PDGM: A Neural Network Approach to Solve Path-Dependent Partial Differential Equations

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Mathematical Finance Colloquium, USC

February 3, 2020

Literature Review

- Path-Dependent Partial Differential Equations (**PPDEs**) and Numerical Methods:
  - B. Dupire [2009].
  - I. Ekren, N. Touzi and J. Zhang [2014, 2016ab].
  - J. Zhang and J. Zhuo [2014].
  - Z. Ren and X. Tan [2017].
  - etc.
- **Machine Learning** on PDEs:
  - W. E, J. Han, and A. Jentze [2017].
  - M. Raiss [2018, 2019].
  - A. Jacquier and M. Oumgari [2019].
  - J. Sirignano and K. Spiliopoulos [2018].
  - etc.

Outline

- Review The Notion of Path-Dependent Partial Differential Equations (PPDEs), and Functional Itô Calculus.
- Review of Neural Networks.
  - Feed-Forward Neural Network.
  - Recurrent Neural Network.
- PDGM Architecture and Algorithm.
- Numerical Examples.
  - Linear and Nonlinear PPDE.
  - Application in Path Dependent Options.

Functional Itô Calculus

- Time horizon $T > 0$ fixed.
- Denote Λ_t the space of càdlàg paths in $[0, t]$ taking values in $\mathbb{R}^n$ and define $\Lambda = \bigcup_{t \in [0,T]} \Lambda_t$.
- Capital letters will denote elements of Λ (i.e. paths) and lower-case letters will denote spot value of paths.
  - eg. $Y_t \in \Lambda$ means $Y_t \in \Lambda_t$ and $y_s = Y_t(s)$, for $s \leq t$.
- We consider here continuity of functionals as the usual continuity in metric spaces with respect to the metric:

$$d_\Lambda(Y_t, Z_s) = \|Y_{t,s-t} - Z_s\|_\infty + |s - t|,$$

where, without loss of generality, we are assuming $s \geq t$, and

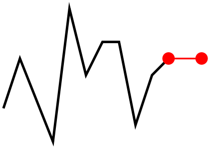
$$\|Y_t\|_\infty = \sup_{u \in [0,t]} |y_u|.$$

The norm $|\cdot|$ is the usual Euclidean norm in the appropriate Euclidean space, depending on the dimension of the path being considered.

Functional Itô Calculus

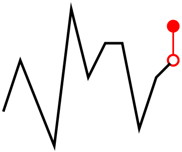
Flat extension of a path.

$$Y_{t,\delta t}(u) = \begin{cases} y_u, & \text{if } 0 \leq u \leq t, \\ y_t, & \text{if } t \leq u \leq t + \delta t. \end{cases}$$



Bumped path.

$$Y_t^h(u) = \begin{cases} y_u, & \text{if } 0 \leq u < t, \\ y_t + h, & \text{if } u = t. \end{cases}$$



Functional Itô Calculus

- A functional is any function $f : \Lambda \longrightarrow \mathbb{R}$. For such objects, we define, when the limits exist, the **time** and **space** functional derivatives, respectively, as

$$\Delta_t f(Y_t) = \lim_{\delta t \to 0^+} \frac{f(Y_{t,\delta t}) - f(Y_t)}{\delta t},$$
$$\Delta_x f(Y_t) = \lim_{h \to 0} \frac{f(Y_t^h) - f(Y_t)}{h}.$$

- Our numerical method is based on the following approximation of the functional derivatives: for a smooth functional $f \in \mathbb{C}^{1,2}$, we use

$$\Delta_t f(Y_t) = \frac{f(Y_{t,\delta t}) - f(Y_t)}{\delta t} + o(\delta t),$$
$$\Delta_x f(Y_t) = \frac{f(Y_t^h) - f(Y_t^{-h})}{2h} + o(h^2),$$
$$\Delta_{xx} f(Y_t) = \frac{f(Y_t^h) - 2f(Y_t) + f(Y_t^{-h})}{h^2} + o(h^2).$$

Theorem (Functional Feynman-Kac Formula; Dupire 2009)

Let x be a process given by the SDE

$$dx_s = \mu(X_s)ds + \sigma(X_s)dw_s.$$

Consider functionals $g : \Lambda_T \longrightarrow \mathbb{R}$, $\lambda : \Lambda \longrightarrow \mathbb{R}$ and $k : \Lambda \longrightarrow \mathbb{R}$ and define the functional f as

$$f(Y_t) = \mathbb{E}\left[e^{-\int_t^T \lambda(X_u)du} g(X_T) + \int_t^T e^{-\int_t^s \lambda(X_u)du} k(X_s)ds \;\middle|\; Y_t \right],$$

for any path $Y_t \in \Lambda$, $t \in [0, T]$. Thus, if $f \in \mathbb{C}^{1,2}$ and k, λ, μ and σ are Λ-continuous, then f satisfies the (linear) PPDE:

$$\Delta_t f(Y_t) + \mu(Y_t)\Delta_x f(Y_t) + \frac{1}{2}\sigma^2(Y_t)\Delta_{xx} f(Y_t) - \lambda(Y_t)f(Y_t) + k(Y_t) = 0,$$

with $f(Y_T) = g(Y_T)$, for any Y_t in the topological support of the stochastic process process x.

Feed-Forward Neural Networks

- A set of **layers** $\mathbb{M}^{\rho}_{d,k}$ with input $x \in \mathbb{R}^d$ in a feed-forward neural network can be defined as

$$\mathbb{M}^{\rho}_{d,k} := \{M : \mathbb{R}^d \to \mathbb{R}^k \ ; \ M(x) = \rho(Ax + b), A \in \mathbb{R}^{k \times d}, b \in \mathbb{R}^k\}.$$

- ρ is some activation function such as

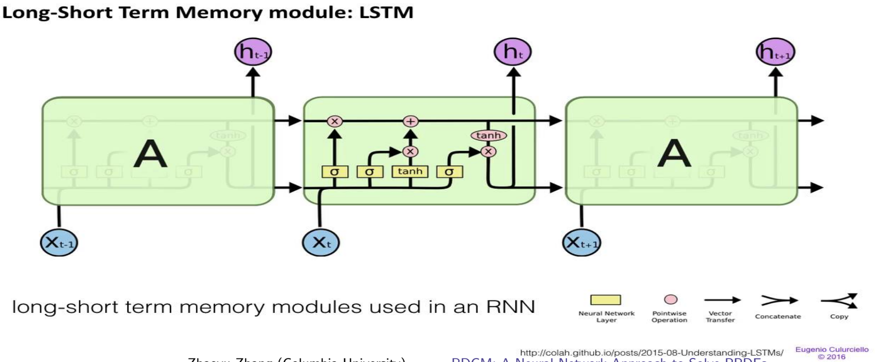
$$\rho_{\tanh}(x) := \tanh(x), \quad \rho_s(x) := \frac{1}{1 + e^{-x}} \text{ and } \rho_{Id}(x) := x.$$

- The set of **feed-forward** neural networks with ℓ hidden layers is defined as a composition of layers:

$$\begin{aligned}\mathbb{NN}^{\ell}_{d_1,d_2} = \{\widetilde{M} : \mathbb{R}^{d_1} \to \mathbb{R}^{d_2} \ ; \ \widetilde{M} &= M_\ell \circ \cdots \circ M_1 \circ M_0, \\ & M_0 \in \mathbb{M}^{\rho}_{d_1,k_1}, M_\ell \in \mathbb{M}^{\rho}_{k_l,d_2}, M_i \in \mathbb{M}^{\rho}_{k_i,k_{i+1}}, \\ & k_i \in \mathbb{Z}^+, i = 1, \ldots, \ell - 1\}.\end{aligned}$$

Recurrent Neural Network

- The recurrent neural network(RNN) is powerful for capturing **long-range dependence** of the data.
- The LSTM network was proposed in Hochreiter & Schmidhuber(1997). It is designed to solve the **shrinking gradient effects** which basic RNN often suffers from.
- The LSTM network is a **chain of cells**. Each LSTM cell composes of a cell state, which contains information, and three gates, which regulate the flow of information.



LSTM Network

- Mathematically, the rule inside ***i*th cell** follows,

$$
\begin{aligned}
\textbf{Forget gate}, \quad & \Gamma_{F_i}(x_i, a_{i-1}) = \rho_s(A_F x_i + U_F a_{i-1} + b_F), \\
\textbf{Input gate}, \quad & \Gamma_{I_i}(x_i, a_{i-1}) = \rho_s(A_I x_i + U_I a_{i-1} + b_I), \\
\textbf{Output gate}, \quad & \Gamma_{O_i}(x_i, a_{i-1}) = \rho_s(A_O x_i + U_O a_{i-1} + b_O), \\
\textbf{Cell state}, \quad & c_i = \Gamma_{F_i} \odot c_{i-1} + \Gamma_{I_i} \odot \rho_{\tanh}(A_C x_i + U_C a_{i-1} + b_C), \\
\textbf{Output vector}, \quad & a_i = \Gamma_{O_i} \odot \rho_{\tanh}(c_i).
\end{aligned}
$$

- The set of **LSTM network** up to time i as

$$
\begin{aligned}
\mathbb{LSTM}_{i,d,k} = \Big\{ M : (\mathbb{R}^d)^i \times \mathbb{R}^k \times \mathbb{R}^k \to \mathbb{R}^k \times \mathbb{R}^k \,;\, M(x_{[0,i]}, a_{-1}, c_{-1}) = (a_i, c_i), \\
c_i = \Gamma_{F_i} \odot c_{i-1} + \Gamma_{I_i} \odot \rho_{\tanh}(A_C x_i + U_C a_{i-1} + b_C), \\
a_i = \Gamma_{O_i} \odot \rho_{\tanh}(c_i), a_{-1} = c_{-1} = 0 \Big\},
\end{aligned}
$$

where $x_{[0,i]} = [x_0, \ldots, x_i]$.

PDGM Architecture

- Time discretization $\{t_i\}_{i=1,\dots,N}$, with $\delta t = t_i - t_{i-1}$.
- Approximate $f(Y_t)$ by a feed-forward neural network

$$f(Y_t) \approx u(Y_{t_i};\theta) = \varphi(t_i, y_{t_i}, a_{t_{i-1}}; \theta^f),$$

  where $t_i \leq t < t_{i+1}$.
- Here $\varphi \in \mathbb{NN}^{\ell}_{k+2,1}$, where a is an output vector from an LSTM network, i.e.

$$a_{t_{i-1}} = \psi(y_{t_0}, \dots, y_{t_{i-1}}; \theta^r),$$

  for some $\psi \in \mathbb{LSTM}_{i-1,1,k}$.
- $\theta = [\theta^f, \theta^r]$ are the neural network's parameters.

PDGM Architecture

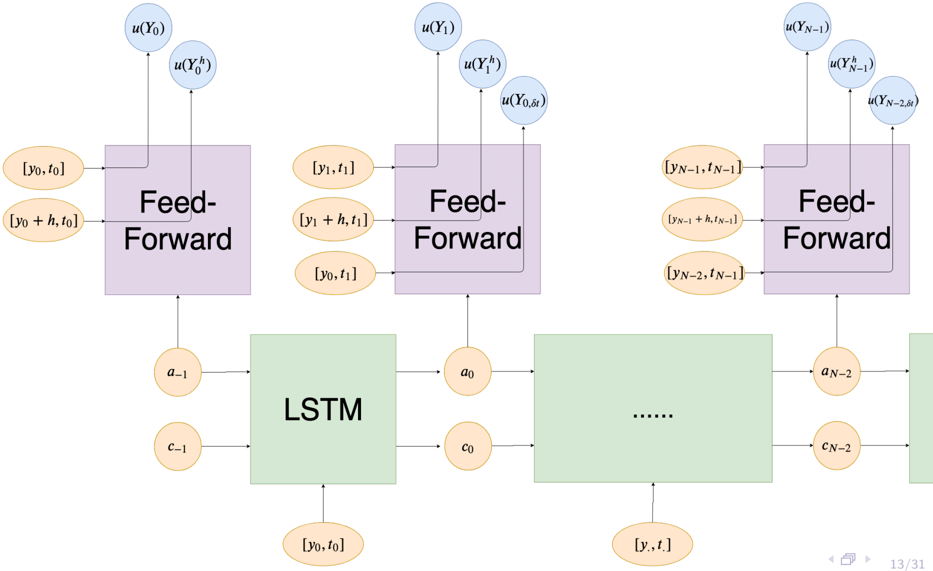
- Neural Network Approximation of the **Solution**.

$$u(Y_{t_i};\theta) = \varphi(t_i, y_{t_i}, a_{t_{i-1}};\theta^f)$$
$$u(Y^h_{t_i};\theta) = \varphi(t_i, y_{t_i}+h, a_{t_{i-1}};\theta^f),$$
$$u(Y_{t_i,\delta t};\theta) = \varphi(t_{i+1}, y_{t_i}, a_{t_i};\theta^f).$$

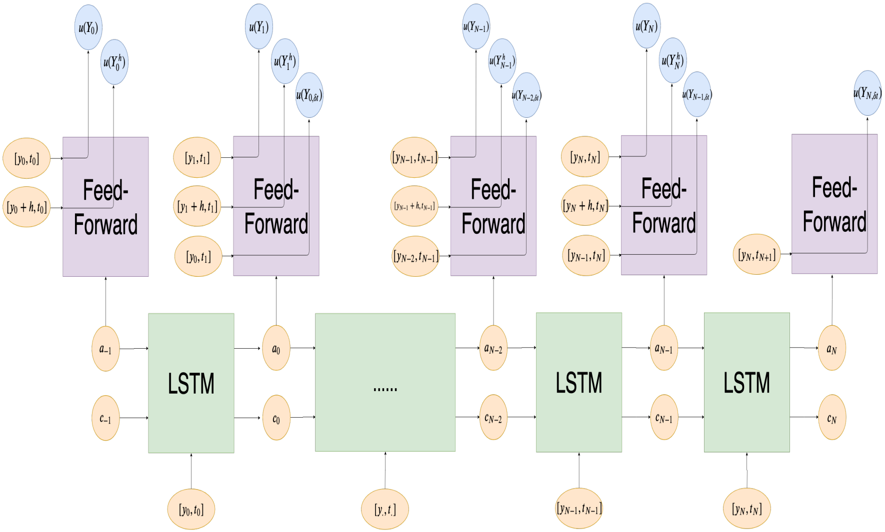
- Neural Network Approximation of the **Derivatives**.

$$\Delta_t^{[\delta t]} u(Y_{t_i};\theta) = \frac{u(Y_{t_i,\delta t};\theta) - u(Y_{t_i};\theta)}{\delta t},$$
$$\Delta_x^{[h]} u(Y_{t_i};\theta) = \frac{u(Y^h_{t_i};\theta) - u(Y_{t_i};\theta)}{h},$$
$$\Delta_{xx}^{[h]} u(Y_{t_i};\theta) = \frac{u(Y^h_{t_i};\theta) - 2u(Y_{t_i};\theta) + u(Y^{-h}_{t_i};\theta)}{h^2}.$$

PDGM Architecture



PDGM Architecture



PDGM Algorithm

- Consider the general class of final-value PPDE problem:

$$\begin{cases} \Delta_t f(Y_t) + \mathcal{L} f(Y_t) = 0, \\ f(Y_T) = g(Y_T). \end{cases}$$

- As an illustration, $\mathcal{L}$ could be given by the linear operator

$$\mathcal{L} f(Y_t) = \mu(Y_t)\Delta_x f(Y_t) + \frac{1}{2}\sigma^2(Y_t)\Delta_{xx} f(Y_t) - \lambda(Y_t) f(Y_t) + k(Y_t).$$

- Given M simulated paths, time and space discretization parameters δt and h, the loss J will be approximated by

$$J_{N,M}(\theta) = \frac{1}{M}\frac{1}{N}\sum_{j=1}^{M}\sum_{i=0}^{N}\left(\Delta_t^{[\delta t]} u(Y_{t_i}^{(j)};\theta) + \mathcal{L}^{[h]} u(Y_{t_i}^{(j)};\theta)\right)^2 + \frac{1}{M}\sum_{j=1}^{M}\left(u(Y_{t_N}^{(j)};\theta) - g(Y_{t_N}^{(j)})\right)^2.$$

Algorithm 1: Path-Dependent DGM - PDGM

initialize discretization parameter δt, mini-batch size M and threshold ϵ;

while $J_{N,M}(\theta) > \epsilon$ **do**

  generate a mini-batch size of M paths $\{(Y_{t_i}^{(j)})_{i=0,\ldots,N}\}_{j=1,\ldots,M}$

  for $i \in \{1, \ldots, N\}$ **do**

    calculate $u(Y_{t_i}^{(j)};\theta)$, $\Delta_t^{[\delta t]} u(Y_{t_i}^{(j)};\theta)$,
    $\Delta_x^{[h]} u(Y_{t_i}^{(j)};\theta)$ and $\Delta_{xx}^{[h]} u(Y_{t_i}^{(j)};\theta)$;

    put them all together to compute $\mathcal{L}^{[h]} u(Y_{t_i}^{(j)};\theta)$;

  end

  calculate the approximated loss function, $J_{N,M}(\theta)$;

  minimize $J_{N,M}(\theta)$, update θ using stochastic gradient descent.

end

Linear Running Integral

- Consider the class PPDEs of the form

$$\begin{cases} \Delta_t f(Y_t) + \frac{1}{2}\Delta_{xx} f(Y_t) = 0, \\ f(Y_T) = g(Y_T). \end{cases}$$

- As an example, let $g(Y_T) = \int_0^T y_u du$. The explicit solution can be calculated as

$$f(Y_t) = \int_0^t y_u du + y_t(T - t).$$

- Training paths in this subsection are 12800 simulated standard Brownian motions paths with $T = 1$ and $N = 100$. We choose mini-batch size $M = 128$.
- We use a single layer LSTM network with 64 units connecting with a deep feed-forward neural network which consists of three hidden layers with 64, 128, 64 respectively.

Linear Running Integral

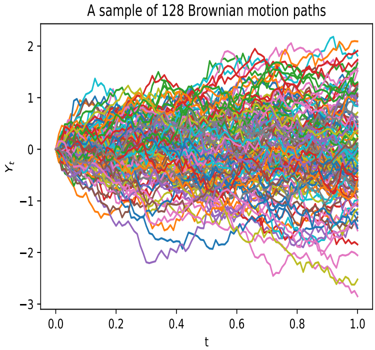


Figure: A sample of 128 Brownian motion paths.

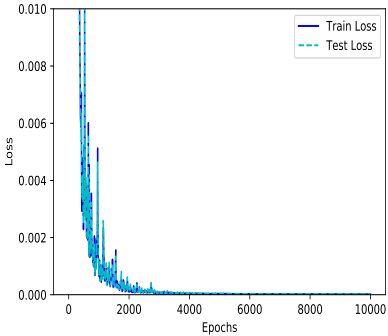
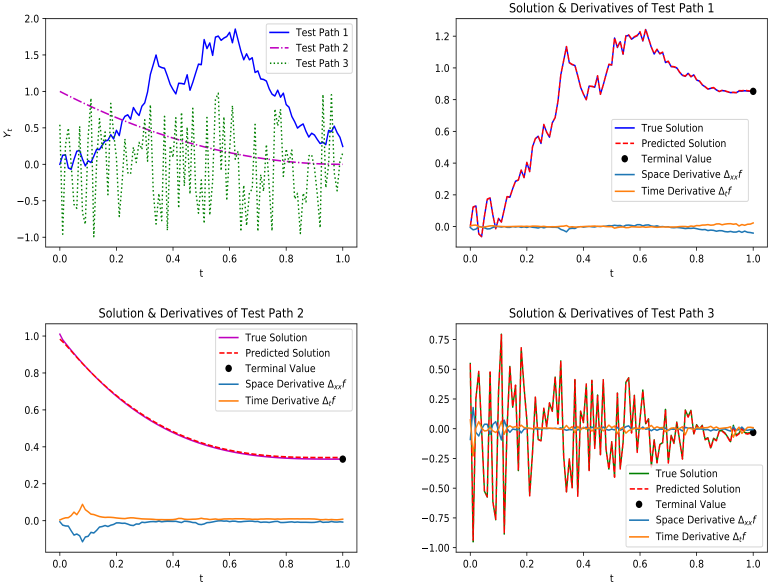


Figure: Train and test losses for the linear running integral example.

Linear Running Integral



Linear Running Integral

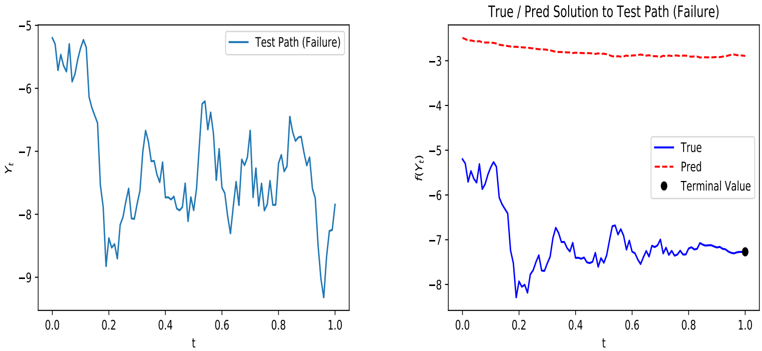
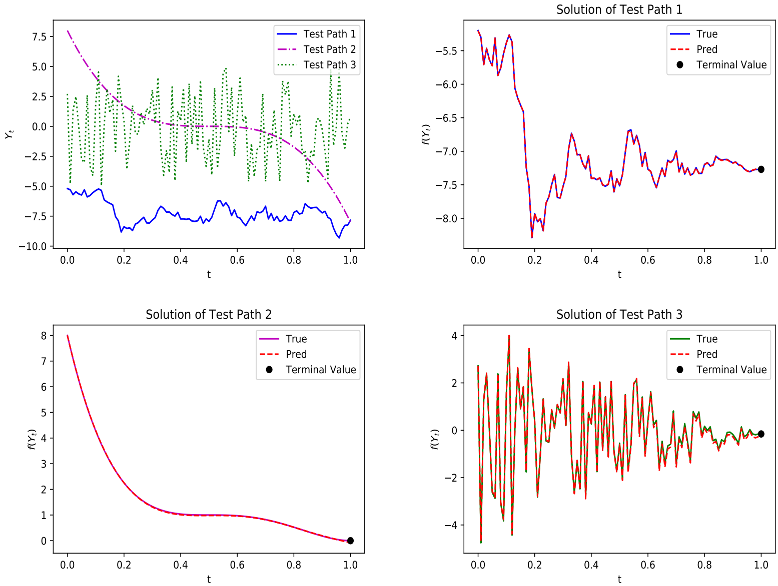


Figure: Prediction failure due to the limitation of training domain.

Linear Running Integral



Non-Linear Example

- Consider PPDE is of the form

$$\begin{cases} \Delta_t f(Y_t) + (\underline{\mu} 1_{\{\Delta_x f(Y_t)>0\}} + \overline{\mu} 1_{\{\Delta_x f(Y_t)<0\}})\Delta_x f(Y_t) \\ + \frac{1}{2}(\underline{\sigma}^2 1_{\{\Delta_{xx} f(Y_t)<0\}} + \overline{\sigma}^2 1_{\{\Delta_{xx} f(Y_t)>0\}})\Delta_{xx} f(Y_t) + \phi(Y_t) = 0, \\ f(Y_T) = \cos(y_T + I_T). \end{cases}$$

with

$$\phi(Y_t) = (y_t + \underline{\mu}) \min(\sin(y_t + I_t), 0) + (y_t + \overline{\mu}) \max(\sin(y_t + I_t), 0) + \frac{\underline{\sigma}^2}{2} \max(\cos(y_t + I_t), 0) + \frac{\overline{\sigma}^2}{2} \min(\cos(y_t + I_t), 0),$$

where $I_t = \int_0^t y_u du$ is the running integral.

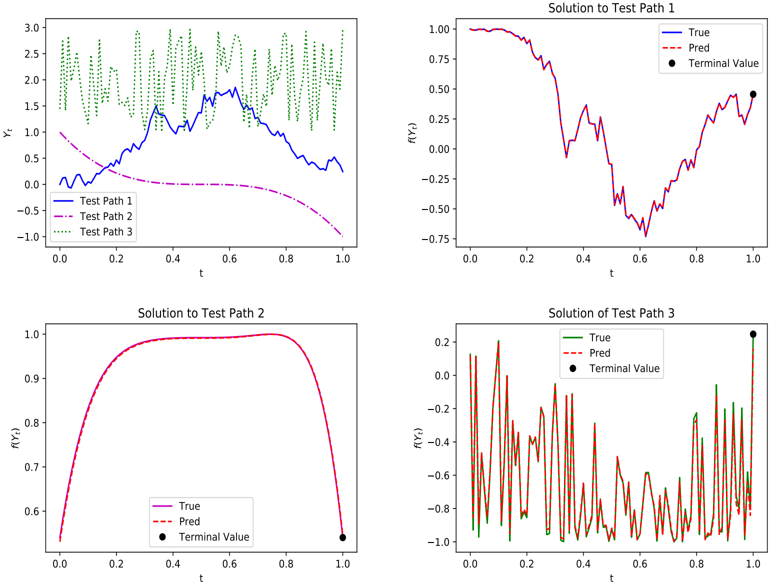
- The closed-formula solution is given by

$$f(Y_t) = \cos(y_t + I_t).$$

Non-Linear Example

- Use standard Brownian motion paths to train the neural network.
- Specify the coefficients to be $\underline{\mu} = -0.2, \overline{\mu} = 0.2, \underline{\sigma} = 0.2,$ and $\overline{\sigma} = 0.3$.
- Loss reaches around 4×10^{-6} after 15000 epochs.
- Test path 1 is a realization of standard Brownian motion path.
  Test path 2 is a smooth path $y_t = (1 - 2t)^3$.
  Test path 3 is is $y_{t_i} \sim U(1, 3),\ i \in \{1, \dots, 100\}$.

Non-Linear Example



Applications in Mathematical Finance

- We will consider the classical Black–Scholes model, where the spot value follows a geometric Brownian Motion with constant parameters

$$dx_t = (r - q)x_t dt + \sigma x_t dw_t.$$

- Under this model, the price of a general path-dependent financial derivative with maturity T and payoff $g : \Lambda_T \longrightarrow \mathbb{R}$ solves the PPDE

$$\begin{cases} \Delta_t f(Y_t) + (r-q) y_t \Delta_x f(Y_t) + \frac{1}{2}\sigma^2 y_t^2 \Delta_{xx} f(Y_t) - rf(Y_t) = 0, \\ f(Y_T) = g(Y_T). \end{cases}$$

  - Geometric Asian Option. $g(Y_T) = \left(\exp\left\{\frac{1}{T}\int_0^T \log y_t dt\right\} - K\right)^+$.
  - Lookback Option. $g(Y_T) = y_T - \inf_{0 \le t \le T} y_t$.
  - Barrier Option. $g(Y_T) = (y_T - K)^+ \mathbb{1}_{\left\{\inf\limits_{0 \le t \le T} y_t > B\right\}}$.

Down-and-Out Call

- We focus on the case of down-and-out call options. More precisely, the option becomes worthless whether the spot value crosses a down barrier $B < S_0$. Otherwise, the payoff is a call with strike $K \geq B$.
- The payoff functional can then be written as

$$g(Y_T) = (y_T - K)^+ 1_{\left\{\inf\limits_{0 \leq t \leq T} y_t > B\right\}}.$$

- A closed-form solution is available:

$$f(Y_t) = \begin{cases} 0, & \text{if } \inf_{0 \leq u \leq t} y_u \leq B, \\ C_{BS}(y_t, T-t) - \left(\frac{y_t}{B}\right)^{1-\lambda} C_{BS}\left(\frac{B^2}{y_t}, T-t\right), & \text{if } \inf_{0 \leq u \leq t} y_u > B, \end{cases}$$

where $C_{BS}(y_t, T-t)$ is the price of a call option with strike K and maturity T at (t, y_t) and $\lambda = \frac{2(r-q)}{\sigma^2}$.

Down-and-Out Call

- The option become valueless when the stock price crosses the barrier. Modify the loss function.
- The loss for a given sample path j at time t_i is

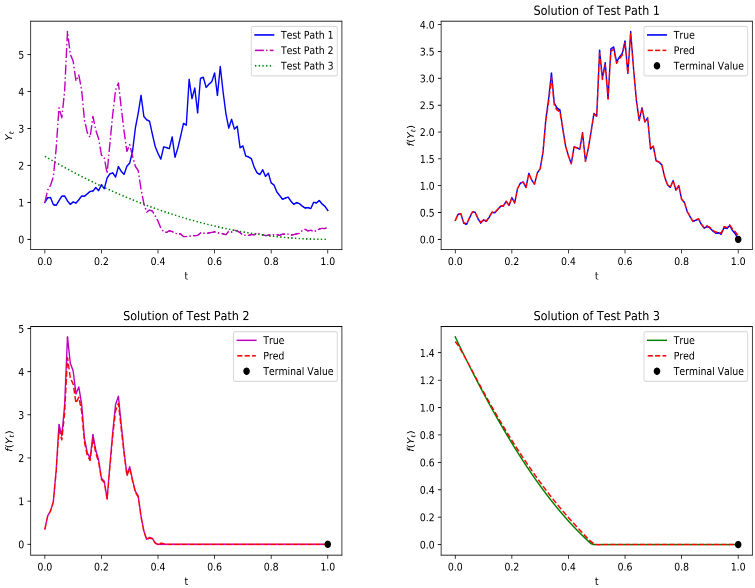
$$J_{t_i}^{(j)}(\theta) = \begin{cases} |u(Y_{t_i}^{(j)};\theta) - 0| & \text{if } \inf_{0\le i'\le i} Y_{t_{i'}}^{(j)} < B, \\ \left(\Delta_t u(Y_{t_i}^{(j)};\theta) + \mathcal{L}u(Y_{t_i}^{(j)};\theta)\right)^2 & \text{otherwise.} \end{cases}$$

The total loss is calculated as

$$\begin{aligned} J_{N,M}(\theta) = & \frac{1}{M}\frac{1}{N}\sum_{j=1}^{M}\sum_{i=0}^{N} J_{t_i}^{(j)}(\theta) \\ & + \frac{1}{M}\sum_{j=1}^{M}\Bigg[\left(u(Y_{t_N}^{(j)};\theta) - g(Y_{t_N}^{(j)})1_{\left\{\inf\limits_{0\le i\le N} y_{t_i} > B\right\}}\right)^2 \\ & + |u(Y_{t_N}^{(j)};\theta) - 0| 1_{\left\{\inf\limits_{0\le i\le N} y_{t_i} < B\right\}}\Bigg]. \end{aligned}$$

- Then minimize the above loss objective using stochastic gradient descent algorithm and update parameter θ.

Down-and-Out Call ($B = 0.6$ and $K = 0.8$)



Heston Model

- A more complex model. Consider is the well-known Heston model:

$$\begin{cases} dx_t = (r-q)x_t dt + \sqrt{v_t} x_t dw_t, \\ dv_t = \kappa(m - v_t)dt + \xi\sqrt{v_t} dw_t^*, \\ dw_t dw_t^* = \rho dt. \end{cases}$$

- The price at time t of a general path-dependent option with maturity T and payoff $g : \Lambda_T \longrightarrow \mathbb{R}$ can be written as the functional $f(Y_t, v)$ and solves the PPDE

$$\begin{cases} \Delta_t f(Y_t, v) + (r-q)y_t \Delta_x f(Y_t, v) + \dfrac{1}{2} v y_t^2 \Delta_{xx} f(Y_t, v) - rf(Y_t, v) \\ +\kappa(m - v)\partial_v f(Y_t, v) + \frac{1}{2}\xi^2 v \partial_{vv} f(Y_t, v) + \rho\xi v y_t \Delta_x \partial_v f(Y_t, v) = 0 \\ f(Y_T, v) = g(Y_T). \end{cases}$$

- Consider the geometric Asian option, and The generalization of our algorithm to this multidimensional case is straightforward. Specify $r = 0.03, q = 0.01, \kappa = 3, m = 1, \xi = 1, \rho = 0.6, x_0 = v_0 = 1, T = 1.$

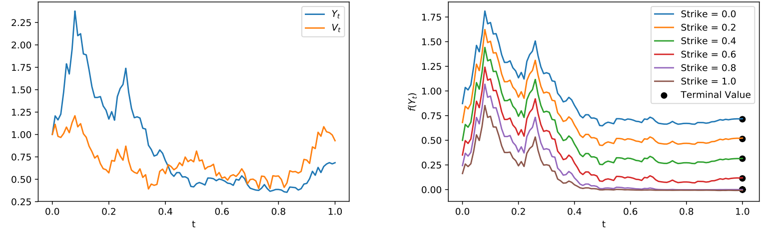


Figure: Given a pair of paths of (Y_t, V_t), Solutions to the Heston model vs strike prices.

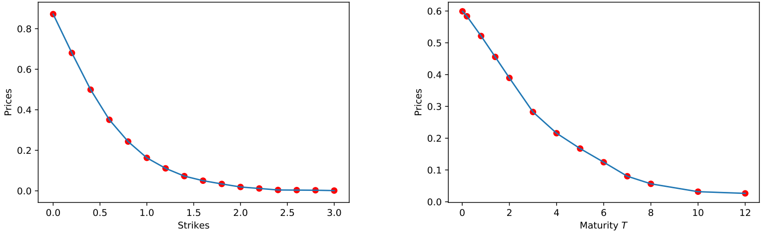


Figure: On the left $(T = 1)$: prices vs strike prices K. On the Right$(K = 0.4)$: prices vs maturity times T.

Summary

- Review Functional Itô Calculus $\Delta_t f(Y_t), \Delta_x f(Y_t), \Delta_{xx} f(Y_t)$.
  PPDE and Functional Feynman-Kac Formula.

- Review of Neural Networks. $\mathbb{NN}^{\ell}_{d_1,d_2}$ and $\mathbb{LSTM}_{i,d,k}$.

- PDGM Architecture and Algorithm.

$$
\begin{aligned}
u(Y_{t_i};\theta) &= \varphi(t_i, y_{t_i}, a_{t_{i-1}}; \theta^f) \\
u(Y^h_{t_i};\theta) &= \varphi(t_i, y_{t_i} + h, a_{t_{i-1}}; \theta^f), \\
u(Y_{t_i,\delta t};\theta) &= \varphi(t_{i+1}, y_{t_i}, a_{t_i}; \theta^f).
\end{aligned}
$$

- Numerical Examples.
  - Linear and Nonlinear PPDE.
  - Down-and-Out Call and Heston Model.