Asset Pricing with Transaction Costs

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Outline

Introduction

Equilibrium Returns

Equilibrium Asset Prices

Outlook
Introduction
Asset Pricing with Transaction Costs

- How are trading costs reflected in asset prices?
  - Liquidity premia in expected returns?
  - Effect of a transaction tax on market volatility?
- Needs to be studied with *equilibrium* models.
  - Prices determined as output by matching supply and demand, rather than modeled as input.
- Equilibrium analyses are already hard without trading costs.
  - Notoriously intractable feedback loop.
  - Trading depends on prices. Prices have to change if market does not clear. Fixed-point problem.
- Intractability is compounded with frictions.
  - Individual optimization becomes much more involved.
  - Representative agent not applicable.
Introduction

Literature

- Numerical solution of discrete-time tree models:

- Additional restrictive modeling assumptions:
  - No risky asset (Vayanos/Vila ‘99, Weston ‘16).
  - Constant asset prices (Lo/Mamaysky/Wang ‘04).
  - Full refund of costs that is not internalized (Davila ‘15).
  - Only one rational optimizer (Garleanu/Pedersen ‘16).

- Recent working paper of Sannikov/Skrzypacz:
  - Private endowments revealed through linear demand schedules.
  - Price impact endogenous like in microstructure literature.
  - Linear-quadratic control arguments suggest stationary linear equilibria should solve system of algebraic equations.
  - Existence and uniqueness?
Introduction
Asset Pricing with Transaction Costs

This talk:

- Equilibrium \textit{returns} with transaction costs.
  - Endogenous expected returns but exogenous volatilities.
  - Global existence, uniqueness, and characterization of equilibrium by matrix Riccati equations.
  - Explicitly solvable examples.
  - Joint work with Bouchard/Fukasawa/Herdegen ‘18.

- Equilibrium \textit{asset prices} with transaction costs.
  - Endogenous returns \textit{and} volatilities.
  - Local existence and uniqueness for similar risk aversions.
  - Explicit asymptotic formulas.
  - Joint work in progress with Herdegen/Possamai.
Equilibrium Returns
Frictionless Benchmark

- Exogenous savings account. Price normalized to one.

- Unit net supply of risky asset with Itô dynamics:
  \[ dS_t = \mu_t dt + \sigma dW_t \]
  - Risky returns \((\mu_t)_{t \in [0, T]}\) to be determined in equilibrium.
  - Exogenous volatility \(\sigma > 0\) as in Zitković ‘12, Choi/Larsen‘15, Kardaras/Xing/Zitković ‘15, Garleanu/Pedersen ‘16.

- Agents \(n = 1, 2\) with partially spanned endowments:
  \[ dY^n_t = \nu^n_t dt + \beta^n_t dW_t + \beta^\perp_{\perp, n} dW^\perp_t \]

- Frictionless wealth dynamics of a trading strategy \((\varphi_t)_{t \in [0, T]}\):
  \[ \varphi_t dS_t + dY^n_t \]
▶ Equilibria are generally intractable even for CARA preferences.
  ▶ Abstract existence results if market is complete (classical), or almost complete (Kardaras/Xing/Zitković ‘15).
  ▶ Some partial very recent existence results for the general incomplete case (Xing/Zitković ‘17).
  ▶ Only few examples that can be solved explicitly (Larsen et al).
▶ Tractability issues exacerbated by trading frictions.
▶ Need simpler frictionless starting point.
▶ Use local mean-variance preferences over changes in wealth:

\[
E \left[ \int_0^T (\varphi_t dS_t + dY^n_t) - \frac{\gamma^n}{2} \int_0^T \langle \varphi_t dS_t + dY^n_t \rangle \right] \rightarrow \text{max!}
\]
Equilibrium Returns
Frictionless Benchmark ct’d

- Optimizers readily determined by pointwise maximization of

$$E \left[ \int_0^T \varphi_t \mu_t + \nu_t^n - \frac{\gamma^n}{2} (\varphi_t \sigma + \beta_t^n)^2 dt \right]$$

- Optimum is Merton portfolio plus mean-variance hedge:

$$\varphi_t^n = \frac{\mu_t}{\gamma^n \sigma^2} - \frac{\beta_t^n}{\sigma}$$

- Myopic. Available in closed form for any risky return.
- Leads to CAPM-equilibrium by summing across agents:

$$\mu_t = \bar{\gamma} \sigma^2 + \bar{\gamma} \sigma (\beta_t^1 + \beta_t^2), \quad \text{where} \quad \bar{\gamma} = \frac{\gamma^1 \gamma^2}{\gamma^1 + \gamma^2}$$
Equilibrium Returns
Adding Transaction Costs

- Optimization criterion with *quadratic* trading costs:

\[
J(\dot{\varphi}) = \mathbb{E} \left[ \int_0^T \varphi_t \mu_t - \frac{\gamma}{2} (\varphi_t \sigma + \beta_t^n)^2 - \frac{\lambda}{2} \dot{\varphi}_t^2 dt \right] \rightarrow \text{max!}
\]

- Linear price impact proportional to trade size *and* speed.
- Standard model in optimal execution (Almgren/Chriss ‘01).
- Recently used in portfolio choice (Garleanu/Pedersen ‘13, ‘16; Almgren/Li ‘16; Moreau/M-K/Soner ‘17).
- Problem is no longer myopic with trading costs. Current position becomes extra state variable.
- But still tractable for single-investor problems:
  - Dynamic programming (Garleanu/Pedersen ‘16).
  - Calculus-of-variations (Bank/Soner/Voss ‘17).
First step towards equilibrium:

- Fix return \((\mu_t)_{t\in[0,T]}\), compute agents’ individual optimizers.

- Necessary and sufficient for optimality: directional derivative

\[
\lim_{\rho \to 0} \frac{1}{\rho} (J(\dot{\varphi} + \rho \dot{\psi}) - J(\dot{\varphi})) \text{ vanishes for any perturbation } \psi:
\]

\[
0 = E_t \left[ \int_0^T \left( \mu_t \int_0^t \dot{\psi}_u du - \gamma^n \sigma (\varphi_t\sigma + \beta^n_t) \int_0^t \dot{\psi}_u du - \lambda \dot{\varphi}_t \dot{\psi}_t \right) dt \right]
\]

- As in Bank/Soner/Voss, rewrite using Fubini’s theorem:

\[
0 = E_t \left[ \int_0^T \left( \int_t^T \left( \mu_u - \gamma^n \sigma (\varphi_u\sigma + \beta^n_u) \right) du - \lambda \dot{\varphi}_t^\top \right) \dot{\psi}_t dt \right]
\]

- Has to hold for any perturbation \(\dot{\psi}_t\).
Whence, tower property of conditional expectation yields:

\[ \dot{\varphi}_t = \frac{1}{\lambda} E_t \left[ \int_t^T \mu_u - \gamma^n \sigma^2 \left( \varphi_u + \frac{\beta^n}{\sigma} \right) \, du \right] \]

\[ = M_t - \frac{1}{\lambda} \int_0^t \left( \mu_u - \gamma^n \sigma^2 \left( \varphi_u + \frac{\beta^n}{\sigma} \right) \right) \, du \]

for a martingale \( M_t \).

Thus, individually optimal strategy solves linear FBSDE:

\[ d\varphi^n_t = \dot{\varphi}^n_t \, dt, \quad \varphi^n_0 = \text{initial condition} \]

\[ d\dot{\varphi}^n_t = dM_t + \frac{\gamma^n \sigma^2}{\lambda} \left( \varphi^n_t - \xi^n_t \right) \, dt, \quad \dot{\varphi}^n_T = 0 \]

where \( \xi^n_t = \frac{\mu_t}{\gamma^n \sigma^2} - \frac{\beta^n_t}{\sigma} \) is the frictionless optimum.
Equilibrium Returns
Linear FBSDEs and Riccati ODEs

- Bank/Soner/Voss ‘17: one-dimensional case can be reduced to Riccati equations using the ansatz

\[ \dot{\varphi}_t = F(t)(\hat{\xi}_t - \varphi_t), \quad \hat{\xi}_t = K_1(t)E_t \left[ \int_t^T K_2(s)\xi_s ds \right] \]

- Higher dimensions lead to coupled but still linear FBSDEs.
  - Many risky assets here. Many agents in equilibrium.

- Matrix version of ansatz still allows to reduce to matrix Riccati ODEs.

- Can be solved in terms of “primary matrix functions”.

- Aggregate individual optimizers into an equilibrium?
For equilibrium, need returns \((\mu_t)_{t\in[0,T]}\) such that

\[
0 = d\varphi^1_t + d\varphi^2_t \\
= \frac{\sigma^2}{\lambda} \left( (\gamma^1 \varphi^1_t + \gamma^2 \varphi^2_t) + (\gamma^1 \beta^1_t + \gamma^2 \beta^2_t) - \frac{2\mu_t}{\sigma^2} \right) dt + dM_t
\]

Since \(\varphi^2_t = 1 - \varphi^1_t\) in equilibrium:

\[
\mu_t = \sigma^2 \left( \frac{\gamma^1 \beta^1_t + \gamma^2 \beta^2_t}{2} + \frac{\gamma^2}{2} + \frac{\gamma^1 - \gamma^2}{2} \varphi^1_t \right)
\]

For agents with the same risk aversion \(\gamma^1 = \gamma^2 = \gamma\):

- Same equilibrium return as without trading costs.
- Agents are not indifferent, but market still clears.
With heterogenous risk aversions $\gamma^1 \neq \gamma^2$:

- Plug back formula for $\mu_t$ into clearing condition for agent 1.
- Again leads to a linear FBSDE:

$$d\varphi^1_t = \dot{\varphi}^1_t dt, \quad \varphi^1_0 = \text{initial position}$$

$$d\dot{\varphi}^1_t = \frac{\sigma^2}{\lambda} \left( \frac{\gamma^1 \beta^1_t - \gamma^2 \beta^2_t}{2} - \frac{\gamma^2}{2} + \frac{\gamma^1 + \gamma^2}{2} \varphi^1_t \right) dt + dM^1_t, \quad \dot{\varphi}^1_T = 0$$

- Solution as for individual optimality (modulo matrix algebra).
- Direct construction also yields uniqueness

In summary:

- Existence of a unique equilibrium return.
- Characterized in terms of matrix Riccati equations. Explicit formulas if conditional expectations of $\beta^1_t, \beta^2_t$ are known.
Equilibrium Returns

Example

- Simplest case (Lo/Mamaysky/Wang ‘04):
  - No aggregate endowments. Individual exposures follow

\[ \beta_1^t = -\beta_2^t = \alpha t + N_t, \]

for a constant \( \alpha \) and a Brownian motion \( N_t \).

- To obtain simpler stationary solutions: \( T = \infty \).

- Well posed with discount rate \( \delta > 0 \): adds one term to FBSDE, but allows to replace terminal with transversality condition.

- Trading rates become constant, discounting becomes exponential.

- (Discounted) conditional expectations of endowment exposures can be readily computed in closed form.

- Leads to explicit dynamics of the equilibrium return.
Equilibrium Returns
Example ct’d

- Ornstein-Uhlenbeck equilibrium dynamics like in reduced-form models (Kim/Omberg ‘96; Bouchaud et al. ‘12):

\[
d\mu_t = \left( \sqrt{\frac{\gamma_1 + \gamma_2}{2}} \frac{\sigma^2}{2\lambda} + \frac{\delta^2}{4} - \frac{\delta}{2} \right) \left( 2 \frac{\gamma_1 - \gamma_2}{\gamma_1 + \gamma_2} \delta \lambda \alpha - \mu_t \right) dt \\
+ \frac{(\gamma_1 - \gamma_2)\sigma^2}{2} dN_t
\]

- Average liquidity premium vanishes for equal risk aversions. Generally proportional to relative difference times impatience.
- Positive premium if more risk averse agent is a net seller.
  - Has stronger motive to trade, therefore provides extra compensation.
- Momentum even for martingale endowments. Induced by sluggishness of frictional portfolios.
Extra condition to pin down equilibrium volatility?

Simplest model: exogenous terminal condition \( S_T = S \).
  - Fundamental value or terminal dividend.

Individual optimization works as before \( (\varphi^n_t = \frac{\mu_t}{\gamma^n_s^2} - \frac{\beta^n_t}{\sigma_t}) \).

Equilibrium return still determined by summing across agents:

\[
\mu_t = \bar{\gamma}\sigma_t^2 + \bar{\gamma}\sigma_t(\beta_1^t + \beta_2^t)
\]

But terminal condition now imposes a \textit{quadratic} BSDE:

\[
dS_t = \left[\bar{\gamma}\sigma_t^2 + \bar{\gamma}\sigma_t(\beta_1^t + \beta_2^t)\right] dt + \sigma_t dW_t, \quad S_T = S
\]

Volatility \( \sigma_t \) (and initial price \( S_0 \)) is part of the solution.
Equilibrium Asset Prices
Extension with Transaction Costs?

- Quadratic BSDE for frictionless volatility has unique solution by standard results, e.g., for bounded $\beta^1 + \beta^2, S$.
- Purely quadratic after switching to measure $\mathbb{P}^\beta$ with density process $\mathcal{E}(-\bar{\gamma} \int_0^t (\beta^1_t + \beta^2_t) dW_t)$. Explicit solution:
  \[ S_t = -\frac{1}{2\bar{\gamma}} E^\beta_t \left[ e^{-2\bar{\gamma}S} \right] \]
- Explicit formulas for terminal conditions produced by affine processes: e.g., if $S = bT + aW_T$, then
  \[ \sigma_t = a, \quad \mu_t = \bar{\gamma}a^2, \quad S_0 = (b - \bar{\gamma}a^2)T \]
- Still tractable with (quadratic) transaction costs?
Calculus-of-variations argument of Bank/Soner/Voss still leads to FBSDE linear in optimal position and trading rate.

But squared volatility is now no longer exogenous.

Terminal condition leads to another coupled BSDE:

\[
\begin{align*}
    d\varphi_t^1 &= \varphi_t^1, \quad \varphi_0^1 = \text{initial position}, \\
    d\dot{\varphi}_t^1 &= \frac{(\gamma^1 + \gamma^2)^2\sigma_t^2}{2\lambda} \left( \frac{\gamma^1 \beta_t^1 - \gamma^2 \beta_t^2}{(\gamma^1 + \gamma^2)\sigma_t} - \frac{\gamma^2}{\gamma^1 + \gamma^2} + \varphi_t^1 \right) + dM_t^1, \quad \dot{\varphi}_T^1 = 0 \\
    dS_t &= \sigma_t^2 \left( \frac{\gamma^1 - \gamma^2}{2} \varphi_t^1 + \frac{\gamma^2}{2} + \frac{\gamma^1 \beta_t^1 + \gamma^2 \beta_t^2}{2\sigma_t} \right) dt + \sigma_t dW_t, \quad S_T = S
\end{align*}
\]

Fully coupled. Bad news.
Equilibrium Asset Prices

Picard Iteration?

Existence? Uniqueness?

- Direct Picard iteration only works if time horizon $T$ is small.
  - Similar to large costs. Almost no trading.
- Exponential weighting does not help due to coupling.
- Way out?
  - Suitable “smallness” condition?
  - Trading rate explodes for small transaction costs.
- Forward-backward system for $(\phi^1, \varphi^1)$: studied in Kohlmann/Tang ‘02 for an exogenous bounded volatility $\sigma$.
- How to use this here?
Equilibrium Asset Prices
Almost Homogenous Risk Aversions

- Coupling disappears for $\gamma^1 = \gamma^2 = \gamma$:

$$dS_t = \left(\frac{\gamma^1 - \gamma^2}{2} \varphi_t \sigma_t^2 + \frac{\gamma^2}{2} (\sigma_t^2 + (\beta^1_t + \beta^2_t)\sigma_t)\right) dt + \sigma_t dW_t$$

- Equilibrium volatility coincides with frictionless counterpart $\bar{\sigma}$.
- For bounded $\bar{\sigma}$: trading strategies determined by linear FBSDE with stochastic coefficients as in Kohlmann/Tang ‘02:

$$d\varphi^1_t = \dot{\varphi}^1_t, \quad \varphi^1_0 = \text{initial position},$$

$$d\dot{\varphi}^1_t = \frac{\gamma \bar{\sigma}_t^2}{\lambda} \left(\frac{\beta^1_t - \beta^2_t}{2\bar{\sigma}_t} - \frac{1}{2} + \varphi^1_t\right) + dM^1_t, \quad \dot{\varphi}^1_T = 0$$

- Solutions in terms of backward stochastic Riccati equation.
- Expansion around this case?
Equilibrium Asset Prices
Almost Homogenous Risk Aversions ct’d

- Idea: Picard iteration only for BSDE for equilibrium price:

\[ dS_t = \sigma_t^2 \left( \frac{\gamma^1 - \gamma^2}{2} \varphi_t^1 + \frac{\gamma^2}{2} + \frac{\gamma^1 \beta^1_t + \gamma^2 \beta^2_t}{2\sigma_t} \right) \, dt + \sigma_t \, dW_t, \quad S_T = S \]

- Construct \( \varphi^1 \) with the volatility from the previous step.
  - Bounded for bounded \( \beta^1, \beta^2, S \).
  - BSDE for \( S \) of quadratic growth. But data is not small.
- Way out: consider difference \( Y \) to frictionless equilibrium:

\[
\begin{align*}
  dY_t &= \left( (\bar{\sigma}_t + Z_t)^2 \frac{\gamma^1 - \gamma^2}{2} (\varphi_t^1 - \bar{\varphi}_t^1) + \bar{\gamma} Z_t^2 + \bar{\gamma} (2\bar{\sigma}_t + \beta^1_t + \beta^2_t)Z_t \right) \, dt \\
  &\quad + Z_t \, dW_t \quad Y_T = 0,
\end{align*}
\]

- Linear drift can be removed by change of measure.
In summary: study Picard Iteration for
\[ dY_t = \left( (\bar{\sigma}_t + Z_t)^2 \frac{\gamma_1 - \gamma_2}{2} (\varphi_t^1 - \bar{\varphi}_t^1) + \bar{\gamma} Z_t^2 \right) dt + Z_t dW_t^Q \quad Y_T = 0 \]
under $Q$ with density process $\mathcal{E}(\int_0^t \bar{\gamma} (2\bar{\sigma}_t + \beta_t^1 + \beta_t^2) dW_t)$.

- Unique solution in $L_\infty \times \mathbb{H}_{\text{BMO}}^2$ as in Tevzadze '08?
  - Extend Kohlmann/Tang '02 from bounded to BMO-volatility by localization.
  - Establish stability estimates for BSRDEs (under $Q$).
  - Gives convergence for bounded $\bar{\sigma}$, sufficiently small $|\gamma_1 - \gamma_2|$.

- Existence and uniqueness for sufficiently similar risk aversions.
- Characterization?
Equilibrium Asset Prices

Asymptotic Expansion

- For small $|\gamma^1 - \gamma^2|$ ($\sim$ small $Z_t$): price correction

$$dY_t = \left( (\bar{\sigma}_t + Z_t)^2 \frac{\gamma^1 - \gamma^2}{2} (\varphi_t^1 - \bar{\varphi}_t^1) + \bar{\gamma}Z_t^2 \right) dt + Z_t dW_t^Q \quad Y_T = 0$$

can be approximated in $L_\infty \times H^2_{BMO}$ by linear BSDE:

$$d\bar{Y}_t = \bar{\sigma}_t^2 \frac{\gamma^1 - \gamma^2}{2} (\varphi_t^1, \bar{\sigma}_t - \bar{\varphi}_t^1) dt + \bar{Z}_t dW_t^Q \quad \bar{Y}_T = 0$$

- Difference $\varphi_t^1, \bar{\sigma} - \bar{\varphi}_t^1$ between frictionless equilibrium and tracking strategy for volatility $\bar{\sigma}$ has decoupled dynamics.

- Explicit price correction in concrete examples:

$$\bar{Y}_t = \frac{\gamma^2 - \gamma^1}{2} E_t^Q \left[ \int_t^T \bar{\sigma}_s^2 (\bar{\varphi}_s^1 - \varphi_s^{1, \bar{\sigma}}) ds \right]$$
Equilibrium Asset Prices

Volatility Correction

- For Brownian target positions $\beta^1 = -\beta^2 = \beta W_t$:
  - $\bar{\sigma}$ is constant.
  - $\varphi_t^1, \bar{\sigma} - \varphi_t^1$ follows Ornstein-Uhlenbeck process.
  - $\bar{Y}_t$ is multiple of OU process plus smooth drift.

- Volatility correction due to small transaction costs $\lambda$ is

$$\sigma \approx \bar{\sigma} \left( 1 - \frac{\gamma_1 - \gamma_2}{\sqrt{2(\gamma_1 + \gamma_2)}} \lambda^{1/2} \beta \right)$$

- Interpretation?

- Recall that

$$\beta = \frac{d\langle Y^1, S \rangle_t}{d\langle S, S \rangle_t}$$
Equilibrium Asset Prices
Volatility Correction ct’d

- Asymptotic volatility correction:

\[
\sigma_t^\lambda \approx \sigma^0 \left( 1 - \lambda^{1/2} \frac{\gamma^1 - \gamma^2}{\sqrt{2(\gamma^1+\gamma^2)}} \frac{d\langle Y^1, S \rangle_t}{d\langle S, S \rangle_t} \right)
\]

- Suppose \( \gamma^1 > \gamma^2 \), \( \beta = \frac{d\langle Y^1, S \rangle_t}{d\langle S, S \rangle_t} > 0 \).

- Then if risky asset increases, agent 1’s exposure also tends to increase. Has to sell to hedge.

- Agent 2 has opposite exposure. Has to buy.

- More risk-averse agent 1 wants to trade faster. To clear market, need to add positive expected return.

- Amplifies price shock. To reach given terminal distribution, have to reduce volatility.
Outlook

Open Problems

- Results with transaction costs beyond bounded inputs?
  - To make Brownian example rigorous, need to stop appropriately.
- Global existence and uniqueness?
- Small-cost asymptotics as in partial equilibrium models?
- Mean-reverting volatility due to illiquidity?
- Other, e.g., proportional trading costs?
- Price impact rather than “tax”?
- Nash competition rather than competitive equilibrium?