Functional portfolio generation

Ting-Kam Leonard Wong

University of Toronto
tkl.wong@utoronto.ca

WCMF-9 @ USC

My collaborators

- Soumik Pal (UW) - discrete time formulation of SPT and functionally generated portfolio
- Christa Cuchiero and Walter Schachermayer (Vienna) - Cover's universal portfolio and SPT
- Peter Baxendale (USC) - random concave functions

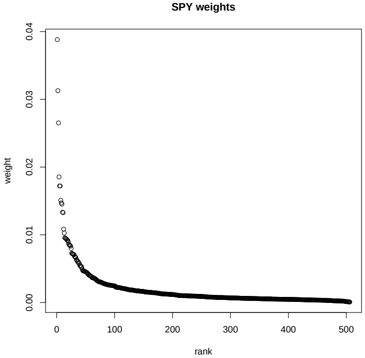
Outline

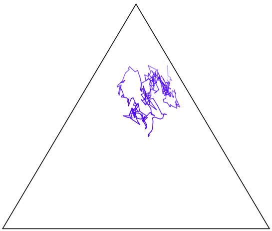
Two parts:

1. Concave functions on the simplex Δ_n have financial applications:
   - Measure of market diversity and volatility
   - Trading strategy
2. Models for generating *random* concave functions

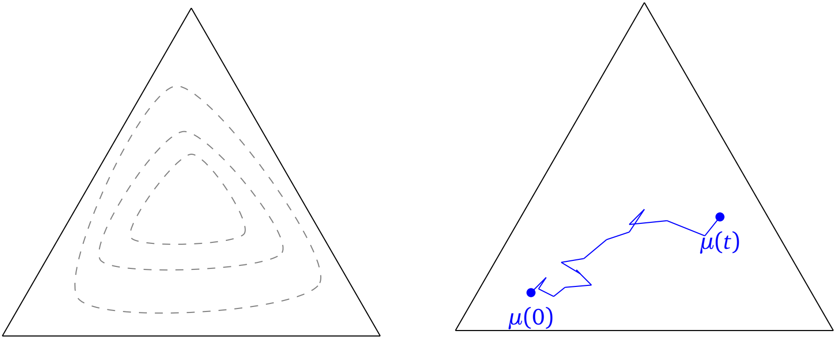
Motivations

- **Stochastic portfolio theory** (Fernholz, Karatzas, ...)
- Macroscopic behaviors of stock markets, and trading strategies that exploit them.
- Mathematical framework: market weight $\mu(t) = (\mu_1(t), \ldots, \mu_n(t))$ as a process in the simplex Δ_n.





Diversity and volatility



$\varphi(\mu(t))$ $\sum_i \mathbf{D}[\mu(t_{i+1})|\mu(t_i)]$

- Can we have portfolios such that

$$\text{portfolio value relative to market} = \text{diversity} + \text{volatility?}$$

Functional portfolio generation

Definition (W. 2018)

A trading strategy is functionally generated, if there exist g, φ and $\mathbf{D}[\cdot|\cdot] \geq 0$ such that the following pathwise decomposition holds:

$$g(V(t)) - g(V(0)) = \varphi(\mu(t)) - \varphi(\mu(0)) + \sum_{s<t} \mathbf{D}[\mu(s+1)|\mu(s)].$$

Here, the relative value of a self-financing strategy η is defined by

$$V(t) - V(0) := \sum_{s<t} \eta(s) \cdot \Delta\mu(s)$$

- Multiplicative generation (Fernholz 1999): $g(x) = \log x$
- Additively generation (Karatzas and Ruf 2017): $g(x) = x$

Characterization

Theorem (W. 2018)

Under suitable regularity conditions:

(i) *Up to a linear transformation g is of the form* $g(x) = \frac{1}{\alpha}\log(C + x)$*, where* $\alpha > 0$*,* $C \geq 0$*, or* $g(x) = x$*.*

(ii) $e^{\alpha\varphi} > 0$ *is* **concave**.

(iii) **D** *is the* $L^{(\alpha)}$**-divergence** *[Pal and W. (2016), W. (2018)]*

$$\mathbf{D}[q|p] = \frac{1}{\alpha}\log(1 + \alpha\nabla\varphi(p)\cdot(q-p)) - (\varphi(q) - \varphi(p)).$$

The trading strategy is given by

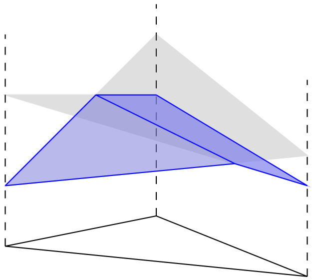
$$\eta_i(t) = \alpha(C + V(t))D_{e_i - \mu(t)}\varphi(\mu(t)) + V(t).$$

Random concave functions

- **Problem:** Construct probability measures ν on

$$\mathscr{C} := \{\psi : \Delta_n \to [0, \infty) \text{ concave}\}.$$

- **Motivation:** ν can serve as the *prior distribution* in Cover's universal portfolio and nonparametric statistics.
- **Main idea:** Let $\Psi = \min_\alpha \ell_\alpha$, the minimum of a random family of affine functions.

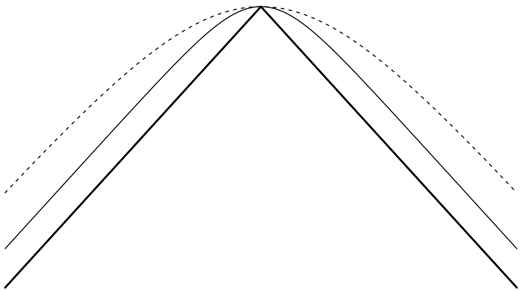


Softmin

- More generally, we consider the soft minimum

$$m_\lambda(x_1, \ldots, x_K) := \frac{-1}{\lambda} \log\left(\frac{1}{K} \sum_{k=1}^{K} e^{-\lambda x_k} \right)$$

which is smooth and concave in x for any $\lambda > 0$.

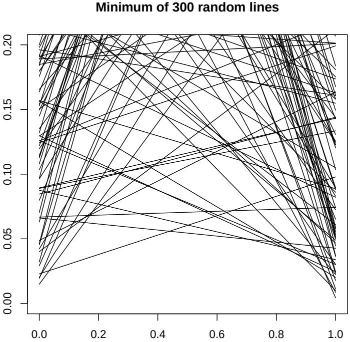


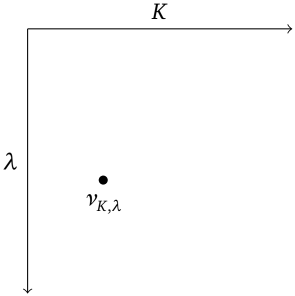
Probabilistic model

- Fix random vector $C \in (0,\infty)^n$ as coefficients of $\ell(p) = C \cdot p$.
- Let $\ell_1, \dots, \ell_K, \dots$ be i.i.d. copies of ℓ.
- Let $\Psi_K = c_K m_{\lambda_K}(\ell_1, \dots, \ell_K)$, $c_K > 0$ suitable scaling constant.

This gives a distribution $\nu = \nu_{K,\lambda_K,\mathrm{Law}(C),c_K}$ on $\mathscr{C}$.

We are interested in the limiting properties as $K \to \infty$.



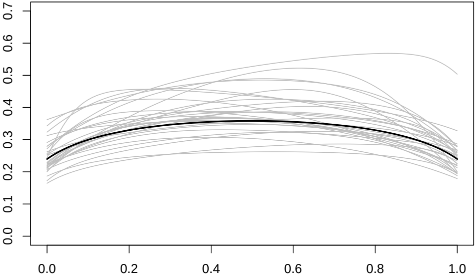


Case 1: $\lambda_K \equiv \lambda < \infty$

Theorem (Baxendale and W. (2018))

Let C be a random vector with values in $(0,\infty)^n$ which generates $\ell(p) = C \cdot p$. Let $\psi(t) = \log \mathbb{E}e^{t\cdot C}$ be its cgf. For $\lambda \in (0,\infty)$ fixed, let $\Psi_K = m_\lambda(\ell_1,\ldots,\ell_K)$. Then, almost surely,

$$\Psi_K \to \Psi_\infty, \quad \Psi_\infty(p) := \frac{-1}{\lambda}\psi(-\lambda p).$$



Case 2: $\lambda_K \equiv \infty$

Conditions on the random vector $C = (C_1, \ldots, C_n)$:

- C has a joint density $f(x)$ on $(0,\infty)^n$.
- f is asymptotically homogeneous of order $\alpha \geq 0$ near the origin, i.e.,

$$f(\lambda x) \sim \lambda^\alpha h(x), \quad \lambda \to 0^+.$$

(And some technical conditions.)

Theorem (Baxendale and W. (2018))

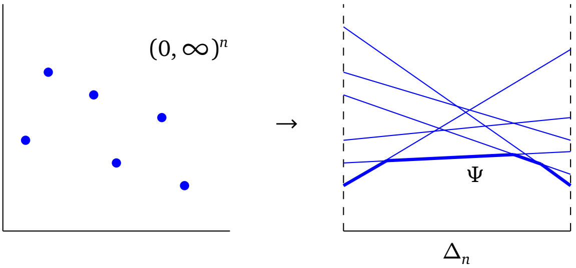
As $K \to \infty$ (with $\lambda_K \equiv \infty$) the law of the random concave function $\Psi_K = K^{1/(n+\alpha)} \min(\ell_1, \ldots, \ell_K)$ converges weakly to a measure ν supported on

$$\mathscr{C}_+ = \{\psi : \Delta_n \to (0,\infty) \text{ concave}\}.$$

Realization by Poisson point process

Consider a point process $N = N(\omega)$ which is a random set in $(0,\infty)^n$. It induces

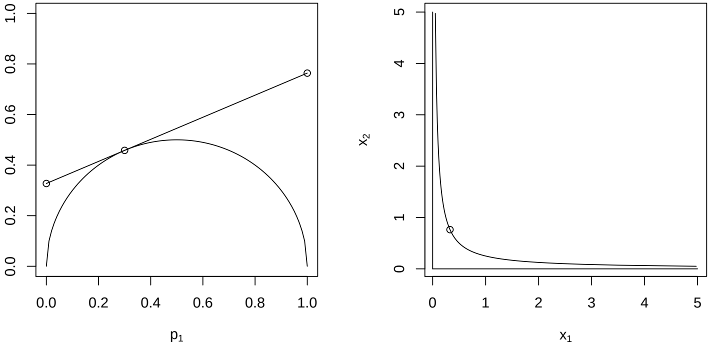
$$\Psi(p) = \inf_{x \in N(\omega)} x \cdot p.$$



Theorem (Baxendale and W. (2018))

Let N be a Poisson point process on $(0,\infty)^n$ with rate measure $m(A) = \int_A h(x)dx$. Then the law of $\Psi(p) = \inf_{x \in N} x \cdot p$ is the limiting measure ν in the previous theorem.

Convex duality

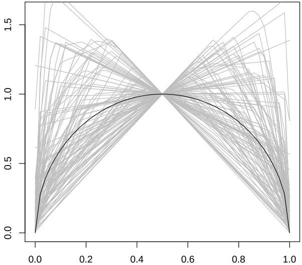


This gives a map $\psi \in \mathscr{C} \mapsto \hat{R}(\psi)$ (lower left part).
In fact, if $\Psi \sim \nu$, then

$$\mathbb{P}(\Psi \geq \psi) = \exp(-m(\hat{R}(\psi))) = \exp\left(-\int_{\hat{R}(\psi)} h(x)dx\right).$$

When intensity measure = $\gamma\times$Lebesgue

- Let $\Psi \sim \nu$, when $h = \text{const} = \gamma$. Then $\mathbb{E}\Psi(p) \propto (p_1 \cdots p_n)^{1/n}$. It generates multiplicatively the equal-weighted portfolio.



- Let π be portfolio generated multiplicatively by $\Phi \sim \nu$. Then for $p \in \Delta_n$, $\pi(p)$ is uniformly distributed in Δ_n.
- Let $\Psi \sim \nu$. For $\psi \in \mathscr{C} \cap C^2$, $\psi|_{\partial\Delta_n} \equiv 0$, we have

$$\mathbb{P}(\Psi \geq \psi) = \exp\left(\frac{-\gamma}{n}\int_{\Delta_n} \psi(p)\det(-D^2\psi(p))dp\right).$$

Open problem

- Let $V_\psi(t)$ be value of portfolio generated by ψ.
- Let $V^*(t) = \sup_\psi V_\psi(t)$.
- Given a prior distribution ν, let

$$\hat{V}(t) = \int V_\psi(t) d\nu(\psi).$$

  This is the value of *Cover's universal portfolio* in the context of SPT where portfolios are functionally generated.
- **Problem:** Prove that

$$\lim_{t\to\infty} \frac{1}{t} \log \frac{V^*(t)}{\hat{V}(t)} = 0$$

  uniformly in the market path and establish convergence rate.