Asset Pricing with Transaction Costs

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Outline

Introduction

Asset Pricing with Transaction Costs

- How are trading costs reflected in asset prices?
  - Liquidity premia in expected returns?
  - Effect of a transaction tax on market volatility?
- Needs to be studied with *equilibrium* models.
  - Prices determined as output by matching supply and demand, rather than modeled as input.
- Equilibrium analyses are already hard without trading costs.
  - Notoriously intractable feedback loop.
  - Trading depends on prices. Prices have to change if market does not clear. Fixed-point problem.
- Intractability is compounded with frictions.
  - Individual optimization becomes much more involved.
  - Representative agent not applicable.

Introduction

Literature

- Numerical solution of discrete-time tree models:
  - Heaton/Lucas '96. Buss/Dumas '15; Buss/Vilkov/Uppal '15.
- Additional restrictive modeling assumptions:
  - No risky asset (Vayanos/Vila '99, Weston '16).
  - Constant asset prices (Lo/Mamaysky/Wang '04).
  - Full refund of costs that is not internalized (Davila '15).
  - Only one rational optimizer (Garleanu/Pedersen '16).
- Recent working paper of Sannikov/Skrzypacz:
  - Private endowments revealed through linear demand schedules.
  - Price impact endogenous like in microstructure literature.
  - Linear-quadratic control arguments suggest stationary linear equilibria should solve system of algebraic equations.
  - Existence and uniqueness?

Introduction

Asset Pricing with Transaction Costs

This talk:

- Equilibrium *returns* with transaction costs.
  - Endogenous expected returns but exogenous volatilities.
  - Global existence, uniqueness, and characterization of equilibrium by matrix Riccati equations.
  - Explicitly solvable examples.
  - Joint work with Bouchard/Fukasawa/Herdegen '18.
- Equilibrium *asset prices* with transaction costs.
  - Endogenous returns *and* volatilities.
  - Local existence and uniqueness for similar risk aversions.
  - Explicit asymptotic formulas.
  - Joint work in progress with Herdegen/Possamai.

Equilibrium Returns

Frictionless Benchmark

- Exogenous savings account. Price normalized to one.
- Unit net supply of risky asset with Itô dynamics:

$$dS_t = \mu_t dt + \sigma dW_t$$

  - Risky returns $(\mu_t)_{t \in [0,T]}$ to be determined in equilibrium.
  - Exogenous volatility $\sigma > 0$ as in Zitković '12, Choi/Larsen'15, Kardaras/Xing/Zitković '15, Garleanu/Pedersen '16.

- Agents $n = 1, 2$ with partially spanned endowments:

$$dY_t^n = \nu_t^n dt + \beta_t^n dW_t + \beta_t^{\perp,n} dW_t^{\perp}$$

- Frictionless wealth dynamics of a trading strategy $(\varphi_t)_{t \in [0,T]}$:

$$\varphi_t dS_t + dY_t^n$$

Equilibrium Returns

Frictionless Benchmark ct'd

- Equilibria are generally intractable even for CARA preferences.
  - Abstract existence results if market is complete (classical), or almost complete (Kardaras/Xing/Zitković '15).
  - Some partial very recent existence results for the general incomplete case (Xing/Zitković '17).
  - Only few examples that can be solved explicitly (Larsen et al).
- Tractability issues exacerbated by trading frictions.
- Need simpler frictionless starting point.
- Use *local* mean-variance preferences over changes in wealth:

$$E\left[\int_0^T (\varphi_t dS_t + dY_t^n) - \frac{\gamma^n}{2}\int_0^T \langle \varphi_t dS_t + dY_t^n \rangle\right] \to \max!$$

Equilibrium Returns

Frictionless Benchmark ct'd

- Optimizers readily determined by pointwise maximization of

$$E\left[\int_0^T \varphi_t\mu_t + \nu_t^n - \frac{\gamma^n}{2}(\varphi_t\sigma + \beta_t^n)^2 dt\right]$$

- Optimum is Merton portfolio plus mean-variance hedge:

$$\varphi_t^n = \frac{\mu_t}{\gamma^n\sigma^2} - \frac{\beta_t^n}{\sigma}$$

- Myopic. Available in closed form for *any* risky return.
- Leads to CAPM-equilibrium by summing across agents:

$$\mu_t = \bar{\gamma}\sigma^2 + \bar{\gamma}\sigma(\beta_t^1 + \beta_t^2), \quad \text{where } \bar{\gamma} = \frac{\gamma^1\gamma^2}{\gamma^1+\gamma^2}$$

Equilibrium Returns

Adding Transaction Costs

- Optimization criterion with *quadratic* trading costs:

$$J(\dot{\varphi}) = E\left[\int_0^T \varphi_t \mu_t - \frac{\gamma^n}{2}(\varphi_t \sigma + \beta_t^n)^2 - \frac{\lambda}{2}\dot{\varphi}_t^2 dt\right] \to \max!$$

- Linear price impact proportional to trade size *and* speed.
- Standard model in optimal execution (Almgren/Chriss '01).
- Recently used in portfolio choice (Garleanu/Pedersen '13, '16; Almgren/Li '16; Moreau/M-K/Soner '17).
- Problem is no longer myopic with trading costs. Current position becomes extra state variable.
- But still tractable for single-investor problems:
  - Dynamic programming (Garleanu/Pedersen '16).
  - Calculus-of-variations (Bank/Soner/Voss '17).

Equilibrium Returns

Individual Optimality with Transaction Costs

- First step towards equilbrium:
  - Fix return $(\mu_t)_{t\in[0,T]}$, compute agents' individual optimizers.
- Necessary and sufficient for optimality: directional derivative $\lim_{\rho\to 0}\frac{1}{\rho}(J(\dot\varphi+\rho\dot\psi)-J(\dot\varphi))$ vanishes for *any* perturbation ψ:

$$0 = E_t\left[\int_0^T \left(\mu_t \int_0^t \dot\psi_u du - \gamma^n\sigma(\varphi_t\sigma+\beta_t^n)\int_0^t \dot\psi_u du - \lambda\dot\varphi_t\dot\psi_t\right)dt\right]$$

- As in Bank/Soner/Voss, rewrite using Fubini's theorem:

$$0 = E_t\left[\int_0^T \left(\int_t^T \left(\mu_u - \gamma^n\sigma(\varphi_u\sigma+\beta_u^n)\right)du - \lambda\dot\varphi_t^\top\right)\dot\psi_t dt\right]$$

- Has to hold for *any* perturbation $\dot\psi_t$.

Equilibrium Returns

Individual Optimality and FBSDEs

- Whence, tower property of conditional expectation yields:

$$\begin{aligned}\dot{\varphi}_t &= \tfrac{1}{\lambda} E_t \left[\int_t^T \mu_u - \gamma^n \sigma^2 \left(\varphi_u + \tfrac{\beta_u^n}{\sigma} \right) du \right] \\ &= M_t - \tfrac{1}{\lambda} \int_0^t \left(\mu_u - \gamma^n \sigma^2 \left(\varphi_u + \tfrac{\beta_u^n}{\sigma} \right) \right) du\end{aligned}$$

  for a martingale M_t.

- Thus, individually optimal strategy solves *linear* FBSDE:

$$\begin{aligned}d\varphi_t^n &= \dot{\varphi}_t^n dt, \quad \varphi_0^n = \text{initial condition} \\ d\dot{\varphi}_t^n &= dM_t + \tfrac{\gamma^n \sigma^2}{\lambda} \left(\varphi_t^n - \xi_t^n \right) dt, \quad \dot{\varphi}_T^n = 0\end{aligned}$$

  where $\xi_t^n = \frac{\mu_t}{\gamma^n \sigma^2} - \frac{\beta_t^n}{\sigma}$ is the frictionless optimum.

Equilibrium Returns

Linear FBSDEs and Riccati ODEs

- Bank/Soner/Voss '17: one-dimensional case can be reduced to Riccati equations using the ansatz

$$\dot{\varphi}_t = F(t)(\hat{\xi}_t - \varphi_t), \quad \hat{\xi}_t = K_1(t)E_t\left[\int_t^T K_2(s)\xi_s ds\right]$$

- Higher dimensions lead to coupled but still linear FBSDEs.
  - Many risky assets here. Many agents in equilibrium.
- Matrix version of ansatz still allows to reduce to matrix Riccati ODEs.
- Can be solved in terms of "primary matrix functions".
- Aggregate individual optimizers into an equilibrium?

Equilibrium Returns

Market Clearing

- For equilibrium, need returns $(\mu_t)_{t\in[0,T]}$ such that

$$\begin{aligned} 0 &= d\dot{\varphi}_t^1 + d\dot{\varphi}_t^2 \\ &= \tfrac{\sigma^2}{\lambda}\left((\gamma^1\varphi_t^1 + \gamma^2\varphi_t^2) + (\gamma^1\beta_t^1 + \gamma^2\beta_t^2) - \tfrac{2\mu_t}{\sigma^2}\right)dt + dM_t \end{aligned}$$

- Since $\varphi_t^2 = 1 - \varphi_t^1$ in equilibrium:

$$\mu_t = \sigma^2\left(\tfrac{\gamma^1\beta_t^1 + \gamma^2\beta_t^2}{2} + \tfrac{\gamma^2}{2} + \tfrac{\gamma^1 - \gamma^2}{2}\varphi_t^1\right)$$

- For agents with the same risk aversion $\gamma^1 = \gamma^2 = \gamma$:
  - Same equilibrium return as without trading costs.
  - Agents are not indifferent, but market still clears.

Equilibrium

Linear FBSDEs

- With heterogenous risk aversions $\gamma^1 \neq \gamma^2$:
  - Plug back formula for μ_t into clearing condition for agent 1.
  - Again leads to a *linear* FBSDE:

$$d\varphi_t^1 = \dot{\varphi}_t^1 dt, \quad \varphi_0^1 = \text{initial position}$$

$$d\dot{\varphi}_t^1 = \frac{\sigma^2}{\lambda}\left(\frac{\gamma^1\beta_t^1 - \gamma^2\beta_t^2}{2} - \frac{\gamma^2}{2} + \frac{\gamma^1+\gamma^2}{2}\varphi_t^1\right)dt + dM_t^1, \quad \dot{\varphi}_T^1 = 0$$

- Solution as for individual optimality (modulo matrix algebra).
  - Direct construction also yields uniqueness
- In summary:
  - Existence of a unique equilibrium return.
  - Characterized in terms of matrix Riccati equations. Explicit formulas if conditional expectations of β_t^1, β_t^2 are known.

Equilibrium Returns

Example

- Simplest case (Lo/Mamaysky/Wang '04):
  - No aggregate endowments. Individual exposures follow
    $$\beta_t^1 = -\beta_t^2 = \alpha t + N_t,$$
    for a constant α and a Brownian motion N_t.
  - To obtain simpler stationary solutions: $T = \infty$.
  - Well posed with discount rate $\delta > 0$: adds one term to FBSDE, but allows to replace terminal with transversality condition.
  - Trading rates become constant, discounting becomes exponential.
- (Discounted) conditional expectations of endowment exposures can be readily computed in closed form.
- Leads to explicit dynamics of the equilibrium return.

Equilibrium Returns

Example ct'd

- Ornstein-Uhlenbeck equilibrium dynamics like in reduced-form models (Kim/Omberg '96; Bouchaud et al. '12):

$$d\mu_t = \left(\sqrt{\frac{\gamma_1+\gamma_2}{2}\frac{\sigma^2}{2\lambda}+\frac{\delta^2}{4}}-\frac{\delta}{2}\right)\left(2\frac{\gamma_1-\gamma_2}{\gamma_1+\gamma_2}\delta\lambda\alpha-\mu_t\right)dt \\ +\frac{(\gamma_1-\gamma_2)\sigma^2}{2}dN_t$$

- Average liquidity premium vanishes for equal risk aversions. Generally proportional to relative difference times impatience.
- Positive premium if more risk averse agent is a net seller.
  - Has stronger motive to trade, therefore provides extra compensation.
- Momentum even for martingale endowments. Induced by sluggishness of frictional portfolios.

Equilibrium Asset Prices

Frictionless Benchmark

- Extra condition to pin down equilibrium volatility?
- Simplest model: exogenous terminal condition $S_T = \mathcal{S}$.
  - Fundamental value or terminal dividend.
- Individual optimization works as before ($\varphi_t^n = \frac{\mu_t}{\gamma^n \sigma_t^2} - \frac{\beta_t^n}{\sigma_t}$).
- Equilibrium return still determined by summing across agents:

$$\mu_t = \bar{\gamma}\sigma_t^2 + \bar{\gamma}\sigma_t(\beta_t^1 + \beta_t^2)$$

- But terminal condition now imposes a *quadratic* BSDE:

$$dS_t = \left[\bar{\gamma}\sigma_t^2 + \bar{\gamma}\sigma_t(\beta_t^1 + \beta_t^2)\right] dt + \sigma_t dW_t, \quad S_T = \mathcal{S}$$

- Volatility σ_t (and initial price S_0) is part of the solution.

Equilibrium Asset Prices

Extension with Transaction Costs?

- Quadratic BSDE for frictionless volatility has unique solution by standard results, e.g., for bounded $\beta^1 + \beta^2, \mathcal{S}$.
- Purely quadratic after switching to measure $\mathbb{P}^\beta$ with density process $\mathcal{E}(-\bar{\gamma} \int_0^\cdot (\beta_t^1 + \beta_t^2) dW_t)$. Explicit solution:

$$S_t = -\tfrac{1}{2\bar{\gamma}} E_t^\beta \left[e^{-2\bar{\gamma}\mathcal{S}} \right]$$

- Explicit formulas for terminal conditions produced by affine processes: e.g., if $\mathcal{S} = bT + aW_T$, then

$$\sigma_t = a, \quad \mu_t = \bar{\gamma} a^2, \quad S_0 = (b - \bar{\gamma} a^2) T$$

- Still tractable with (quadratic) transaction costs?

Equilibrium Asset Prices

Extension with Transaction Costs ct'd

- Calculus-of-variations argument of Bank/Soner/Voss still leads to FBSDE linear in optimal position and trading rate.
- But squared volatility is now no longer exogenous.
- Terminal condition leads to another coupled BSDE:

$$d\varphi^1_t = \dot{\varphi}^1_t,\ \varphi^1_0 = \text{initial position},$$

$$d\dot{\varphi}^1_t = \frac{(\gamma^1+\gamma^2)\sigma^2_t}{2\lambda}\left(\frac{\gamma^1\beta^1_t - \gamma^2\beta^2_t}{(\gamma^1+\gamma^2)\sigma_t} - \frac{\gamma^2}{\gamma^1+\gamma^2} + \varphi^1_t\right) + dM^1_t,\ \dot{\varphi}^1_T = 0$$

$$dS_t = \sigma^2_t\left(\frac{\gamma^1-\gamma^2}{2}\varphi^1_t + \frac{\gamma^2}{2} + \frac{\gamma^1\beta^1_t + \gamma^2\beta^2_t}{2\sigma_t}\right)dt + \sigma_t dW_t,\ S_T = \mathcal{S}$$

- Fully coupled. Bad news.

Equilibrium Asset Prices

Picard Iteration?

Existence? Uniqueness?

- Direct Picard iteration only works if time horizon T is small.
  - Similar to large costs. Almost no trading.
- Exponential weighting does not help due to coupling.
- Way out?
  - Suitable "smallness" condition?
  - Trading rate explodes for small transaction costs.
- Forward-backward system for $(\dot{\varphi}^1, \varphi^1)$: studied in Kohlmann/Tang '02 for an exogenous bounded volatility σ.
- How to use this here?

Equilibrium Asset Prices

Almost Homogenous Risk Aversions

- Coupling disappears for $\gamma^1 = \gamma^2 = \gamma$:

$$dS_t = \left(\frac{\gamma^1 - \gamma^2}{2} \varphi_t^1 \sigma_t^2 + \frac{\gamma}{2}(\sigma_t^2 + (\beta_t^1 + \beta_t^2)\sigma_t) \right) dt + \sigma_t dW_t$$

- Equilibrium volatility coincides with frictionless counterpart $\bar{\sigma}$.
- For bounded $\bar{\sigma}$: trading strategies determined by linear FBSDE with stochastic coefficients as in Kohlmann/Tang '02:

$$d\varphi_t^1 = \dot{\varphi}_t^1, \quad \varphi_0^1 = \text{initial position},$$
$$d\dot{\varphi}_t^1 = \frac{\gamma \bar{\sigma}_t^2}{\lambda} \left(\frac{\beta_t^1 - \beta_t^2}{2\bar{\sigma}_t} - \frac{1}{2} + \varphi_t^1 \right) + dM_t^1, \quad \dot{\varphi}_T^1 = 0$$

- Solutions in terms of backward *stochastic* Riccati equation.
- Expansion around this case?

Equilibrium Asset Prices

Almost Homogenous Risk Aversions ct'd

- Idea: Picard iteration only for BSDE for equilibrium price:

$$dS_t = \sigma_t^2 \left(\frac{\gamma^1 - \gamma^2}{2} \varphi_t^1 + \frac{\gamma^2}{2} + \frac{\gamma^1 \beta_t^1 + \gamma^2 \beta_t^2}{2\sigma_t} \right) dt + \sigma_t dW_t, \ S_T = \mathcal{S}$$

- Construct φ^1 with the volatility from the previous step.
  - Bounded for bounded $\beta^1, \beta^2, \mathcal{S}$.
  - BSDE for S of quadratic growth. But data is not small.
- Way out: consider difference Y to frictionless equilibrium:

$$dY_t = \left((\bar{\sigma}_t + Z_t)^2 \frac{\gamma^1 - \gamma^2}{2} (\varphi_t^1 - \bar{\varphi}_t^1) + \bar{\gamma} Z_t^2 + \bar{\gamma}(2\bar{\sigma}_t + \beta_t^1 + \beta_t^2) Z_t \right) dt$$
$$+ Z_t dW_t \quad Y_T = 0,$$

- Linear drift can be removed by change of measure.

Equilibrium Asset Prices

Picard Iteration

- In summary: study Picard Iteration for

$$dY_t = \left((\bar{\sigma}_t + Z_t)^2 \frac{\gamma^1 - \gamma^2}{2} (\varphi^1_t - \bar{\varphi}^1_t) + \bar{\gamma} Z^2_t \right) dt + Z_t dW^Q_t \quad Y_T = 0$$

  under Q with density process $\mathcal{E}(\int_0^\cdot \bar{\gamma}(2\bar{\sigma}_t + \beta^1_t + \beta^2_t) dW_t)$.
- Unique solution in $\mathbb{L}_\infty \times \mathbb{H}^2_{\mathrm{BMO}}$ as in Tevzadze '08?
  - Extend Kohlmann/Tang '02 from bounded to BMO-volatility by localization.
  - Establish stability estimates for BSRDEs (under Q).
  - Gives convergence for bounded $\bar{\sigma}$, sufficiently small $|\gamma^1 - \gamma^2|$.
- Existence and uniqueness for sufficiently similar risk aversions.
- Characterization?

Equilibrium Asset Prices

Asymptotic Expansion

- For small $|\gamma^1 - \gamma^2|$ ($\rightsquigarrow$ small Z_t): price correction

$$dY_t = \left((\bar{\sigma}_t + Z_t)^2 \tfrac{\gamma^1-\gamma^2}{2}(\varphi^1_t - \bar{\varphi}^1_t) + \bar{\gamma} Z_t^2\right) dt + Z_t dW^Q_t \quad Y_T = 0$$

can be approximated in $\mathbb{L}_\infty \times \mathbb{H}^2_{\mathrm{BMO}}$ by linear BSDE:

$$d\bar{Y}_t = \bar{\sigma}^2_t \tfrac{\gamma^1-\gamma^2}{2}(\varphi^{1,\bar{\sigma}}_t - \bar{\varphi}^1_t)dt + \bar{Z}_t dW^Q_t \quad \bar{Y}_T = 0$$

- Difference $\varphi^{1,\bar{\sigma}}_t - \bar{\varphi}^1_t$ between frictionless equilibrium and tracking strategy for volatility $\bar{\sigma}$ has decoupled dynamics.
- Explicit price correction in concrete examples:

$$\bar{Y}_t = \frac{\gamma^2 - \gamma^1}{2} E^Q_t \left[\int_t^T \bar{\sigma}^2_s (\bar{\varphi}^1_s - \varphi^{1,\bar{\sigma}}_s) ds\right]$$

Equilibrium Asset Prices

Volatility Correction

- For Brownian target positions $\beta^1 = -\beta^2 = \beta W_t$:
  - $\bar{\sigma}$ is constant.
  - $\varphi_t^{1,\bar{\sigma}} - \bar{\varphi}_t^1$ follows Ornstein-Uhlenbeck process.
  - $\bar{Y}_t$ is multiple of OU process plus smooth drift.
- Volatility correction due to small transaction costs λ is

$$\sigma \approx \bar{\sigma}\left(1 - \frac{\gamma^1 - \gamma^2}{\sqrt{2(\gamma^1 + \gamma^2)}}\lambda^{1/2}\beta\right)$$

- Interpretation?
- Recall that

$$\beta = \frac{d\langle Y^1, S\rangle_t}{d\langle S, S\rangle_t}$$

Equilibrium Asset Prices

Volatility Correction ct'd

- Asymptotic volatility correction:

$$\sigma_t^\lambda \approx \sigma^0 \left(1 - \lambda^{1/2} \frac{\gamma^1 - \gamma^2}{\sqrt{2(\gamma^1+\gamma^2)}} \frac{d\langle Y^1, S\rangle_t}{d\langle S, S\rangle_t}\right)$$

- Suppose $\gamma^1 > \gamma^2$, $\beta = \frac{d\langle Y^1, S\rangle_t}{d\langle S, S\rangle_t} > 0$.
- Then if risky asset increases, agent 1's exposure also tends to increase. Has to sell to hedge.
- Agent 2 has opposite exposure. Has to buy.
- More risk-averse agent 1 wants to trade faster. To clear market, need to add positive expected return.
- Amplifies price shock. To reach given terminal distribution, have to reduce volatility.

Outlook

Open Problems

- Results with transaction costs beyond bounded inputs?
  - To make Brownian example rigorous, need to stop appropriately.
- Global existence and uniqueness?
- Small-cost asymptotics as in partial equilibrium models?
- Mean-reverting volatility due to illiquidity?
- Other, e.g., proportional trading costs?
- Price impact rather than "tax"?
- Nash competition rather than competitive equilibrium?