\gamma}{n} \int_{\Delta_n} \psi(p) \det(-D^2\psi(p)) dp \right).$$
Open problem

- Let $V_\psi(t)$ be value of portfolio generated by $\psi$.
- Let $V^*(t) = \sup_\psi V_\psi(t)$.
- Given a prior distribution $\nu$, let

$$\hat{V}(t) = \int V_\psi(t) d\nu(\psi).$$

This is the value of Cover’s universal portfolio in the context of SPT where portfolios are functionally generated.

- **Problem:** Prove that

$$\lim_{t \to \infty} \frac{1}{t} \log \frac{V^*(t)}{\hat{V}(t)} = 0$$

uniformly in the market path and establish convergence rate.