On Fairness of Systemic Risk Measures

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Outline

- Brief introduction to convex risk measures and general capital requirements

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- Brief introduction to Systemic Risk Measures
    - Aggregation functions
    - First aggregate, then inject capital
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    - **Not only inject cash but also random (***scenario-dependent***) capital injection** (*BFFM2018 Mathematical Finance*)

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- Brief introduction to Systemic Risk Measures
  - Aggregation functions
  - First aggregate, then inject capital
  - First inject capital, then aggregate
  - **Not only inject cash but also random (***scenario-dependent***) capital injection** (*BFFM2018 Mathematical Finance*)
- On Fairness of Systemic Risk Measures (*BFFM this paper*)
  - Some basic conceptual questions on fairness and their solutions
  - Technical results

Coherent / Convex Risk Measures

Artzner, Delbaen, Eber and Heath (1999); Föllmer and Schied (2002); Frittelli and Rosazza Gianin (2002)

A monetary risk measure is a map

$$\eta : \mathcal{L}^0(\mathbb{R}) \to \mathbb{R}$$

that represents the **minimal (extra) capital needed to secure a financial position** with payoff $X \in \mathcal{L}^0(\mathbb{R})$, i.e. the minimal amount $m \in \mathbb{R}$ that must be added to X in order to make the resulting payoff at time T acceptable:

$$\eta(X) \triangleq \inf\{m \in \mathbb{R} \mid X + m \in \mathbb{A}\},$$

where the acceptance set $\mathbb{A} \subseteq \mathcal{L}^0(\mathbb{R})$ is assumed to be monotone.

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where the acceptance set $\mathbb{A} \subseteq \mathcal{L}^0(\mathbb{R})$ is assumed to be monotone. The characterizing feature of these monetary maps is **cash additivity**:

$$\eta(X + m) = \eta(X) - m, \quad \text{for all } m \in \mathbb{R}.$$

Under the assumption that the set $\mathbb{A}$ is convex (resp. is a convex cone) the maps η are convex (resp. convex and positively homogeneous) and are called **convex (resp. coherent) risk measures**.

$$\eta(X) \triangleq \inf\{m \in \mathbb{R} \mid X + m1 \in \mathbb{A}\}, \quad \mathbb{A} \subseteq \mathcal{L}^0(\mathbb{R})$$

Why should we consider **only** "money" as safe capital ?

One should be more liberal and **permit the use of other financial assets** (other than the bond $:= 1$), in an appropriate set $\mathcal{C}$ of *safe* instruments, **to hedge the position** X.

Definition

The general capital requirement is

$$\eta(X) \triangleq \inf\{\pi(Y) \in \mathbb{R} \mid Y \in \mathcal{C}, X + Y \in \mathbb{A}\},$$

for some evaluation functional $\pi : \mathcal{C} \to \mathbb{R}$.

Systemic Risk Measure

Consider a system of N interacting financial institutions and a vector $\mathbf{X} = (X^1, \ldots, X^N) \in \mathcal{L}^0(\mathbb{R}^N) := \mathcal{L}^0(\Omega, \mathcal{F}; \mathbb{R}^N)$ of associated risk factors (future values of positions) at a given future time horizon T.

- In this paper we are interested in **real-valued** systemic risk measures:

$$\rho : \mathcal{L}^0(\mathbb{R}^{\mathbb{N}}) \to \overline{\mathbb{R}}$$

that evaluates the risk $\rho(\mathbf{X})$ of the complete financial system $\mathbf{X}$.

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- Initially, many of the SRM in the literature were of the form

$$\rho(\mathbf{X}) = \eta(\Lambda(\mathbf{X})),$$

where $\eta : \mathcal{L}^0(\mathbb{R}) \to \overline{\mathbb{R}}$ is a univariate risk measure and

$$\Lambda : \mathbb{R}^N \to \mathbb{R}$$

is an aggregation rule that aggregates the N-dimensional risk factor $\mathbf{X}$ into a univariate risk factor $\Lambda(\mathbf{X})$ representing the total risk in the system.

Examples of aggregation rule

- $\Lambda(\mathbf{x}) = \sum_{n=1}^{N} x_n, \quad \mathbf{x} = (x_1, \ldots, x_N) \in \mathbb{R}^N.$
- $$\Lambda(\mathbf{x}) = \sum_{n=1}^{N} -x_n^- \quad \text{or} \quad \Lambda(\mathbf{x}) = \sum_{n=1}^{N} -(x_n - d_n)^-, \ d_n \in \mathbb{R}$$
  takes into account the lack of cross-subsidization between financial institutions
- $$\Lambda(\mathbf{x}) = \sum_{n=1}^{N} -\exp\left(-\alpha_n x_n^-\right), \quad \alpha_n \in \mathbb{R}_+$$
  $$\Lambda(\mathbf{x}) = \sum_{n=1}^{N} -\exp\left(-\alpha_n x_n\right), \quad \alpha_n \in \mathbb{R}_+$$
- $$\Lambda(\mathbf{x}) = \sum_{n=1}^{N} u_n(x_n)$$
  where $u_n : \mathbb{R} \to \mathbb{R}$ are utility functions.

First aggregate, then inject cash

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If η is a convex (cash additive) risk measure then we can rewrite such ρ as

$$\rho(\mathbf{X}) \triangleq \inf\{m \in \mathbb{R} \mid \Lambda(\mathbf{X}) + m \in \mathbb{A}\}.$$

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after aggregating individual risks

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- For an axiomatic approach for this type of SRM see Chen Iyengar Moallemi 2013 and Kromer Overbeck Zilch 2013 and the references therein. Acharya et al. 2010, Adrian Brunnermeier 2011, Cheridito Brunnermeier 2014, Gauthier Lehar Souissi 2010, Hoffmann Meyer-Brandis Svindland 2014, Huang Zhou Zhu 2009, Lehar 2005.

In BFFM18: First inject cash, then aggregate

$$\rho(\mathbf{X}) \triangleq \inf\{\sum_{n=1}^{N} m_n \in \mathbb{R} \mid \mathbf{m} = (m_1, ..., m_N) \in \mathbb{R}^N;\ \Lambda(\mathbf{X}+\mathbf{m}) \in \mathbb{A}\}\,.$$

- The amount m_n is added to the financial position X^n before the corresponding total loss $\Lambda(\mathbf{X}+\mathbf{m})$ is computed.
- $\rho(X)$ **is the minimal capital that secures the aggregated system by injecting the capital into the single institutions before aggregating the individual risks.**

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- Also applied to shortfall systemic risk measures by Armenti-Crepey-Drapeau-Papapantoleon (2018).
- ρ delivers at the same time a measure of total systemic risk and a potential **ranking** of the institutions in terms of systemic riskiness.

Second feature of BFFM18: Random allocation

We allow to add to $\mathbf{X}$ not only a vector $\mathbf{m} = (m_1, ..., m_N) \in \mathbb{R}^N$ of cash but a random vector

$$\mathbf{Y} \in \mathcal{C}_{\mathbb{R}} := \{\mathbf{Y} \in L^0(\mathbb{R}^N) \mid \sum_{n=1}^{N} Y^n \in \mathbb{R}\},$$

so that $\rho(\mathbf{X})$ is the minimal cash $\sum_{n=1}^{N} Y^n \in \mathbb{R}$ needed today to secure the system by distributing the capital at time T among $(X^1, ..., X^N)$:

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- In general the allocation $Y^i(\omega)$ to institution i does not need to be decided today but depends on the scenario ω realized at time T.
- For $\mathcal{C} = \mathbb{R}^N$ the situation corresponds to the previous case where the distribution is already determined today.
- For $\mathcal{C} = \mathcal{C}_{\mathbb{R}}$ the distribution can be chosen freely depending on the scenario ω realized in T (including negative amounts, i.e. **full cross-subsidization**).

Grouping Example

For a partition of $\{1, ..., N\}$ in h groups we consider the set $\mathcal{C}^{(h)} \subseteq \mathcal{C}_{\mathbb{R}}$

$$\mathcal{C}^{(h)} = \left\{ \mathbf{Y} \in \mathcal{L}^0(\mathbb{R}^N) \mid \exists\, d = (d_1, \cdots, d_h) \in \mathbb{R}^h : \sum_{i \in I_m} Y^i = d_m, m = 1, .., h \right\}$$

- the values $(d_1, \cdots, d_h)$ may change, but the number of elements in each of the h groups I_m is fixed.
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- $\mathcal{C}^{(h)}$ is a linear space containing $\mathbb{R}^N$.
- $h = N$, (exactly N groups), then $\mathcal{C}^{(h)} = \mathbb{R}^N$ corresponds to the deterministic case;
- $h = 1$ (only one group) then $\mathcal{C}^{(h)} = \mathcal{C}_{\mathbb{R}}$. Completely arbitrary random injection $\mathbf{Y}$ with the only requirement $\sum_{n=1}^{N} Y^n \in \mathbb{R}$

Dependence can be taken into account

- Allowing **random** allocations $\mathbf{Y} \in \mathcal{C} \subseteq \mathcal{C}_{\mathbb{R}}$, that might differ from scenario to scenario, the systemic risk measure will take the dependence structure of the components of $\mathbf{X}$ into account even though acceptable positions might be defined in terms of the marginal distributions of X^n, $n = 1, ..., N$, only.
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- This fact allows to considerably reduce the total systemic risk.
- For example: $\Lambda(\mathbf{x}) = \sum_{n=1}^{N} u_n(x^n)$, $u_n : \mathbb{R} \to \mathbb{R}$ and $\mathbf{x} \in \mathbb{R}^N$.

$$\mathbb{A} := \left\{ Z \in \mathcal{L}^0(\mathbb{R}) \mid \mathbb{E}[Z] \geq B \right\}, B \in \mathbb{R}.$$

- $\mathbf{X} + \mathbf{Y} \in \mathcal{L}^0(\mathbb{R}^N)$ is acceptable if and only if $\Lambda(\mathbf{X} + \mathbf{Y}) \in \mathbb{A}$, i.e.

$$\mathbb{E}\left[\sum_{n=1}^{N} u_n(X^n + Y^n) \right] \geq B$$

- If $\mathcal{C} = \mathbb{R}^N$ (i.e. $\mathbf{Y} = \mathbf{m} \in \mathbb{R}^N$) then $\rho(\mathbf{X})$ depends on the marginal distributions of $\mathbf{X}$ only.

Fairness. Definitions and Assumptions

$$\rho(\mathbf{X}) := \inf_{\mathbf{Y}\in\mathcal{C}} \left\{ \sum_{n=1}^{N} Y^n \mid \mathbb{E}\left[\sum_{n=1}^{N} u_n(X^n + Y^n) \right] \geq B \right\},$$

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2. $u_n : \mathbb{R} \to \mathbb{R}$ is increasing, strictly concave, differentiable and satisfies the Inada conditions

$$u_n'(-\infty) \triangleq \lim_{x\to-\infty} u_n'(x) = +\infty,\ u_n'(+\infty) \triangleq \lim_{x\to+\infty} u_n'(x) = 0.$$

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3. $B < \sum_{n=1}^{N} u_n(+\infty)$

$\rho = \rho_B$ is well defined and has good properties

$$\rho(\mathbf{X}) := \inf_{\mathbf{Y} \in \mathcal{C}} \left\{ \sum_{n=1}^{N} Y^n \mid \mathbb{E}\left[\sum_{n=1}^{N} u_n(X^n + Y^n) \right] \geq B \right\}$$

Proposition

The map $\rho : M^\Phi \to \mathbb{R} \cup \{+\infty\}$ is finitely valued, monotone decreasing, convex, continuous and subdifferentiable on the Orlicz Heart $M^\Phi = dom(\rho)$.

$$\rho : M^\Phi \to \mathbb{R}$$

where we skip the details on the Orlicz setting

Optimal Allocation and Risk Allocation

$$\rho(\mathbf{X}) := \inf_{\mathbf{Y}\in\mathcal{C}} \left\{ \sum_{n=1}^{N} Y^n \mid \mathbb{E}\left[\sum_{n=1}^{N} u_n(X^n + Y^n)\right] \geq B \right\}$$

Definition

(i) We say that the scenario dependent allocation $\mathbf{Y_X} = (Y^n_{\mathbf{X}})_n \in \mathcal{C}$ is an *optimal allocation* to $\rho(\mathbf{X})$, if

$$\mathbb{E}\left[\sum_{n=1}^{N} u_n(X^n + Y^n_{\mathbf{X}})\right] \geq B \quad \text{and} \quad \rho(\mathbf{X}) = \sum_{n=1}^{N} Y^n_{\mathbf{X}}.$$

(ii) We say that a vector $(\rho^n(\mathbf{X}))_n \in \mathbb{R}^N$ is a *risk allocation* for $\rho(\mathbf{X})$ if

$$\rho(\mathbf{X}) = \sum_{n=1}^{N} \rho^n(\mathbf{X}).$$

Key qualitative questions on fairness

1. When is the *systemic valuation* $\rho(\mathbf{X})$ *and its random allocation* $\mathbf{Y_X} = (Y^n_{\mathbf{X}})_n \in \mathcal{C}_{\mathbb{R}}$ fair from the point of view of the *whole system*?

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2. When is a *risk allocation* $(\rho^n(\mathbf{X}))_n \in \mathbb{R}^N$ of $\rho(\mathbf{X})$ fair from the point of view of the *whole system*?
3. When are the *systemic allocation* $\mathbf{Y_X} = (Y^n_{\mathbf{X}})_n \in \mathcal{C}_{\mathbb{R}}$ and the *risk allocation* $(\rho^n(\mathbf{X}))_n \in \mathbb{R}^N$ associated to $\rho(\mathbf{X})$, fair from the point of view of *each individual bank*?

Preview of the answer

Let $\mathbf{Y_X} = (Y^1_{\mathbf{X}}, \cdots, Y^N_{\mathbf{X}})$ be an optimal random allocation for $\rho(\mathbf{X})$

$$\rho(\mathbf{X}) = \sum_{n=1}^{N} Y^n_{\mathbf{X}}.$$

Let $\mathbf{Q_X} = (Q^1_{\mathbf{X}}, \cdots, Q^N_{\mathbf{X}})$ be the optimal solution of the **dual problem** for $\rho(\mathbf{X})$, then

$$\rho^n(\mathbf{X}) = \mathbb{E}_{Q^n_{\mathbf{X}}}\left[Y^n_{\mathbf{X}}\right], \quad n = 1, \cdots, N,$$

is a fair risk allocation.

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Technical questions:

a) What is the dual representation of $\rho(\mathbf{X})$?

b) When an optimal solution $\mathbf{Q_X}$ to the dual problem exists?

c) When an optimal random allocation $\mathbf{Y_X}$ exists?

Dual representation

Theorem

For any $\mathbf{X} \in M^{\Phi}$,

$$\rho(\mathbf{X}) = \max_{\mathbf{Q}\in\mathcal{D}} \left\{ \sum_{n=1}^{N} \mathbb{E}_{Q^n}[-X^n] - \alpha_{\Lambda,B}(\mathbf{Q}) \right\}, \tag{1}$$

$$\alpha_{\Lambda,B}(\mathbf{Q}) = \sup_{\mathbf{Z}\in M^{\Phi}} \left\{ \sum_{n=1}^{N} \mathbb{E}_{\mathbf{Q}^n}[-Z^n] \mid \sum_{n=1}^{N} \mathbb{E}[u_n(Z^n)] \geq B \right\},$$

$$\mathcal{D} := dom(\alpha_{\Lambda,B}) \cap \left\{ \frac{d\mathbf{Q}}{d\mathbb{P}} \in L_+^{\Phi^*} \mid Q^n(\Omega) = 1, \sum_{n=1}^{N} (\mathbb{E}_{Q^n}[Y^n] - Y^n) \leq 0 \ \forall \mathbf{Y} \in \mathcal{C} \right\}$$

The maximizer $\mathbf{Q_X} = (Q_{\mathbf{X}}^1, \cdots, Q_{\mathbf{X}}^N)$ *in (1) exists and is unique.*

Penalty function

For $\alpha_{\Lambda,B}(\mathbf{Q}) < +\infty$, we have

$$\alpha_{\Lambda,B}(\mathbf{Q}) = \sum_{n=1}^{N} \mathbb{E}\left[\frac{dQ^n}{d\mathbb{P}} v_n'\left(\widehat{\lambda}\frac{dQ^n}{d\mathbb{P}}\right)\right],$$

where where $\widehat{\lambda} > 0$ is the unique solution of the equation

$$-B + \sum_{n=1}^{N} \mathbb{E}\left[v_n\left(\lambda\frac{dQ^n}{d\mathbb{P}}\right)\right] - \lambda \sum_{n=1}^{N} \mathbb{E}\left[\frac{dQ^n}{d\mathbb{P}} v_n'\left(\lambda\frac{dQ^n}{d\mathbb{P}}\right)\right] = 0.$$

Here, v_n is the convex conjugate function $v_n(y) := \sup_{x\in\mathbb{R}}\{u_n(x) - xy\}$. Note that $\widehat{\lambda}$ will depend on B, $(u_n)_{n=1,\cdots,N}$ and $\left(\frac{dQ_n}{d\mathbb{P}}\right)_{n=1,\cdots,N}$.

Optimal allocation existence result

Theorem

Let $\mathcal{C} = \mathcal{C}_0 \cap M^\Phi$ and suppose that $\mathcal{C}_0 \subseteq \mathcal{C}_\mathbb{R}$ is closed for the convergence in probability and that it is closed under truncation. For any $\mathbf{X} \in M^\Phi$ there exists $\mathbf{Y}_\mathbf{X} \in \mathcal{C}_0 \cap L^1(\mathbb{P}; \mathbf{Q}_\mathbf{X})$ such that

$$\sum_{n=1}^N Y^n_\mathbf{X} \in \mathbb{R}, \quad \mathbb{E}\left[\sum_{n=1}^N u_n(X^n + Y^n_\mathbf{X})\right] \geq B, \quad \sum_{n=1}^N \left(\mathbb{E}_{Q^n_\mathbf{X}}\left[Y^n_\mathbf{X}\right] - Y^n_\mathbf{X}\right) = 0,$$

$$\begin{aligned}\rho(\mathbf{X}) &= \inf\left\{\sum_{n=1}^N Z^n \mid \mathbf{Z} \in \mathcal{C}_0 \cap M^\Phi, \mathbb{E}\left[\Lambda(\mathbf{X}+\mathbf{Y})\right] \geq B\right\} = \sum_{n=1}^N Y^n_\mathbf{X} \\ &= \inf\left\{\sum_{n=1}^N Z^n \mid \mathbf{Z} \in \mathcal{C}_0 \cap L^1(\mathbb{P}; \mathbf{Q}_\mathbf{X}), \mathbb{E}\left[\Lambda(\mathbf{X}+\mathbf{Y})\right] \geq B\right\} := \widetilde{\rho}(\mathbf{X}),\end{aligned}$$

so that $\mathbf{Y}_\mathbf{X}$ is the optimal solution to the extended problem $\widetilde{\rho}(\mathbf{X})$.

Back to the fairness questions

- The above results answer the technical questions raised before.

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    - Associated risk allocations: $\rho^n(\mathbf{X}) = \mathbb{E}_{Q^n_{\mathbf{X}}}\left[Y^n_{\mathbf{X}}\right]$ such that $\sum_{n=1}^N \rho^n(\mathbf{X}) = \rho(\mathbf{X})$

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- Recall:

$$\rho(\mathbf{X}) = \rho_B(\mathbf{X}) = \inf_{\mathbf{Y}\in\mathcal{C}} \left\{ \sum_{n=1}^N Y^n \mid \mathbb{E}\left[\sum_{n=1}^N u_n(X^n + Y^n)\right] \geq B \right\}$$

To better understand the fairness properties, we introduce an associated systemic utility maximization problem:

Systemic utility maximization problem

Consider the related optimization problem:

$$\pi_A(\mathbf{X}) := \sup_{\mathbf{Y}\in\mathcal{C}} \left\{ \mathbb{E}\left[\sum_{n=1}^{N} u_n(X^n + Y^n)\right] \mid \sum_{n=1}^{N} Y^n \le A \right\},$$

- $\sum_{n=1}^{N} u_n(X^n + Y^n)$ is the aggregated utility of the system after allocating $\mathbf{Y}$,
- $\pi(\mathbf{X})$ is the maximal expected utility of the system over all random allocations $\mathbf{Y} \in \mathcal{C}$ such that the aggregated budget constraint $\sum_{n=1}^{N} Y^n \le A$ holds.

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Proposition

$$B = \pi_A(\mathbf{X}) \textit{ if and only if } A = \rho_B(\mathbf{X}),$$

and in these cases, the two problems $\pi_A(\mathbf{X})$ *and* $\rho_B(\mathbf{X})$ *have the same unique optimal solution* $\mathbf{Y}_{\mathbf{X}}$.

First answer to first question

Fairness from the perspective of the society/regulator.

- The requirement $\mathbf{Y} \in \mathcal{C} \subseteq \mathcal{C}_{\mathbb{R}}$ (that implies $\sum_{n=1}^{N} Y^n \in \mathbb{R}$) is important from the society's perspective as it guarantees that the cash amount $\rho_B(\mathbf{X})$ determined today is sufficient to cover the allocations $\mathbf{Y}$ at time T in any possible scenario.
- $\sum_{n=1}^{N} Y^n = \rho_B(\mathbf{X})$ means that the system clears and no additional external injections (or withdrawals) are necessary at time T.
- In that sense, the requirement $\mathbf{Y} \in \mathcal{C} \subseteq \mathcal{C}_{\mathbb{R}}$ is fair from the society/regulator's perspective.

Once a level $A = \rho_B(\mathbf{X})$ of total systemic risk has been determined, the optimal allocation $\mathbf{Y_X}$ of ρ_B maximizes the expected system utility among all random allocations of value less or equal to A.

$$B = \pi_A(\mathbf{X}) \text{ if and only if } A = \rho_B(\mathbf{X}),$$

Risk Allocations

- Once the total systemic risk has been identified as $\rho(\mathbf{X})$, the second essential question is how to allocate the total risk to the individual institutions, i.e. how to compute $(\rho^1(\mathbf{X}), \cdots, \rho^N(\mathbf{X})) \in \mathbb{R}^N$ s.t.

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$$\sum_{n=1}^{N} \rho^n(\mathbf{X}) = \rho(\mathbf{X}).$$

- For deterministic allocations $\mathbf{Y} \in \mathbb{R}^N$, i.e. $\mathcal{C} = \mathbb{R}^N$, the optimal deterministic $\mathbf{Y}_{\mathbf{X}}$ is a canonical risk allocation $\rho^n(\mathbf{X}) := Y^n_{\mathbf{X}} \in \mathbb{R}^N$.

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- For deterministic allocations $\mathbf{Y} \in \mathbb{R}^N$, i.e. $\mathcal{C} = \mathbb{R}^N$, the optimal deterministic $\mathbf{Y_X}$ is a canonical risk allocation $\rho^n(\mathbf{X}) := Y^n_{\mathbf{X}} \in \mathbb{R}^N$.
- For general (random) allocations $\mathbf{Y} \in \mathcal{C} \subseteq \mathcal{C}_{\mathbb{R}}$, we consider risk allocations of the form:

$$\rho^n(\mathbf{X}) := \mathbb{E}_{Q^n}[Y^n_{\mathbf{X}}] \;\; \text{for } n = 1, \cdots, N,$$

where $\mathbf{Q} = (Q^1, \cdots, Q^N)$ is a vector of probability measures s.t.:

$$\sum_{n=1}^{N} \mathbb{E}_{Q^n}[Y^n_{\mathbf{X}}] = \rho(\mathbf{X}).$$

Which $Q = (Q^1, ..., Q^N)$?

- Besides providing a ranking in terms of systemic riskiness, a risk allocation $\rho^n(\mathbf{X})$ can be interpreted as a capital requirement for institution n in order to fund the total amount $\rho(\mathbf{X})$ of cash needed.
- The vector $\mathbf{Q}$ allows for the monetary interpretation of a systemic pricing operator to determine the price (or cost) of (future) random allocations of the individual institutions:

$$\rho^n(\mathbf{X}) = \mathbb{E}_{Q^n}[Y^n_{\mathbf{X}}]$$

can be understood as a *systemic risk valuation* of $Y^n_{\mathbf{X}}$.

- **We want to identify fairness criteria, acceptable both by the society and by the individual financial institutions, for such systemic valuation measures and their corresponding risk allocations.**

Associated Problems, given $Q = (Q^1, ..., Q^N)$

Suppose that a valuation (or cost) operator $\mathbf{Q} = (Q^1, \cdots, Q^N)$ is given.

$$\rho_B^{\mathbf{Q}}(\mathbf{X}) := \inf_{\mathbf{Y}\in M^\Phi} \left\{ \sum_{n=1}^N \mathbb{E}_{Q^n}[Y^n] \mid \mathbb{E}\left[\sum_{n=1}^N u_n(X^n + Y^n)\right] \geq B \right\},$$

$$\pi_A^{\mathbf{Q}}(\mathbf{X}) := \sup_{\mathbf{Y}\in M^\Phi} \left\{ \mathbb{E}\left[\sum_{n=1}^N u_n(X^n + Y^n)\right] \mid \sum_{n=1}^N \mathbb{E}_{Q^n}[Y^n] \leq A \right\}.$$

- Notice that the allocation $\mathbf{Y}$ is **not required to belong to** $\mathcal{C}_{\mathbb{R}}$ (that is adding up to a deterministic quantity).

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- Notice that the allocation $\mathbf{Y}$ is **not required to belong to** $\mathcal{C}_{\mathbb{R}}$ (that is adding up to a deterministic quantity).
- Similarly to the previous Proposition, we prove that

$$B = \pi_A^{\mathbf{Q}}(\mathbf{X}) \text{ if and only if } A = \rho_B^{\mathbf{Q}}(\mathbf{X}),$$

and the two problems $\pi_A^{\mathbf{Q}}(\mathbf{X})$ and $\rho_B^{\mathbf{Q}}(\mathbf{X})$ have the same unique optimal solution.

Optimal solution to the associated problem $\rho_B^{\mathbf{Q}}$

Theorem

Suppose that $\alpha_{\Lambda,B}(\mathbf{Q}) < +\infty$. Then the random vector $\mathbf{Y_Q}$ given by

$$Y_{\mathbf{Q}}^n := -X^n - v_n'\left(\lambda^* \frac{dQ^n}{d\mathbb{P}}\right), n = 1, \dots, N$$

satisfies $Y_{\mathbf{Q}} \in L^1(\mathbf{Q})$, $u_n(X^n + Y_{\mathbf{Q}}^n) \in L^1(\mathbb{P})$, $\mathbb{E}\left[\Lambda(\mathbf{X}+\mathbf{Y})\right] = B$,

$$\begin{aligned} \rho_B^{\mathbf{Q}}(\mathbf{X}) &= \inf_{\mathbf{Y}\in M^\Phi}\left\{\sum_{n=1}^N \mathbb{E}_{Q^n}\left[Y^n\right] \mid \mathbb{E}\left[\Lambda(\mathbf{X}+\mathbf{Y})\right] \geq B\right\} = \sum_{n=1}^N \mathbb{E}_{Q^n}\left[Y_{\mathbf{Q}}^n\right] \\ &= \inf_{\mathbf{Y}\in L^1(\mathbf{Q})}\left\{\sum_{n=1}^N \mathbb{E}_{Q^n}\left[Y^n\right] \mid \mathbb{E}\left[\Lambda(\mathbf{X}+\mathbf{Y})\right] \geq B\right\} := \widehat{\rho}_B^{\mathbf{Q}}(\mathbf{X}), \end{aligned}$$

so that $\mathbf{Y_Q}$ is the optimal solution to the extended problem $\widehat{\rho}_B^{\mathbf{Q}}(\mathbf{X})$.
If $\mathbf{Q} = \mathbf{Q_X}$ then $\mathbf{Y_{Q_X}} = \mathbf{Y_X}$.

On the fair probability measure $\mathbf{Q_X}$

- The specific choice of a systemic valuation $\mathbf{Q} = (Q^1, \cdots, Q^N)$ is the central question of this paper and we will prove that the fair valuation is obtained by using $\mathbf{Q_X}$.
- We also show that

$$\rho_B(\mathbf{X}) = \max_{\frac{d\mathbf{Q}}{d\mathbb{P}} \in \mathcal{D}} \rho_B^{\mathbf{Q}}(\mathbf{X}) = \rho_B^{\mathbf{Q_X}}(\mathbf{X}),$$

  which means that ρ_B is the most conservative among those risk measures $\rho_B^{\mathbf{Q}}$ defined through fair valuation operators $\frac{d\mathbf{Q}}{d\mathbb{P}} \in \mathcal{D}$.
- The probability measure $\mathbf{Q_X}$ plays, in the theory of systemic risk measure, an analogous role played by the minimax martingale measure in the theory of contingent claim valuation in incomplete markets, see Frittelli and Bellini 2002 for details.

The optimizer of the dual problem provides the risk allocation $\rho^n(\mathbf{X}) = \mathbb{E}_{Q^n_{\mathbf{X}}}[Y^n_{\mathbf{X}}]$

Theorem

The optimizer $\mathbf{Q_X} = (Q^1_{\mathbf{X}}, \cdots Q^N_{\mathbf{X}})$ *of the dual problem of* $\rho_B(\mathbf{X})$*, provides a risk allocation* $(\mathbb{E}_{Q^1_{\mathbf{X}}}[Y^1_{\mathbf{X}}], \cdots, (\mathbb{E}_{Q^N_{\mathbf{X}}}[Y^N_{\mathbf{X}}])$*,*

$$\sum_{n=1}^{N} \mathbb{E}_{Q^n_{\mathbf{X}}}[Y^n_{\mathbf{X}}] = \rho_B(\mathbf{X}),$$

satisfying

$$\rho_B(\mathbf{X}) = \rho_B^{Q_{\mathbf{X}}}(\mathbf{X}),$$
$$\pi_A(\mathbf{X}) = \pi_A^{Q_{\mathbf{X}}}(\mathbf{X}).$$

Furthermore, $\mathbf{Y_X}$ *is the unique optimal solution for* $\rho_B(\mathbf{X})$ *and* $\rho_B^{Q_{\mathbf{X}}}(\mathbf{X})$ *and for* $\pi_A(\mathbf{X})$ *and* $\pi_A^{Q_{\mathbf{X}}}(\mathbf{X})$ *(if* $A = \rho_B(\mathbf{X})$*)*

Fairness from the perspective of the society/regulator

$$\rho_B^{\mathbf{Q_X}}(\mathbf{X}) := \inf_{\mathbf{Y} \in M^{\Phi}} \left\{ \sum_{n=1}^{N} \mathbb{E}_{Q_{\mathbf{X}}^n} [Y^n] \mid \mathbb{E}\left[\sum_{n=1}^{N} u_n(X^n + Y^n) \right] \geq B \right\}$$

- Given a systemic risk valuation $\mathbf{Q}$, one is naturally led to the specification $\rho_B^{\mathbf{Q}}(\mathbf{X})$ for a systemic risk measure.

Fairness from the perspective of the society/regulator

$$\rho_B^{\mathbf{Q}_\mathbf{X}}(\mathbf{X}) := \inf_{\mathbf{Y} \in M^\Phi} \left\{ \sum_{n=1}^{N} \mathbb{E}_{Q_\mathbf{X}^n}\left[Y^n\right] \mid \mathbb{E}\left[\sum_{n=1}^{N} u_n(X^n + Y^n)\right] \geq B \right\}$$

- Given a systemic risk valuation $\mathbf{Q}$, one is naturally led to the specification $\rho_B^{\mathbf{Q}}(\mathbf{X})$ for a systemic risk measure.
- Note, however, that in $\rho_B^{\mathbf{Q}}(\mathbf{X})$ the clearing condition $\sum_{n=1}^N Y^n = \rho(\mathbf{X})$ is not guaranteed since the optimization is performed over all $\mathbf{Y} \in M^\Phi$.

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- Note, however, that in $\rho_B^{\mathbf{Q}}(\mathbf{X})$ the clearing condition $\sum_{n=1}^N Y^n = \rho(\mathbf{X})$ is not guaranteed since the optimization is performed over all $\mathbf{Y} \in M^\Phi$.
- Using the valuation with $\mathbf{Q_X}$ is then fair from the society/regulator's point of view since, by the previous Theorem, the optimal allocation of $\rho_B^{\mathbf{Q_X}}(\mathbf{X})$ fulfills the clearing condition $\mathbf{Y} \in \mathcal{C}_{\mathbb{R}}$, and is in fact the same as the optimal allocation of the original systemic risk measure $\rho_B(\mathbf{X})$.

$$\sum_{n=1}^N Y_{\mathbf{X}}^n = \rho_B(\mathbf{X}) = \rho_B^{\mathbf{Q_X}}(\mathbf{X}) = \sum_{n=1}^N \mathbb{E}_{Q_{\mathbf{X}}^n}[Y_{\mathbf{X}}^n]$$

Fairness from the perspective of the individual institutions

- The essential question for a financial institution is whether its allocated share of the total systemic risk determined by the risk allocation $(\mathbb{E}_{Q^1_{\mathbf{X}}}[Y^1_{\mathbf{X}}], \cdots, (\mathbb{E}_{Q^N_{\mathbf{X}}}[Y^N_{\mathbf{X}}])$ is fair.
- For the banks, the clearing condition $\mathbf{Y} \in \mathcal{C}_{\mathbb{R}}$ is not too relevant.

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- For the banks, the clearing condition $\mathbf{Y} \in \mathcal{C}_{\mathbb{R}}$ is not too relevant.
- Instead, given a vector $\mathbf{Q} = (Q^1, \cdots, Q^N)$ of valuation measures, the systemic risk measure $\rho_B^{\mathbf{Q}}(\mathbf{X})$ is more relevant.
- Thus, by choosing $\mathbf{Q} = \mathbf{Q_X}$, the requirements from both the society and the banks are reconciled as seen from $\rho_B(\mathbf{X}) = \rho_B^{Q_{\mathbf{X}}}(\mathbf{X})$.

A simple reformulation

Fix $\mathbf{Q} = (Q^1, ..., Q^N)$ such that $\frac{dQ^n}{d\mathbb{P}} \in L^{\Phi^*}(\mathbb{R})$. Then:

$$M^\Phi = \left\{ \mathbf{Y} = \mathbf{a} + \mathbf{Z} \mid \mathbf{a} \in \mathbb{R}^N \text{ and } \mathbf{Z} \in M^\Phi \text{ such that } \mathbb{E}_{Q^n}[Z^n] = 0 \ \forall n \right\}.$$

(take $\mathbf{Y} \in M^\Phi$ and let $a^n := \mathbb{E}_{Q^n}[Y^n]$ and $Z^n := Y^n - a^n \in M^\Phi$)

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(take $\mathbf{Y} \in M^\Phi$ and let $a^n := \mathbb{E}_{Q^n}[Y^n]$ and $Z^n := Y^n - a^n \in M^\Phi$)

$$\begin{aligned}
\pi_A^Q(\mathbf{X}) &= \sup_{\mathbf{Y} \in M^\Phi} \left\{ \mathbb{E}\left[\sum_{n=1}^N u_n(X^n + Y^n) \right] \mid \sum_{n=1}^N \mathbb{E}_{Q^n}[Y^n] = A \right\} \\
&= \sup_{\sum_{n=1}^N a^n = A, \ \mathbb{E}_{Q^n}[Z^n]=0 \ \forall n} \left\{ \mathbb{E}\left[\sum_{n=1}^N u_n(X^n + a^n + Z^n) \right] \right\} \\
&= \sup_{\sum_{n=1}^N a^n = A,} \ \sum_{n=1}^N \sup_{\mathbb{E}_{Q^n}[Y^n] = a_n} \mathbb{E}\left[u_n(X^n + Y^n) \right]
\end{aligned}$$

Fairness from the perspective of the individual institutions

- With the choice $\mathbf{Q} = \mathbf{Q_X}$, we have:

$$\pi_A(\mathbf{X}) = \pi_A^{\mathbf{Q_X}}(\mathbf{X}) = \sup_{\sum_{n=1}^N a^n = A,} \sum_{n=1}^N \sup_{\mathbb{E}_{Q^n_{\mathbf{X}}}[Y^n]=a_n} \mathbb{E}\left[u_n(X^n + Y^n)\right],$$

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- Choosing $A = \rho_B(\mathbf{X})$, we obtain by the fact that $\mathbf{Y_X}$ is the optimal solution of $\pi_A^{\mathbf{Q_X}}(\mathbf{X})$, that $\mathbb{E}_{Q_{\mathbf{X}^n}}\left[Y^n_{\mathbf{X}}\right] = a_n$, $\sum_{n=1}^N \mathbb{E}_{Q_{\mathbf{X}^n}}\left[Y^n_{\mathbf{X}}\right] = A$ and

$$\pi_A(\mathbf{X}) = \pi_A^{\mathbf{Q_X}}(\mathbf{X}) = \sum_{n=1}^N \sup_{\mathbb{E}_{Q^n_{\mathbf{X}}}[Y^n] = \mathbb{E}_{Q^n_{\mathbf{X}}}[Y^n_{\mathbf{X}}]} \mathbb{E}\left[u_n(X^n + Y^n)\right].$$

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- This means that by using $\mathbf{Q_X}$ for valuation, the system utility maximization in $\pi_A(\mathbf{X})$ reduces to individual utility maximization problems for the banks without the "systemic" constraint $\mathbf{Y} \in \mathcal{C}$:

$$\forall n, \quad \sup_{Y^n} \left\{\mathbb{E}\left[u_n(X^n+Y^n)\right] \mid \mathbb{E}_{Q^n_{\mathbf{X}}}[Y^n] = \mathbb{E}_{Q^n_{\mathbf{X}}}[Y^n_{\mathbf{X}}]\right\}.$$

Fair valuation from the perspective of the single institution

$$\forall n, \quad \sup_{Y^n} \left\{ \mathbb{E}\left[u_n(X^n + Y^n)\right] \mid \mathbb{E}_{Q^n_{\mathbf{X}}}[Y^n] = \mathbb{E}_{Q^n_{\mathbf{X}}}[Y^n_{\mathbf{X}}] \right\}.$$

- The optimal allocation $Y^n_{\mathbf{X}}$ and its value $\mathbb{E}_{Q^n_{\mathbf{X}}}[Y^n_{\mathbf{X}}]$ can thus be considered fair by the n^{th} bank, as $Y^n_{\mathbf{X}}$ maximizes the individual expected utility of bank n among all random allocations (not constrained to be in $\mathcal{C}_{\mathbb{R}}$) with value $\mathbb{E}_{Q^n_{\mathbf{X}}}[Y^n_{\mathbf{X}}]$.
- This finally argues for the fairness of the risk allocation $(\mathbb{E}_{Q^1_{\mathbf{X}}}[Y^1_{\mathbf{X}}], \cdots, \mathbb{E}_{Q^N_{\mathbf{X}}}[Y^N_{\mathbf{X}}])$ as fair valuation of the optimal allocation $(Y^1_{\mathbf{X}}, \cdots, Y^N_{\mathbf{X}})$.

Additional fairness properties

- We now present two additional fairness properties:
  - Monotonicity of the risk allocations with respect to $\mathcal{C}$
  - Marginal risk contribution

Fairness Property: Monotonicity

- It is clear that if $\mathcal{C}_1 \subseteq \mathcal{C}_2 \subseteq \mathcal{C}_{\mathbb{R}}$, then $\rho_1(\mathbf{X}) \geq \rho_2(\mathbf{X})$ for the corresponding systemic risk measures

$$\rho_i(\mathbf{X}) := \inf \left\{ \sum_{n=1}^{N} Y^n \mid \mathbf{Y} \in \mathcal{C}_i,\ \Lambda(\mathbf{X} + \mathbf{Y}) \in \mathbb{A} \right\}, \quad i = 1, 2.$$

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- The two extreme cases occur for:
  - $\mathcal{C}_1 := \mathbb{R}^N$ (the deterministic case)
  - $\mathcal{C}_2 := \mathcal{C}_{\mathbb{R}}$ (the unconstraint scenario dependent case). Hence we know that when going from deterministic to scenario-dependent allocations the total systemic risk decreases.
- It is desirable that each institution profits from this decrease in total systemic risk ($\rho_1(\mathbf{X}) \geq \rho_2(\mathbf{X})$) in the sense that also its individual risk allocation decreases:

$$\rho_1^n(\mathbf{X}) \geq \rho_2^n(\mathbf{X}) \text{ for each } n = 1, \ldots, N.$$

The opposite would clearly be perceived as unfair.

Marginal risk contribution

- The risk measure ρ keeps the cash additivity property:

$$\rho(\mathbf{X} + \mathbf{m}) = \rho(\mathbf{X}) - \sum_{n=1}^{N} m^n \quad \forall \mathbf{m} \in \mathbb{R}^N \text{ and } \forall \mathbf{X},$$

  which is a *global* property.
- Its local version is:

$$\frac{d}{d\varepsilon}\rho(\mathbf{X}+\varepsilon\mathbf{m})|_{\varepsilon=0} = -\sum_{n=1}^{N} m^n \quad \text{for } \mathbf{m} \in \mathbb{R}^N.$$

- The expression

$$\frac{d}{d\varepsilon}\rho(\mathbf{X}+\varepsilon\mathbf{m})|_{\varepsilon=0}$$

  represents the sensitivity of the risk of $\mathbf{X}$ with respect to the impact $\mathbf{m} \in \mathbb{R}^N$ and was named by Armenti Crepey Drapeau Papapantoleon 2018, the *marginal risk contribution*.
- Such property can not be directly generalized if $\mathbf{m} \in \mathbb{R}^N$ is replaced by random vectors $\mathbf{V}$.

On Marginal Risk Contribution in our Setting

Proposition

Let $\mathbf{V} \in M^{\Phi}$ *and* $\mathbf{X} \in M^{\Phi}$. *Let* $\mathbf{Q_X}$ *be the optimal solution to the dual problem associated to* $\rho(\mathbf{X})$ *and assume that* $\rho(\mathbf{X}+\varepsilon\mathbf{V})$ *is differentiable with respect to* ε *at* $\varepsilon = 0$, *and* $\frac{d\mathbf{Q}_{\mathbf{X}+\varepsilon\mathbf{V}}}{d\mathbb{P}} \to \frac{d\mathbf{Q_X}}{d\mathbb{P}}$ *in* $\sigma^*(L^{\Phi^*}, M^{\Phi})$, *as* $\varepsilon \to 0$. *Then,*

$$\frac{d}{d\varepsilon}\rho(\mathbf{X}+\varepsilon\mathbf{V})|_{\varepsilon=0} = -\sum_{n=1}^{N} \mathbb{E}_{Q^n_{\mathbf{X}}}[V^n].$$

- This generalization holds because we are computing the expectation with respect to the systemic probability measure $\mathbf{Q_X}$.
- A relevant example where the assumptions of this Proposition hold is provided is the grouping and exponential setting that we present next.

Dual representation in the Grouping Example

For a partition $\mathbf{n}$ and for $\mathcal{C}^{(\mathbf{n})}$:

$$\rho(\mathbf{X}) = \max_{\mathbf{Q}\in\mathcal{D}} \left\{ \sum_{m=1}^{h} \mathbb{E}_{Q^m}[-\overline{X}_m] - \alpha_{\Lambda,B}(\mathbf{Q}) \right\},$$

with

$$\mathcal{D} := dom(\alpha_{\Lambda,B}) \cap \left\{ \frac{d\mathbf{Q}}{d\mathbb{P}} \in L_+^{\Phi^*} \mid Q^i = Q^j := Q^m \ \forall i,j \in I_m,\ \ Q^m(\Omega) = 1 \right\}$$

and

$$\overline{X}_m := \sum_{k\in I_m} X^k.$$

Exponential Setting

- $\mathcal{C} = \mathcal{C}^{(h)} \subseteq \mathcal{C}_{\mathbb{R}}$, for a partition of $\{1, \ldots., N\}$ in h groups.
- $u_n(x) = -e^{-\alpha_n x}$, $\alpha_n > 0$, $B < 0 = \sum_{n=1}^{N} u_n(+\infty)$.

$$\phi_n(x) := -u_n(-|x|) + u_n(0) = e^{\alpha_n |x|} - 1$$

$$M^{\phi_n} = M^{\exp} := \left\{ X \in L^0(\mathbb{R}) \mid \mathbb{E}[e^{c|X|}] < +\infty \text{ for all } c > 0 \right\},$$

$$\alpha_{\Lambda,B}(\mathbf{Q}) = \sum_{n=1}^{N} \mathbb{E}\left[\frac{dQ^n}{d\mathbb{P}} v_n'\left(\widehat{\lambda}\frac{dQ^n}{d\mathbb{P}}\right)\right] = \sum_{n=1}^{N} \frac{1}{\alpha_n}\left(H\left(Q^n, \mathbb{P}\right) + \ln\left(-\frac{B}{\beta\alpha_n}\right)\right)$$

with $\beta := \sum_{n=1}^{N} \frac{1}{\alpha_n}$ and $H\left(Q^n, \mathbb{P}\right) := \mathbb{E}\left[\frac{dQ^n}{d\mathbb{P}} \ln\left(\frac{dQ^n}{d\mathbb{P}}\right)\right]$.

In the exponential setting

- Existence, uniqueness and explicit formulation of the optimal solution $\mathbf{Y_X} \in M^{\exp}$ to $\rho_B(\mathbf{X})$, $\rho_B^{\mathbf{Q_X}}(\mathbf{X})$, $\pi_A(\mathbf{X})$ and $\pi_A^{\mathbf{Q_X}}(\mathbf{X})$ (for $A := \rho_B(\mathbf{X})$)

In the exponential setting

- Existence, uniqueness and explicit formulation of the optimal solution $\mathbf{Y_X} \in M^{\exp}$ to $\rho_B(\mathbf{X})$, $\rho_B^{\mathbf{Q_X}}(\mathbf{X})$, $\pi_A(\mathbf{X})$ and $\pi_A^{\mathbf{Q_X}}(\mathbf{X})$ (for $A := \rho_B(\mathbf{X})$)
- Existence, uniqueness and explicit formulation of the optimal solution $\mathbf{Q_X}$ to the dual problem associated to $\rho_B(\mathbf{X})$:

$$\frac{dQ^m_{\mathbf{X}}}{d\mathbb{P}} := \frac{e^{-\frac{1}{\beta_m}\overline{X}_m}}{\mathbb{E}\left[e^{-\frac{1}{\beta_m}\overline{X}_m}\right]} \quad m = 1, \cdots, h.$$

with $\beta_m = \sum_{k \in I_m} \frac{1}{\alpha_k}$, $\overline{X}_m = \sum_{k \in I_m} X^k$.

In the exponential setting

- Existence, uniqueness and explicit formulation of the optimal solution $\mathbf{Y_X} \in M^{\exp}$ to $\rho_B(\mathbf{X})$, $\rho_B^{\mathbf{Q_X}}(\mathbf{X})$, $\pi_A(\mathbf{X})$ and $\pi_A^{\mathbf{Q_X}}(\mathbf{X})$ (for $A := \rho_B(\mathbf{X})$)
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  with $\beta_m = \sum_{k \in I_m} \frac{1}{\alpha_k}$, $\overline{X}_m = \sum_{k \in I_m} X^k$.
- Monotonicity property: $\rho^n_1(\mathbf{X}) := Y^n_1 \geq \mathbb{E}_{Q^n_2}[Y^n_2] := \rho^n_2(\mathbf{X})$

In the exponential setting

- Existence, uniqueness and explicit formulation of the optimal solution $\mathbf{Y_X} \in M^{\exp}$ to $\rho_B(\mathbf{X})$, $\rho_B^{\mathbf{Q_X}}(\mathbf{X})$, $\pi_A(\mathbf{X})$ and $\pi_A^{\mathbf{Q_X}}(\mathbf{X})$ (for $A := \rho_B(\mathbf{X})$)
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with $\beta_m = \sum_{k\in I_m} \frac{1}{\alpha_k}$, $\overline{X}_m = \sum_{k\in I_m} X^k$.

- Monotonicity property: $\rho^n_1(\mathbf{X}) := Y^n_1 \geq \mathbb{E}_{Q^n_2}[Y^n_2] := \rho^n_2(\mathbf{X})$
- Total Marginal Risk Contribution:

$$\frac{d}{d\varepsilon}\rho(\mathbf{X}+\varepsilon\mathbf{V})|_{\varepsilon=0} = -\sum_{n=1}^{N} \mathbb{E}_{Q^n_{\mathbf{X}}}[V^n].$$

Sensitivity Analysis

Local causal responsibility:

$$\frac{d}{d\varepsilon}\mathbb{E}_{Q^m_{\mathbf{X}}}[Y^n_{\mathbf{X}+\varepsilon\mathbf{V}}]\Big|_{\varepsilon=0} = \mathbb{E}_{Q^m_{\mathbf{X}}}[-V^n], \quad n \in I_m,$$

Sensitivity of the penalty function:

$$\frac{d}{d\varepsilon}\alpha_{\Lambda,B}(\mathbf{Q}_{\mathbf{X}+\varepsilon\mathbf{V}})\Big|_{\varepsilon=0} = \sum_{m=1}^{h}\frac{1}{\beta_m}COV_{Q^m_{\mathbf{X}}}[\overline{V}_m,\overline{X}_m],$$

Marginal risk allocation of institution $n \in I_m$ (recall $\rho^n(\mathbf{X}+\varepsilon\mathbf{V}) = \mathbb{E}_{Q^m_{\mathbf{X}+\epsilon\mathbf{V}}}[Y^n_{\mathbf{X}+\varepsilon\mathbf{V}}]$)

$$\begin{aligned}\frac{d}{d\varepsilon}\mathbb{E}_{Q^m_{\mathbf{X}+\epsilon\mathbf{V}}}[Y^n_{\mathbf{X}+\varepsilon\mathbf{V}}]\Big|_{\varepsilon=0} &= \mathbb{E}_{Q^m_{\mathbf{X}}}[-V^n] + \frac{1}{\beta_m}COV_{Q^m_{\mathbf{X}}}[\overline{V}_m, X^n] \\ &\quad -\frac{1}{\alpha_n}\frac{1}{\beta_m}\frac{1}{\beta_m}COV_{Q^m_{\mathbf{X}}}[\overline{V}_m,\overline{X}_m]\end{aligned}$$

On marginal risk allocation of Bank n

$$\frac{d}{d\varepsilon}\mathbb{E}_{Q^m_{\mathbf{X}+\epsilon\mathbf{V}}}[Y^n_{\mathbf{X}+\varepsilon\mathbf{V}}]\Big|_{\varepsilon=0} = \mathbb{E}_{Q^m_{\mathbf{X}}}[-V^n] + \frac{1}{\beta_m}COV_{Q^m_{\mathbf{X}}}[\overline{V}_m, X^n] - \frac{1}{\alpha_n}\frac{1}{\beta_m}\frac{1}{\beta_m}COV_{Q^m_{\mathbf{X}}}[\overline{V}_m, \overline{X}_m]$$

- $\mathbb{E}_{Q^m_{\mathbf{X}}}[-V^n]$ is the contribution to the marginal risk allocation of bank n due only to the increment V^n of bank n (regardless of any systemic influence).
- When summing up we get the marginal risk allocation of the whole group.

On marginal risk allocation of Bank n

Take $\mathbf{V} = V^j \mathbf{e}_j$ with $j \neq n$, with j and n in the same group m. Then:

$$\frac{d}{d\varepsilon}\mathbb{E}_{Q^m_{\mathbf{X}+\varepsilon\mathbf{V}}}[Y^n_{\mathbf{X}+\varepsilon\mathbf{V}}]\Big|_{\varepsilon=0} = \frac{1}{\beta_m} COV_{Q^m_{\mathbf{X}}}[V^j, X^n] - \frac{1}{\alpha_n}\frac{1}{\beta_m}\frac{1}{\beta_m} COV_{Q^m_{\mathbf{X}}}[V^j, \overline{X}_m]$$

- The first component does not depend on the "systemic relevance" α_n of bank n
- but it depends on the dependence structure between (V^j, X^n).
- If $\frac{1}{\beta_m} COV_{Q^m_{\mathbf{X}}}[V^j, X^n] < 0$, the systemic risk evaluation $Q^m_{\mathbf{X}}$ attributes negative correlation to (V^j, X^n), then, from the systemic perspective this is good: a decrement in bank j is balanced by bank n, and viceversa, and the risk allocation of bank n should decrease.
- Bank n takes advantage of this, as its risk allocation is reduced.
- Since the overall marginal risk allocation of the group m is fixed (equal to $\mathbb{E}_{Q^m_{\mathbf{X}}}[-\overline{V}_m] = \mathbb{E}_{Q^m_{\mathbf{X}}}[-V^j]$), someone else has to pay for such advantage to bank n. This is the last term.

On marginal risk allocation of Bank n

Take $\mathbf{V} = V^j \mathbf{e}_j$ with $j \neq n$, with j and n in the same group m. Then:

$$\frac{d}{d\varepsilon}\mathbb{E}_{Q^m_{\mathbf{X}+\varepsilon\mathbf{V}}}[Y^n_{\mathbf{X}+\varepsilon\mathbf{V}}]\Big|_{\varepsilon=0} = \frac{1}{\beta_m} COV_{Q^m_{\mathbf{X}}}[V^j, X^n] - \frac{1}{\alpha_n}\frac{1}{\beta_m}\frac{1}{\beta_m} COV_{Q^m_{\mathbf{X}}}[V^j, \overline{X}_m]$$

- The third component does not depend on n, except for the systemic relevance α_n.
- It does not depend on the specific bank X^n but only on the aggregate group $\overline{X}_m$.
- It is a true systemic component
- Once the systemic component $-\frac{1}{\beta_m}\frac{1}{\beta_m} COV_{Q^m_{\mathbf{X}}}[V^j, \overline{X}_m]$ is determined, it is distributed among the various banks according to $\frac{1}{\alpha_n}$.
- This term must compensate for the possible risk reduction term (the other term), as the overall risk allocation to group m is determined by $\mathbb{E}_{Q^m_{\mathbf{X}}}[-\overline{V}_m] = \mathbb{E}_{Q^m_{\mathbf{X}}}[-V^j]$.

Take home message

Fairness is addressed by solving the dual problem

See the paper
On Fairness of Systemic Risk Measures
F. Biagini, J.-P. F., M. Frittelli, and T. Meyer-Brandis, 2018
for the details

Thanks for your attention