Imperfect Competition Among Liquidity Providers

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Motivation

LO specifies a price and a quantity \rightarrow price elastic demand

MO specifies only a quantity \rightarrow price inelastic demand

Trade means MO executes against LO
Motivation

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MO specifies only a quantity → price inelastic demand

Trade means MO executes against LO
Motivation

- LO specifies a price and a quantity

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Motivation

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- MO specifies only a quantity
Motivation

- LO specifies a price and a quantity
- MO specifies only a quantity
- Trade means MO executes against LO
Motivation

- LO specifies a price and a quantity $\rightarrow$ **price elastic demand**
- MO specifies only a quantity
- Trade means MO executes against LO
Motivation

- LO specifies a price and a quantity → price elastic demand
- MO specifies only a quantity → price inelastic demand
- Trade means MO executes against LO
Related Work

- Grossman-Miller (1985)
- Kyle (1989)
- Almgren & Chriss (2001), Cartea & Jaimungal
- Garleanu & Pedersen (2016)
- Rostek & Wretka (2015)
- Sannikov & Skrzypacz (2016)
**Related Work**

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4 Equilibrium Analysis
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The Model

Single asset in zero net supply
Smooth trading on an infinite horizon

Two types of traders:
N Market Makers (MMs)
Liquidity Traders (LTs)

Trades occur at a uniform price $p_t$ to be determined endogenously
The Model

- Single asset in zero net supply
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The Model

\[ S_t = S_t^\prime dt \]
\[ dF_t = -\psi F_t dt + \sigma F_t dB_t \]
\[ \psi > 0 \]

MMs inventories evolve according to

\[ dX_{nt} = q_{nt} dt \]

where \( q_{nt} \) will be determined endogenously

MMs cash evolves according to

\[ dC_{nt} = -q_{nt} p_t dt \]
The Model

- LTs inventory is $-S_t$, where

\[ dS_t = F_t \, dt \]

\[ dF_t = -\psi F_t \, dt + \sigma_F \, dB_t^F, \quad \psi > 0 \]
The Model

- LTs inventory is \(-S_t\), where

\[
dS_t = F_t dt
\]

\[
dF_t = -\psi F_t dt + \sigma_F dB_t^F, \quad \psi > 0
\]

- MMs inventories evolve according to

\[
dx_t^n = q_t^n dt
\]
The Model

- LTs inventory is $-S_t$, where
  \[ dS_t = F_t dt \]
  \[ dF_t = -\psi F_t dt + \sigma F_t dB_t, \quad \psi > 0 \]

- MMs inventories evolve according to
  \[ dX^n_t = q^n_t dt \]
  where $q^n_t$ will be determined endogenously
The Model

- LTs inventory is $-S_t$, where
  
  $$dS_t = F_t dt$$
  
  $$dF_t = -\psi F_t dt + \sigma F dB_t^F, \quad \psi > 0$$

- MMs inventories evolve according to
  
  $$dX^n_t = q^n_t dt$$

  where $q^n_t$ will be determined endogenously

- MMs cash evolves according to
  
  $$dC^n_t = -q^n_t p_t dt$$
MM Objectives

MMs valuation of the stock is exogenously given as

\[ D_t \]

where

\[ dD_t = \mu dt + \sigma D dB \]

\[ \{B_D t\} \]

is independent of \[ \{B_F t\} \]

MMs marked-to-market wealth is

\[ W_n t = C_n t + X_n t D_t \]

MMs want to maximize

\[ \mathbb{E} \left[ \int_0^\infty e^{-\rho t} \left( dW_n t - \gamma^2 d\langle W_n \rangle_t \right) \right] = \mathbb{E} \left[ \int_0^\infty e^{-\rho t} \left( -q_n t (p_t - D_t) + \mu X_n t - \gamma \sigma^2 X_n^2 t \right) dt \right] \]
MM Objectives

- MMs valuation of the stock is exogenously given as $D_t$ where

\[ dD_t = \mu dt + \sigma_D dB_t^D \]

and \( \{B_t^D\} \) is independent of \( \{B_t^F\} \)
MM Objectives

- MMs valuation of the stock is exogenously given as $D_t$ where
  \[ dD_t = \mu dt + \sigma_D dB^D_t \]
  and $\{B^D_t\}$ is independent of $\{B^F_t\}$
- MMs marked-to-market wealth is
  \[ W_t^n = C_t^n + X_t^n D_t \]
MM Objectives

- MMs valuation of the stock is exogenously given as $D_t$ where

$$dD_t = \mu dt + \sigma_D dB^D_t$$

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- MMs marked-to-market wealth is

$$W^n_t = C^n_t + X^n_tD_t$$

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MM Objectives

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- MMs marked-to-market wealth is
  
  $$W^n_t = C^n_t + X^n_t D_t$$

- MMs want to maximize
  
  $$\mathbb{E} \left[ \int_0^\infty e^{-\rho t} \left( dW^n_t - \frac{\gamma}{2} d\langle W^n \rangle_t \right) \right]$$
MM Objectives

- MMs valuation of the stock is exogenously given as $D_t$ where
  \[ dD_t = \mu dt + \sigma_D dB_t^D \]
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- MMs marked-to-market wealth is
  \[ W^n_t = C^n_t + X^n_t D_t \]
- MMs want to maximize
  \[
  \mathbb{E} \left[ \int_0^\infty e^{-\rho t} \left( dW^n_t - \frac{\gamma}{2} d\langle W^n \rangle_t \right) \right] = \\
  \mathbb{E} \left[ \int_0^\infty e^{-\rho t} \left( - q^n_t (\rho_t - D_t) + \mu X^n_t - \frac{\gamma \sigma_D^2}{2} (X^n_t)^2 \right) dt \right]
  \]
The Trading Mechanism

At time $t$, each MM submits a demand schedule of the form $\alpha_n t - \beta n t q_n t = p_t$ and $q_n t$ are then given implicitly by the $N + 1$ equations $\alpha_n t - \beta n t q_n t = p_t q_1 t + \cdots + q_N t = F_t$.

Price and trades are set in Nash equilibrium.
At time $t$ each MM submits a demand schedule of the form

$$\alpha_t^n - \beta_t^n q_t^n = p_t$$
At time $t$ each MM submits a demand schedule of the form

$$\alpha^n_t - \beta^n_t q^n_t = p_t$$

$p_t$ and $q^n_t$ are then given implicitly by the $N+1$ equations

$$\alpha^n_t - \beta^n_t q^n_t = p_t$$
$$q^1_t + \cdots + q^N_t = F_t$$
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$$q^1_t + \cdots + q^N_t = F_t$$

Price and trades are set in Nash equilibrium
Equilibrium Concept

A profile $\alpha_1^t, \beta_1^t, \ldots, \alpha_N^t, \beta_N^t$ is admissible if $\beta_n^t > 0$ a.s.

$$E\left[e^{-\rho t}(X_n^t)^2\right] \to 0 \text{ as } t \to \infty$$

$\alpha_n^t, \beta_n^t \in \sigma(\{D_s\}_{0 \leq s \leq t}, \{p_s\}_{0 \leq s < t}, \{X_n^s\}_{0 \leq s \leq t}, \{q_n^s\}_{0 \leq s < t}, S_0)$

A profile is linear and symmetric if $\exists a < 0, e > 0, b, c, \xi \in \mathbb{R}$ s.t.

$$\alpha_n^t = aX_n^t + bD_t + cS_t + \xi$$

$$\beta_n^t = e$$

Will focus on linear symmetric Nash equilibrium
A profile \( \alpha^1_t, \beta^1_t, \ldots, \alpha^N_t, \beta^N_t \) is admissible if
Equilibrium Concept

A profile $\alpha_1^t, \beta_1^t, \ldots, \alpha_N^t, \beta_N^t$ is **admissible** if

- $\beta_t^n > 0$ a.s.
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A profile is linear and symmetric if $\exists a < 0, e > 0$, and $b, c, \xi \in \mathbb{R}$ s.t.

$$\alpha^n_t = aX^n_t + bD_t + cS_t + \xi$$
$$\beta^n_t = e$$
Equilibrium Concept

- A profile $\alpha^n_1, \beta^n_1, \cdots, \alpha^n_N, \beta^n_N$ is **admissible** if
  - $\beta^n_t > 0$ a.s.
  - $\mathbb{E}[e^{-\rho t}(X^n_t)^2] \to 0$ as $t \to \infty$
  - $\alpha^n_t, \beta^n_t \in \sigma\left(\{D_s\}_{0 \leq s \leq t}, \{p_s\}_{0 \leq s < t}, \{X^n_s\}_{0 \leq s \leq t}, \{q^n_s\}_{0 \leq s < t}, S_0\right)$

- A profile is **linear and symmetric** if $\exists$ $a < 0$, $e > 0$, and $b, c, \xi \in \mathbb{R}$ s.t.

  $$\begin{align*}
  \alpha^n_t &= aX^n_t + bD_t + cS_t + \xi \\
  \beta^n_t &= e
  \end{align*}$$

- Will focus on linear symmetric Nash equilibrium
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Explicit Equilibrium

Theorem

If \( N \geq 3 \) then there is a unique linear symmetric Nash equilibrium.

In equilibrium the price is 
\[
p_t = D_t + \mu \rho - \gamma N \sigma^2 D S_t - \gamma N N - 2 \sigma^2 D \rho + \psi (1 + \rho) F_t
\]
and trading rates are 
\[
q_{nt} = -\kappa (X_{nt} - S_t N) + \frac{1}{N} F_t
\]
where \( \kappa = \rho (N - 2) \rho + \psi \rho + \delta \delta = \sqrt{\rho^2 + 2 \rho (\rho + \psi) (N - 2)} \).
Explicit Equilibrium

Theorem

If $N \geq 3$ then there is a unique linear symmetric Nash equilibrium.
Theorem

If \( N \geq 3 \) then there is a unique linear symmetric Nash equilibrium. In equilibrium the price is

\[
p_t = D_t + \frac{\mu}{\rho} - \frac{1}{\rho} \frac{\gamma}{N} \sigma_D^2 S_t - \frac{\gamma}{N} \frac{N-1}{N-2} \frac{\sigma_D^2}{\rho} \left( \frac{1}{\delta} + \frac{1}{\rho} \right) F_t
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If $N \geq 3$ then there is a unique linear symmetric Nash equilibrium. In equilibrium the price is

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and trading rates are

$$q_t^n = -\kappa \left( X_t^n - \frac{S_t}{N} \right) + \frac{1}{N} F_t$$
Explicit Equilibrium

**Theorem**

If $N \geq 3$ then there is a unique linear symmetric Nash equilibrium. In equilibrium the price is

$$p_t = D_t + \frac{\mu}{\rho} - \frac{1}{\rho} \frac{\gamma}{N} \sigma_D^2 S_t - \frac{\gamma}{N} \frac{N - 1}{N - 2} \frac{\sigma_D^2}{\rho + \psi} \left( \frac{1}{\delta} + \frac{1}{\rho} \right) F_t$$

and trading rates are

$$q^n_t = -\kappa \left( X^n_t - \frac{S_t}{N} \right) + \frac{1}{N} F_t$$

where

$$\kappa = \rho (N - 2) \frac{\rho + \psi}{\rho + \delta}$$

$$\delta = \sqrt{\rho^2 + 2\rho (\rho + \psi)(N - 2)}$$
Sketch of Proof

On equilibrium:
\[ \hat{p}_t = a N S_t + b D_t + c S_t + \xi - e N F_t \]
\[ \hat{q}_n t = a e (\hat{X}_n t - S_{nt}) + 1 N F_t \]

Off equilibrium:
\[ p_t = a_{N-1} (S_t - X_t) + b D_t + c S_t + \xi - e_{N-1} (F_t - q_t) \]
\[ p_t = \alpha_t - \beta_t q_t \]
On equilibrium

\[ \hat{p}_t = \frac{a}{N} S_t + bD_t + cS_t + \xi - \frac{e}{N} F_t \]

\[ \hat{q}^n_t = \frac{a}{e} \left( \hat{X}^n_t - \frac{S_t}{N} \right) + \frac{1}{N} F_t \]
Sketch of Proof

- On equilibrium

\[ \hat{p}_t = \frac{a}{N} S_t + bD_t + cS_t + \xi - \frac{e}{N} F_t \]
\[ \hat{q}_t^n = \frac{a}{e} \left( \hat{X}_t^n - \frac{S_t}{N} \right) + \frac{1}{N} F_t \]

- Off equilibrium

\[ p_t = \frac{a}{N - 1} (S_t - X_t) + bD_t + cS_t + \xi - \frac{e}{N - 1} (F_t - q_t) \]
\[ p_t = \alpha_t - \beta_t q_t \]
Optimize directly over \( \{ q_t \} \)
Sketch of Proof

- Optimize directly over \( \{q_t\} \)
- Almgren-Chriss type problem
Sketch of Proof

- Optimize directly over \( \{q_t\} \)

- Almgren-Chriss type problem

\[
p_t = \frac{a}{N-1} S_t + b D_t + c S_t + \xi - \frac{e}{N-1} F_t - \frac{a}{N-1} X_t + \frac{e}{N-1} q_t
\]
Sketch of Proof

- Optimize directly over $\{q_t\}$

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\]

\[
= M_t + \Lambda X_t + \lambda q_t
\]
Sketch of Proof

- Optimize directly over \( \{q_t\} \)

- Almgren-Chriss type problem
  
  \[
  p_t = \frac{a}{N-1}S_t + bD_t + cS_t + \xi - \frac{e}{N-1}F_t - \frac{a}{N-1}X_t + \frac{e}{N-1}q_t \\
  = M_t + \Lambda X_t + \lambda q_t
  \]

- Dynamic Programming, HJB, Feedback Controller
Sketch of Proof

- Optimize directly over \( \{q_t\} \)

- Almgren-Chriss type problem

\[
p_t = \frac{a}{N-1} S_t + b D_t + c S_t + \xi - \frac{e}{N-1} F_t - \frac{a}{N-1} X_t + \frac{e}{N-1} q_t \\
= M_t + \Lambda X_t + \lambda q_t
\]

- Dynamic Programming, HJB, Feedback Controller

\[
\frac{2e}{N-1} \hat{q} = V_x - \left[ \left( \frac{a}{N-1} + c \right) s - \frac{a}{N-1} x + (b-1) d + \xi - \frac{e}{N-1} f \right]
\]
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Interpreting the Price

\[
pt = Dt + \mu - 1\rho - \gamma N \sigma^2 D - \gamma N - 1 N - 2 \sigma^2 D + \psi \left(1 + 1\rho\right)
\]

First term is MM's current asset valuation
Second term is a premium for expected valuation growth
Third term is a discount to compensate MM for bearing risk
Fourth term captures liquidity

Definition
Price Impact = \gamma N - 1 N - 2 \sigma^2 D + \psi \left(1 + 1\rho\right)

Liquidity = \frac{1}{Price Impact}
Interpreting the Price

\[ \rho_t = D_t + \frac{\mu}{\rho} - \frac{1}{\rho} \frac{\gamma}{N} \sigma_D^2 S_t - \frac{\gamma}{N} \frac{N - 1}{N - 2} \frac{\sigma_D^2}{\rho} + \psi \left( \frac{1}{\delta} + \frac{1}{\rho} \right) F_t \]

First term is MMs current asset valuation
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Interpreting the Price

\[ p_t = D_t + \frac{\mu}{\rho} - \frac{1}{\rho} \frac{\gamma}{N} \sigma_D^2 S_t - \frac{\gamma}{N} \frac{N-1}{N-2} \frac{\sigma_D^2}{\rho} + \psi \left( \frac{1}{\delta} + \frac{1}{\rho} \right) F_t \]

- First term is MMs current asset valuation
Interpreting the Price

\[ p_t = D_t + \frac{\mu}{\rho} - \frac{1}{\rho} \gamma \frac{\sigma^2 D}{N} S_t - \gamma \frac{N - 1}{N} \frac{\sigma^2 D}{N - 2 \rho} + \psi \left( \frac{1}{\delta} + \frac{1}{\rho} \right) F_t \]

- First term is MMs current asset valuation
- Second term is a premium for expected valuation growth
Interpreting the Price

\[ p_t = D_t + \frac{\mu}{\rho} - \frac{1}{\rho} \frac{\gamma}{N} \sigma_D^2 S_t - \frac{\gamma}{N} \frac{N - 1}{N - 2} \frac{\sigma_D^2}{\rho \delta} + \psi \left( \frac{1}{\delta} \right) F_t \]

- First term is MMs current asset valuation
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\[ p_t = D_t + \frac{\mu}{\rho} - \frac{1}{\rho} \frac{\gamma}{N} \sigma_D^2 S_t - \frac{\gamma}{N} \frac{N - 1}{N - 2} \frac{\sigma_D^2}{\rho} + \psi \left( \frac{1}{\delta} + \frac{1}{\rho} \right) F_t \]

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- First term is MMs current asset valuation
- Second term is a premium for expected valuation growth
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- Fourth term captures liquidity

**Definition**

\[ \text{Price Impact} := \frac{\gamma}{N} \frac{N-1}{N-2} \frac{\sigma_D^2}{\rho+\psi} \left( \frac{1}{\delta} + \frac{1}{\rho} \right) \]

\[ \text{Liquidity} := \frac{1}{\text{Price Impact}} \]
Comparative Statics for Liquidity

Proposition

1. $\frac{\partial}{\partial \gamma} \text{Liquidity} < 0$. Liquidity is decreasing in market makers' risk aversion.

2. $\frac{\partial}{\partial \sigma} \text{Liquidity} < 0$. Liquidity is decreasing in fundamental volatility.

3. If $\gamma_N$ is held fixed then $\frac{\partial}{\partial N} \text{Liquidity} > 0$. Liquidity is increasing in market maker competition.

4. $\frac{\partial}{\partial \psi} \text{Liquidity} > 0$. Liquidity is decreasing in order flow risk.
Proposition

1. \( \frac{\partial}{\partial \gamma} \text{Liquidity} < 0. \) 
   \text{Liquidity is decreasing in market makers’ risk aversion.}

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Comparative Statics for Liquidity

Proposition

1. \( \frac{\partial}{\partial \gamma} \text{Liquidity} < 0. \)
   
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2. \( \frac{\partial}{\partial \sigma_D} \text{Liquidity} < 0. \)
   
   Liquidity is decreasing in fundamental volatility.
Comparative Statics for Liquidity

Proposition

1. \( \frac{\partial}{\partial \gamma} \text{Liquidity} < 0 \).
   
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2. \( \frac{\partial}{\partial \sigma_D} \text{Liquidity} < 0 \).
   
   *Liquidity is decreasing in fundamental volatility.*

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   *Liquidity is increasing in market maker competition.*
Comparative Statics for Liquidity

Proposition

1. \( \frac{\partial}{\partial \gamma} \text{Liquidity} < 0. \)
   Liquidity is decreasing in market makers’ risk aversion.

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   Liquidity is increasing in market maker competition.

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   Liquidity is decreasing in order flow risk.
Limiting Cases

\[ \text{Price Impact} \to 0 \quad \text{as} \quad \psi \to \infty \]

\[ \text{Price Impact} \to \gamma_0 \rho + \psi \quad \text{as} \quad N \to \infty \]

With \( \gamma_N = \gamma_0 \) fixed.

In both cases the price is asymptotic to

\[ D_t + \mu \rho - E_t \left[ \int_0^\infty e^{-\rho (T-t)} \gamma_0 \sigma^2 D S_T \, dT \right] \]
Limiting Cases

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Limiting Cases

- **Price Impact** → 0 as $\psi \to \infty$

- **Price Impact** → $\frac{\gamma_0}{\rho} \frac{\sigma_D^2}{\rho + \psi}$ as $N \to \infty$ with $\frac{\gamma}{N} = \gamma_0$ fixed
Limiting Cases

- Price Impact $\rightarrow 0$ as $\psi \rightarrow \infty$

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Interpreting the Trading Rates

\[ q_n(t) = -\kappa (X_n t - S t N) + 1 N F t \]

MMs aggregate time \( t \) inventory is \( S t N \).

The efficient allocation is for each MM to hold \( S t N \).

Second term says that new supply shocks are absorbed efficiently.

First term says that existing misallocations move towards efficiency.

\[ X_n t = S t N + e^{-\kappa t} (X_n 0 - S 0 N) \]

Definition

Rate of Convergence to Efficiency

\[ \kappa = \rho (N - 2) \rho + \psi \rho + \delta \]

Angad Singh (Caltech)
Interpreting the Trading Rates

\[ q_t^n = -\kappa \left( X_t^n - \frac{S_t}{N} \right) + \frac{1}{N} F_t \]

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**Definition**

*Rate of Convergence to Efficiency* := \( \kappa := \rho (N - 2) \frac{\rho + \psi}{\rho + \delta} \)
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