Optimal Dividend Distribution Under Drawdown and Ratcheting Constraints on Dividend Rates

Bahman Angoshtari
University of Washington, AMath
Joint work with Erhan Bayraktar and Virginia Young

9th Western Conference on Mathematical Finance
University of Southern California, November 2018
How do companies decide on their dividend policy?

Tradeoff: Paying out more dividends would increase a company’s worth in the eyes of its shareholders, while doing so would also reduce future reserves that are essential during financial hardships.
How do companies decide on their dividend policy?

Tradeoff: Paying out more dividends would increase a company’s worth in the eyes of its shareholders, while doing so would also reduce future reserves that are essential during financial hardships

Classical optimal dividend literature starts with De Finetti (1957)

See Avanzi (2009) for a survey


- Surplus is a Brownian motion with drift
- The objective is to maximize the expected discounted dividend payments until bankruptcy
How do companies decide on their dividend policy?

Tradeoff: Paying out more dividends would increase a company’s worth in the eyes of its shareholders, while doing so would also reduce future reserves that are essential during financial hardships.

Classical optimal dividend literature starts with De Finetti (1957).

See Avanzi (2009) for a survey.


- Surplus is a Brownian motion with drift.
- The objective is to maximize the expected discounted dividend payments until bankruptcy.
- Case I: dividend rate is restricted to $[0, C_0]$
  Pay dividends at rate 0 if surplus is lower than some value $X^*$ and at rate $C_0$ if surplus is greater than $X^*$
- Case II: dividend rate is unrestricted
  Payout any surplus in excess of a barrier $b$. 

Surplus is a Brownian motion with drift.
Following a barrier strategy results in a volatile all-or-nothing path for dividend stream

Shareholders and analysts react negatively (and arguably overreact) when the rate of dividend payment decreases

We propose a way to smooth the rate of dividend payments by requiring that the rate of dividend payments to never fall below a fraction of its historical maximum rate

This requirement is different from most drawdown constraints in the literature, which apply to the surplus or wealth itself e.g. Grossman and Zhou (1993), Cvitanić and Karatzas (1995), and Elie and Touzi (2008)
Problem Setup

- An optimal consumption problem, until bankruptcy, with a drawdown constraint on consumption
An optimal consumption problem, until bankruptcy, with a drawdown constraint on consumption

A company is deciding on its investment and dividend policies

Investment policy:
- For simplicity, we assume the number of shares to be fixed
  only bonds are issued or bought back
- \( \pi_t \): total asset at \( t \)
- \( X_t \): total equity at \( t \), also called the “surplus”
- \( \pi_t - X_t \): total debt
Dividend policy: dividend is paid at rate $C_t$ $$/yr from the surplus
Total dividend paid over $[t, t + \varepsilon]$ is $\int_t^{t+\varepsilon} C_u \, du$
Dividend policy: dividend is paid at rate $C_t$ $$/yr from the surplus. Total dividend paid over $[t, t + \varepsilon]$ is $\int_{t}^{t+\varepsilon} C_u du$

Dividend policy is subject to two constraints

(i) The dividend rate must be higher than the interest rate

$$c_t := C_t - rX_t > 0; \quad t \geq 0$$

(ii) The drawdown constraint: $c_t \geq \alpha z_t; \quad t \geq 0$

$$z_t := \max \left\{ z, \sup_{0 \leq s < t} c_s \right\}, \quad z > 0 \text{ and } 0 < \alpha \leq 1$$

Dividend rate is allowed to increase beyond its historical peak in which case $c_t > z_t$
Surplus evolves according to

\[ dX_t = \pi_t \frac{dI_t}{I_t} + \left( r(X_t - \pi_t) - C_t \right) dt = \left( \mu \pi_t - c_t \right) dt + \sigma \pi_t dW_t, \]

where \( \{I_t\} \), the "intrinsic value" of the company, is a GBM

\[ \frac{dI_t}{I_t} = (\mu + r) dt + \sigma dW_t \]

\((W_t)_{t \geq 0}\) is a Brownian motion on \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})\)
Problem Setup

- Surplus evolves according to

\[ dX_t = \pi_t \frac{dI_t}{I_t} + \left( r(X_t - \pi_t) - C_t \right) dt = \left( \mu \pi_t - c_t \right) dt + \sigma \pi_t dW_t, \]

where \( \{I_t\} \), the “intrinsic value” of the company, is a GBM

\[ \frac{dI_t}{I_t} = (\mu + r)dt + \sigma dW_t \]

\( (W_t)_{t \geq 0} \) is a Brownian motion on \( (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P}) \)

- \((\pi_t, c_t)_{t \geq 0}\) is admissible if:
  
  (i) \((\pi_t)_{t \geq 0}\) is \((\mathcal{F}_t)\)-progressively measurable and satisfies \( \pi_t \geq 0 \) and \( \int_0^{\infty} \pi_t^2 dt < \infty \), \( \mathbb{P} \)-almost surely
  
  (ii) \((c_t)_{t \geq 0}\) is \((\mathcal{F}_t)\)-adapted, non-negative, and right-continuous with left limits; and

  (iii) \( c_t \geq \alpha z_t, t \geq 0 \), where \( z_t := \max \left\{ z, \sup_{0 \leq s < t} c_s \right\} \)

- \( C(\alpha, z) \): the set all admissible policies
Objective:

\[
\sup_{(\pi_t, c_t) \in \mathcal{C}(\alpha, z)} \mathbb{E} \left[ \int_0^\tau e^{-\delta t} \frac{c_t^{1-p}}{1-p} \, dt \right].
\]

\[
\tau = \tau^{X(\pi_t, c_t)} := \inf \{ t \geq 0 : X_t \leq 0 \} \text{ is the time of bankruptcy}
\]

For \( \alpha, z > 0 \), we have \( c_t > 0 \) and thus bankruptcy is “not avoidable”

\[
P(\tau < \infty) > 0
\]

\( \delta > 0 \): subjective time preference, larger values indicates more impatient shareholders
Problem Setup

- $p$ is the constant relative risk aversion, satisfying $\frac{1}{\frac{2\sigma^2}{\mu^2} \delta + 1} < p < 1$, reason:

- $0 \leq p \leq \frac{1}{\frac{2\sigma^2}{\mu^2} \delta + 1}$ leads to infinite expectation, e.g. Merton (1969)
$p$ is the constant relative risk aversion, satisfying $\frac{1}{\frac{2\sigma^2}{\mu^2} \delta + 1} < p < 1$, reason:

- $0 \leq p \leq \frac{1}{\frac{2\sigma^2}{\mu^2} \delta + 1}$ leads to infinite expectation, e.g. Merton (1969)

- $p \geq 1$: How to “penalize” bankruptcy? Two suggestions:

$$\int_{0}^{\tau} e^{-\delta t} \frac{c_{t}^{1-p}}{1-p} \, dt \rightarrow \tau < \infty \text{ is rewarded! Immediate liquidation is optimal}$$

$$\int_{0}^{\infty} e^{-\delta t} \frac{c_{t}^{1-p}}{1-p} \, dt \rightarrow \tau < \infty \text{ is penalized by } -\infty \text{, the problem is infeasible}$$

Note that $\frac{c_{t}^{1-p}}{1-p} \rightarrow \infty$ as $p \rightarrow 1$
The optimal dividend policy depends on \((X_t)\) and \((z_t)\) or, more specifically, the value of the “surplus-to-historical peak” ratio \(X_t/z_t\).
Optimal Dividend Policy

The optimal dividend policy depends on \((X_t)\) and \((z_t)\) or, more specifically, the value of the “surplus-to-historical peak” ratio \(X_t/z_t\)

There exist constants \(0 < w_{α} < w_1 < w^*\) such that:

(a) If \(X_t < w_{α} z_t\), then \(c_t = αz_t\)

(b) If \(w_{α} z_t < X_t < w_1 z_t\), then \(c_t = c^*(X_t, z_t) \in (αz_t, z_t)\), for some function \(c^*(x, z)\)

(c) If \(w_1 z_t \leq X_t < w^* z_t\), then \(c_t = z_t\)

(d) If \(X_t > w^* z_t\), then \(c_t = \frac{X_t}{w^*} > z_t\)

In this case, the historical peak has a jump at \(t\), that is, \(\lim_{s \to t^+} z_s = \frac{X_t}{w^*} > z_t\)

(e) Along the line \(x = w^* z\), the company increases its dividend rate via singular control to keep \(X_t \leq w^* z_t\)
Optimal Dividend Policy

Five regions in the $zx$-plane for the optimal dividend policy

(a): $c^* = \alpha z$

(b): $\alpha z < c^* < z$

(c): $c^* = z$

(d): $c^* = \frac{x}{w^*}$

(e): $x = w^* z$
(z_t) can have a jump only at time \( t = 0 \) and only if rule (d) is applicable, that is, \( X_0 > w^* z_0 \)

Afterwards, the process \((X_t, z_t)_{t \geq 0}\) will be kept in the domain
\[ D = \{(x, z) : 0 \leq x \leq w^* z, z > 0\} \]

In particular, \((z_t)\) is only allowed to increase via singular control in order to keep \((X_t, z_t)\) inside \( D \)
(z_t) can have a jump only at time $t = 0$ and only if rule (d) is applicable, that is, $X_0 > w^* z_0$

Afterwards, the process $(X_t, z_t)_{t \geq 0}$ will be kept in the domain
$$D = \{(x, z) : 0 \leq x \leq w^* z, z > 0\}$$

In particular, $(z_t)$ is only allowed to increase via singular control in order to keep $(X_t, z_t)$ inside $D$

As a consequence of the optimal policy, $M_t^* = w^* z_t; \quad t > 0$

where $M_t^* := \max \left\{ M_0, \max_{0 \leq s < t} X_s^* \right\}$ and $M_0^* = w^* z_0$

The running max of (the optimally controlled) surplus is proportional to the historical consumption peak
Plots of the Optimal Policy

\[ \pi^*(x, 1) \]  Plot of \( \pi^*(x, 1) \) vs. \( x \)

\[ \pi^*(x, z) \]  Plot of \( \pi^*(x, 1) \) and \( \pi^*(x, 3) \) vs. \( x \)

\[ c^*(x, 1) \]  Plot of \( c^*(x, 1) \) vs. \( x \)

\[ c^*(x, z) \]  Plot of \( c^*(x, 1) \) and \( c^*(x, 3) \) vs. \( x \)

\[ \pi^* = \frac{\mu}{\sigma^2 w^* U_{a}} \]

\[ c^* = \frac{x}{w^*} \]
Value function: $V(x, z) = \sup_{(\pi_t, c_t) \in C(\alpha, z)} E^x \left[ \int_0^\tau e^{-\delta t} \frac{c_t^{1-p}}{1-p} \, dt \right]$

HJB equation:

$$
\begin{align*}
\delta v &= \max_{\pi \in \mathbb{R}} \left[ \mu \pi v_x + \frac{1}{2} \sigma^2 \pi^2 v_{xx} \right] + \max_{\alpha z \leq c \leq z} \left[ \frac{c^{1-p}}{1-p} - cv_x \right] \\
v(0, z) &= 0 \\
v_z(w^* z, z) &= 0 = v_{xz}(w^* z, z)
\end{align*}
$$

The last conditions are the “smooth-pasting” and “super-contact” conditions.

See, for example, Dixit (1991) and Dumas (1991)
• $V(x, z)$ is homogeneous of degree $1 - p$ with respect to $x$ and $z$
  \[ V(\beta x, \beta z) = \beta^{1-p} V(x, z); \quad \beta > 0 \]

• Using the ansatz $V(x, z) = z^{1-p} U(x/z)$ leads to

\[ \begin{cases} 
    \delta U = \max_{\hat{\pi} \in \mathbb{R}} \left[ \mu \hat{\pi} U_w + \frac{1}{2} \sigma^2 \hat{\pi}^2 U_{ww} \right] + \max_{\alpha \leq \hat{\epsilon} \leq 1} \left[ \frac{\hat{\epsilon}^{1-p}}{1-p} - \hat{\epsilon} U_w \right] \\
    U(0) = 0 \\
    (1 - p)U(w^*) - w^* U_w(w^*) = 0 \\
    pU_w(w^*) + w^* U_{ww}(w^*) = 0
\end{cases} \]

• Once we obtained $\hat{\pi}^*$ and $\hat{\epsilon}^*$, we get $\pi^*$ and $c^*$ via
  \[ \pi^*(x, z) = \hat{\pi}^*(x/z) z \quad \text{and} \quad c^*(x, z) = \hat{\epsilon}^*(x/z) z \]
Assuming $U$ is increasing and concave with respect to $w$

$$
\frac{1}{\kappa} \frac{U_w^2}{U_{ww}} + \delta U = \begin{cases} 
\frac{\alpha^{1-p}}{1-p} - \alpha U_w, & 0 \leq w \leq w_\alpha \\
\frac{p}{1-p} \left(U_w(w)\right)^{-\frac{1-p}{p}}, & w_\alpha < w < w_1 \\
\frac{1}{1-p} - U_w, & w_1 \leq w \leq w^*
\end{cases}
$$

Here, $\kappa := \frac{2\sigma^2}{\mu^2}$ and $w_\alpha$ and $w_1$ are free boundaries satisfying

$$U_w(w_\alpha) = \alpha^{-p} \text{ and } U_w(w_1) = 1$$

We can linearize the equation by applying the Legendre transform
Define $y_0 := U_w(0) \geq \alpha^{-p}$, $y^* := U_w(w^*) \leq 1$, and

$$\hat{U}(y) := \sup_{0 \leq w \leq w^*}\{U(w) - wy\}; \quad y^* \leq y \leq y_0$$

$\hat{U}$ satisfies:

$$y^2\hat{U}_{yy} + \kappa\delta y\hat{U}_y - \kappa\delta\hat{U} = \begin{cases} \\
\kappa\left(\alpha y - \frac{\alpha^{1-p}}{1-p}\right), & \alpha^{-p} \leq y \leq y_0 \\
-\frac{\kappa p}{1-p} y^{-\frac{1-p}{p}}, & 1 < y < \alpha^{-p} \\
\kappa\left(y - \frac{1}{1-p}\right), & y^* \leq y \leq 1 \\
\end{cases}$$

$\hat{U}(y_0) = 0 = \hat{U}_y(y_0)$

$(1 - p)\hat{U}(y^*) + py^*\hat{U}_y(y^*) = 0$

$\hat{U}_y(y^*) + py^*\hat{U}_{yy}(y^*) = 0$
Lemma ($y_0$ and $y^*$)

There exist unique constants $\eta^* = y_0\alpha^p > 1$ and $0 < y^* < 1$ that solves the system:

\[
\begin{align*}
\ln \frac{\eta^\alpha}{y} + \frac{\alpha}{\eta(1 - p)} - \frac{1}{y} &= \alpha(1 + p) - 1 \\
\alpha^{1-p(1+\kappa\delta)} \left(p(1 + \kappa\delta) - 1\right) \left(\frac{\kappa}{1 + \kappa\delta} \eta^{1+\kappa\delta} - \frac{1}{\delta(1 - p)} \eta^{\kappa\delta}\right) \\
+ \left(\frac{\kappa}{1 + \kappa\delta} y^{1+\kappa\delta} - \frac{1}{\delta} y^{\kappa\delta}\right) &= \frac{\alpha^{1-p(1+\kappa\delta)} - 1}{\delta(1 + \kappa\delta)}
\end{align*}
\]
Proposition ($\hat{U}$)

$\hat{U}$ is given by

$$\hat{U}(y) = \begin{cases} 
C_1 y + C_2 y^{-\kappa \delta} + \frac{\kappa \alpha}{1 + \kappa \delta} y \ln y + \frac{\alpha^{1-p}}{\delta (1 - p)}, & \alpha^{-p} \leq y \leq y_0 \\
C_3 y + C_4 y^{-\kappa \delta} + \frac{\kappa}{1 - p} \frac{p^3}{p(1 + \kappa \delta) - 1} y^{-\frac{1-p}{p}}, & 1 < y < \alpha^{-p} \\
C_5 y + C_6 y^{-\kappa \delta} + \frac{\kappa}{1 + \kappa \delta} y \ln y + \frac{1}{\delta (1 - p)}, & y^* \leq y \leq 1
\end{cases}$$

with $C_1, \ldots, C_6$ given in the next slide.

Moreover, $\hat{U}$ is strictly decreasing and strictly convex with continuous second derivative on $(y^*, y_0)$. 
Solution: $\hat{U}$

\begin{align*}
C_1 &= -\frac{\kappa\alpha}{1 + \kappa\delta} \left( \ln \eta^* - p \ln \alpha + \frac{1}{\eta^*(1 - p)} + \frac{1}{1 + \kappa\delta} \right) \\
C_2 &= \frac{\alpha^{1 - p(1 + \kappa\delta)}}{1 + \kappa\delta} \left( \frac{\kappa}{1 + \kappa\delta} (\eta^*)^{1 + \kappa\delta} - \frac{1}{\delta(1 - p)} (\eta^*)^{\kappa\delta} \right) > 0 \\
C_3 &= -\frac{\kappa\alpha}{1 + \kappa\delta} \left( \ln \eta^* + \frac{1}{\eta^*(1 - p)} - (1 + p) \right) \\
C_4 &= \frac{\alpha^{1 - p(1 + \kappa\delta)}}{1 + \kappa\delta} \left( \frac{\kappa}{1 + \kappa\delta} (\eta^*)^{1 + \kappa\delta} - \frac{1}{\delta(1 - p)} (\eta^*)^{\kappa\delta} - \frac{1}{\delta(1 + \kappa\delta)(\alpha(1 + \kappa\delta) - 1)} \right) < 0 \\
C_5 &= -\frac{\kappa}{1 + \kappa\delta} \left( \alpha \ln \eta^* + \frac{\alpha}{\eta^*(1 - p)} + (1 - \alpha)(1 + p) + \frac{1}{1 + \kappa\delta} \right) \\
C_6 &= \frac{\alpha^{1 - p(1 + \kappa\delta)}}{1 + \kappa\delta} \left( \frac{\kappa}{1 + \kappa\delta} (\eta^*)^{1 + \kappa\delta} - \frac{1}{\delta(1 - p)} (\eta^*)^{\kappa\delta} \right) - \frac{\alpha^{1 - p(1 + \kappa\delta)} - 1}{\delta(1 + \kappa\delta)^2 (p(1 + \kappa\delta) - 1)} > 0
\end{align*}
We find $U(w)$ by reversing the Legendre transform

$$U(w) = \hat{U}(y) - y\hat{U}_y(y), \text{ where } y \in [y^*, y_0] \text{ uniquely solves } \hat{U}_y(y) = -w$$

We then find the (candidate) value function $V(x, z) = z^{1-p}U(x/z)$
We find $U(w)$ by reversing the Legendre transform

$$U(w) = \hat{U}(y) - y \hat{U}_y(y), \text{ where } y \in [y^*, y_0] \text{ uniquely solves } \hat{U}_y(y) = -w$$

We then find the (candidate) value function $V(x, z) = z^{1-p} U(x/z)$

The critical values $w_\alpha$, $w_1$, and $w^*$ are given by

$$w_\alpha = \frac{\kappa \alpha}{1 + \kappa \delta} \left\{ \ln y^* + \left( \frac{\kappa \delta}{1 + \kappa \delta} - \frac{1}{\eta^*(1-p)} \right) \left( \eta^* \right)^{1+\kappa \delta} - 1 \right\}$$

$$w_1 = \frac{\kappa}{1 + \kappa \delta} \left\{ \ln y^* + p + \left( \frac{1}{y^*} - \frac{\kappa \delta}{1 + \kappa \delta} \right) \left( 1 + \frac{(y^*)^{1+\kappa \delta}}{p(1 + \kappa \delta) - 1} \right) \right\}$$

$$w^* = \frac{\kappa p}{p(1+\kappa \delta) - 1} \left\{ \frac{1}{y^*} - (1 - p) \right\}$$
For $0 \leq x \leq w^* z$: Let $y \in [y^*, y_0]$ be the unique solution of $\hat{U}_y(y) = -x/z$

$$\pi^*(x, z) = -\frac{\mu}{\sigma^2} \frac{z U_w(x/z)}{U_{ww}(x/z)} = \frac{\mu}{\sigma^2} z y \hat{U}_{yy}(y)$$

$$c^*(x, z) = \begin{cases} 
\alpha z, & 0 \leq x \leq w_\alpha z, \\
y - \frac{1}{p} z, & w_\alpha z < x < w_1 z, \\
z, & w_1 z \leq x < w^* z. 
\end{cases}$$

For $x > w^* z$:

$$\pi^*(x, z) = -\frac{\mu}{\sigma^2} \frac{U_w(w^*)}{w^* U_{ww}(w^*)} x = \frac{\mu}{\sigma^2} \frac{y^*}{w^*} \hat{U}_{yy}(y^*) x$$

$$c^*(x, z) = \frac{x}{w^*}$$
To verify the solution, we need to show that, for all $x, z \geq 0$, $\pi \in \mathbb{R}$, and $c \geq \alpha z$

(i) $v_z(x, z) \leq 0$

(ii) $\frac{1}{2} \sigma^2 \pi^2 v_{xx}(x, z) + (\mu \pi - c)v_x(x, z) - \delta v(x, z) + \frac{c^{1-p}}{1-p} \leq 0$

(iii) the "transversality condition:"

$$\liminf_{n \to \infty} \mathbb{E}^x \left( e^{-\delta \tau_n} v(X_{\tau_n}, z_{\tau_n}) \right) = 0$$

$\{\tau_n\}_{n=1}^{\infty}$ is a sequence of bounded stopping times satisfying $\tau_n \to \infty$ $\mathbb{P}$-a.s.

(iv) $\max \left[ v_z, \frac{1}{2} \sigma^2 (\pi^*)^2 v_{xx} + (\mu \pi^* - c^*)v_x + \frac{(c^*)^{1-p}}{1-p} - \delta v \right] = 0; \quad x, z > 0$

(v) The following SDE has a unique strong solution

$$dX_t^* = \left( \mu \pi^* (X_t^*, z_t^*) - c^* (X_t^*, z_t^*) \right) dt + \sigma \pi^* (X_t^*, z_t^*) dW_t; \quad t \geq 0,$$

$$z_t^* = \max \left\{ z, \sup_{0 \leq s < t} c^* (X_s^*, z_s^*) \right\}; \quad t \geq 0,$$

$$X_0^* = x,$$

and $(\pi^* (X_t^*, z_t^*), c^* (X_t^*, z_t^*))$ is admissible
Thank you for your attention!


Verification


\[ \mu = 0.08, \sigma = 0.2, \delta = 0.2, \alpha = 0.5, p = 0.8, w_\alpha = 3.7030, w_1 = 5.5947, \text{ and } w^* = 11.2992 \]
Plot of the Optimal Policy

$\pi^*(x, 1)$ Plot of $\pi^*(x, 1)$ vs. $x$

$\pi^*(x, z)$ Plot of $\pi^*(x, 1)$ and $\pi^*(x, 3)$ vs. $x$

$c^*(x, 1)$ Plot of $c^*(x, 1)$ vs. $x$

$c^*(x, z)$ Plot of $c^*(x, 1)$ and $c^*(x, 3)$ vs. $x$
The free boundaries \( w_\alpha, w_1, \) and \( w^* \) vs. \( \alpha \)

\[ \mu = 0.0800, \quad \sigma = 0.2000, \quad \kappa = 12.5000, \quad \delta = 0.2000 \]
Sensitivity of the Value function w.r.t. $\alpha$

Plot of $V(x, 1)$ for three values of $\alpha$

$\alpha = 0.00$
$\alpha = 0.50$
$\alpha = 1.00$

$\mu = 0.0800$, $\sigma = 0.2000$, $\kappa = 12.5000$, $\delta = 0.2000$
Sensitivity of the Optimal Policy w.r.t. $\alpha$

**Plot of $\pi^*(x,1)$ vs. $x$ and $0 < \alpha < 1$**

**Plot of $c^*(x,1)$ vs. $x$ and $0 < \alpha < 1$**

**Plot of $\pi^*(x,1)$ for three values of $\alpha$**

**Plot of $c^*(x,1)$ for three values of $\alpha$**

---

**Parameters**

$\mu = 0.0800$, $\sigma = 0.2000$, $\kappa = 12.5000$, $\delta = 0.2000$
Sensitivity of the Value Function w.r.t. $p$

Plot of $V(x, 1)$ vs. $x$ and $0.286 \approx \frac{1}{1+\kappa \delta} < p < 1$