Set-Valued Risk Measures and Bellman's Principle

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March 7, 2016

Outline

1. Set-valued risk measures
2. Time consistency
3. Recursive algorithm
4. Computation
5. Examples

Set-Valued Risk Measures: Primal Representation

2. Set-valued risk measures: Setup

- Probability space $(\Omega, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$
- d assets (may include different currencies)
- Portfolio vectors in physical units (numéraire free), i.e. number of units in d assets
- Claim: $X \in L^p := L^p_T(\mathbb{R}^d)$ payoff (in physical units) at time T
- Eligible portfolios $M = \mathbb{R}^m \times \{0\}^{d-m}$, linear subspace of $\mathbb{R}^d$ of portfolios that can be used to compensate risk (e.g. Dollars & Euros)

2.1 Set-valued risk measures: Primal representation

- $M_t := L_t^p(M), \quad M_{t,+} := M_t \cap L_{t,+}^p$
- $\mathcal{P}(\mathcal{Z}; C) := \{A \subseteq \mathcal{Z} \mid A = A + C\}$
- $\mathcal{G}(\mathcal{Z}; C) := \{A \subseteq \mathcal{Z} \mid A = \operatorname{cl}\operatorname{co}(A + C)\}$

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Conditional Set-Valued Risk Measure

A set-valued function $R_t : L^p \to \mathcal{P}(M_t; M_{t,+})$ is a conditional risk measure if

1. Finite at zero: $\emptyset \neq R_t(0) \neq M_t$;
2. M_t translative: $R_t(X + m) = R_t(X) - m$ for any $m \in M_t$;
3. L^p_+ monotone: if $X - Y \in L^p_+$ then $R_t(X) \supseteq R_t(Y)$.

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- ***Normalized***: for every $X \in L^p_t : R_t(X) = R_t(X) + R_t(0)$.
- Normalized version:
  $\bar{R}_t(X) := R_t(X) -^{\cdot} R_t(0) = \{u \in M_t \mid R_t(0) + u \subseteq R_t(X)\}$.

- ***(Conditionally) convex***: for all $X, Y \in L^p$, for all $0 \leq \lambda \leq 1$ ($\lambda \in L_t^\infty(\mathbb{R})$ such that $0 \leq \lambda \leq 1$)

$$R_t(\lambda X + (1-\lambda)Y) \supseteq \lambda R_t(X) + (1-\lambda)R_t(Y).$$

- ***(Conditionally) positive homogeneous***: for all $X \in L^p$, for all $\lambda > 0$ ($\lambda \in L_t^\infty(\mathbb{R}_{++})$)

$$R_t(\lambda X) = \lambda R_t(X).$$

- ***(Conditionally) coherent***: if it is (conditionally) convex and (conditionally) positive homogeneous.

2.1 Set-valued risk measures: Primal representation

- ***K*-compatible**: for some set $K \subseteq L^p$ if there exists a risk measure $\tilde{R}$ such that $R_t(X) = \bigcup_{k \in K} \tilde{R}_t(X - k)$.

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- K***-compatible***: for some set $K \subseteq L^p$ if there exists a risk measure $\tilde{R}$ such that $R_t(X) = \bigcup_{k \in K} \tilde{R}_t(X - k)$.
- ***Closed***: if the graph of R_t is closed in the product topology, i.e.,

$$\operatorname{graph} R_t := \{(X, u) \in L^p \times M_t \mid u \in R_t(X)\} \text{ is closed.}$$

- ***(Conditionally) convex upper continuous [(c.)c.u.c.]***: if for any closed (conditionally) convex set $D \in \mathcal{G}(M_t; M_{t,-})$ the inverse image

$$R_t^{-1}(D) := \{X \in L^p \mid R_t(X) \cap D \neq \emptyset\}$$

is closed.

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- Acceptance set: $A_t = \{X \in L^p \mid 0 \in R_t(X)\}$
- Risk measure: $R_t(X) = \{u \in M_t \mid X + u \in A_t\}$

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Properties	
Risk measure	**Acceptance set**
(Conditionally) convex	(Conditionally) convex
(Conditionally) coherent	(Conditionally) convex cone
Closed graph	Closed
$B \subseteq L^p$ B-monotone	$A_t + B = A_t$
$C \subseteq M_t$ $R_t(X) : L^p \to \mathcal{P}(M_t; C)$	$A_t + C \subseteq A_t$
$R_t(X) \neq \emptyset \; \forall X \in L^p$	$L^p = A_t + M_t$
$R_t(X) \neq M_t \; \forall X \in L^p$	$L^p = (L^p \backslash A_t) + M_t$

Set-Valued Risk Measures:
Dual Representation

2.2 Set-valued risk measures: Dual representation

- Dual variables:

$$\mathcal{W}_t := \left\{(\mathbb{Q}, w) \in \mathcal{M}^d \times \left(M_{t,+}^+ \backslash M_t^\perp\right) \mid \mathbb{Q} = \mathbb{P}|_{\mathcal{F}_t},\ w_t^T(\mathbb{Q}, w) \in L_+^q\right\};$$

where $w_t^s(\mathbb{Q}, w) = w \cdot \mathbb{E}\left[\frac{d\mathbb{Q}}{d\mathbb{P}}\middle| \mathcal{F}_s\right] = \left(w_1 \mathbb{E}\left[\frac{d\mathbb{Q}_1}{d\mathbb{P}}\middle| \mathcal{F}_s\right], \ldots, w_d \mathbb{E}\left[\frac{d\mathbb{Q}_d}{d\mathbb{P}}\middle| \mathcal{F}_s\right]\right)^{\mathsf{T}}$.

- Halfspace: $G_t(w) := \left\{u \in L_t^p \mid \mathbb{E}\left[w^{\mathsf{T}} u\right] \geq 0\right\}$
- Conditional Halfspace: $\Gamma_t(w) := \left\{u \in L_t^p \mid w^{\mathsf{T}} u \geq 0 \text{ a.s.}\right\}$

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where $w_t^s(\mathbb{Q}, w) = w \cdot \mathbb{E}\left[\frac{d\mathbb{Q}}{d\mathbb{P}}\middle| \mathcal{F}_s\right] = \left(w_1 \mathbb{E}\left[\frac{d\mathbb{Q}_1}{d\mathbb{P}}\middle| \mathcal{F}_s\right], ..., w_d \mathbb{E}\left[\frac{d\mathbb{Q}_d}{d\mathbb{P}}\middle| \mathcal{F}_s\right]\right)^{\mathsf{T}}$.

- Halfspace: $G_t(w) := \left\{u \in L_t^p \mid \mathbb{E}\left[w^{\mathsf{T}} u\right] \geq 0\right\}$
- Conditional Halfspace: $\Gamma_t(w) := \left\{u \in L_t^p \mid w^{\mathsf{T}} u \geq 0 \text{ a.s.}\right\}$

- Set subtraction: $A \mathbin{-^{\cdot}} B := \{m \in M_t \mid B + m \subseteq A\}$

- $G_t(w) := \left\{u \in L_t^p \mid \mathbb{E}\left[w^\mathsf{T} u\right] \geq 0\right\}$

Convex Risk Measures

A function $R_t : L^p \to \mathcal{G}(M_t; M_{t,+})$ is a ***closed convex risk measure*** if and only if

$$R_t(X) = \bigcap_{(\mathbb{Q},w)\in\mathcal{W}_t} \left[\left(\mathbb{E}^{\mathbb{Q}}\left[-X\right|\mathcal{F}_t] + G_t(w)\right) \cap M_t -^{\cdot} \beta_t(\mathbb{Q},w)\right],$$

where β_t is the minimal penalty function given by

$$\beta_t^{\min}(\mathbb{Q},w) = \bigcap_{Y\in A_t} \left(\mathbb{E}^{\mathbb{Q}}\left[-Y\right|\mathcal{F}_t] + G_t(w)\right) \cap M_t.$$

- $G_t(w) := \left\{u \in L^p_t \mid \mathbb{E}\left[w^\mathsf{T} u\right] \geq 0\right\}$

Coherent Risk Measures

A function $R_t : L^p \to \mathcal{G}(M_t; M_{t,+})$ is a ***closed coherent risk measure*** if and only if

$$R_t(X) = \bigcap_{(\mathbb{Q},w)\in\mathcal{W}_t^{\max}} \left(\mathbb{E}^{\mathbb{Q}}\left[-X\middle|\mathcal{F}_t\right] + G_t\left(w\right)\right) \cap M_t,$$

where $\mathcal{W}_t^{\max}$ is the maximal set of dual variables given by

$$\mathcal{W}_t^{\max} = \left\{(\mathbb{Q}, w) \in \mathcal{W}_t \mid w_t^T(\mathbb{Q}, w) \in A_t^+\right\}.$$

2.2 Set-valued risk measures: Dual representation

- $\Gamma_t(w) := \left\{u \in L_t^p \mid w^\mathsf{T} u \geq 0 \text{ a.s.}\right\}$

Conditionally Convex and Coherent Risk Measures

A function $R_t : L^p \to \mathcal{G}(M_t; M_{t,+})$ is a ***closed conditionally convex risk measure*** if and only if

$$R_t(X) = \bigcap_{(\mathbb{Q},w)\in\mathcal{W}_t} \left[\left(\mathbb{E}^{\mathbb{Q}}\left[-X\middle|\mathcal{F}_t\right] + \Gamma_t(w)\right) \cap M_t \mathbin{\dot{-}} \alpha_t(\mathbb{Q},w)\right],$$

where α_t is the conditional penalty function given by

$$\alpha_t(\mathbb{Q},w) = \bigcap_{Y\in A_t} \left(\mathbb{E}^{\mathbb{Q}}\left[Y\middle|\mathcal{F}_t\right] + \Gamma_t(w)\right) \cap M_t.$$

R_t is additionally ***conditionally coherent*** if and only if

$$R_t(X) = \bigcap_{(\mathbb{Q},w)\in\mathcal{W}_t^{\max}} \left(\mathbb{E}^{\mathbb{Q}}\left[-X\middle|\mathcal{F}_t\right] + \Gamma_t\left(w\right)\right) \cap M_t.$$

Time Consistency:
Multiportfolio Time Consistency

Multiportfolio Time Consistency

A dynamic risk measure $(R_t)_{t=0}^T$ is ***multiportfolio time consistent*** if the relation

$$R_s(X) \subseteq \bigcup_{Y \in \mathbb{Y}} R_s(Y) \Rightarrow R_t(X) \subseteq \bigcup_{Y \in \mathbb{Y}} R_t(Y)$$

for any times $t < s$, any $X \in L^p$ and any $\mathbb{Y} \subseteq L^p$

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- Multiportfolio time consistency implies "time consistency" defined by
$$R_s(X) \subseteq R_s(Y) \Rightarrow R_t(X) \subseteq R_t(Y)$$
for any times $t < s$ and $X, Y \in L^p$.

Multiportfolio Time Consistency

If $(R_t)_{t=0}^T$ is normalized ($R_t(X) = R_t(X) + R_t(0)$ for every X and t) then the following are equivalent:

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- $A_t = A_s + A_{t,s}$ where $A_{t,s} := A_t \cap M_s$.

- If discrete time $t, s \in \{0, 1, ..., T\}$ then sufficient to have any of these conditions with $s = t + 1$.

Multiportfolio Time Consistency

If $(R_t)_{t=0}^T$ is normalized, c.u.c., and convex then the following are equivalent:

- $(R_t)_{t=0}^T$ is multiportfolio time consistent;
- $\beta_t(\mathbb{Q}, w) = \mathrm{cl}\left(\beta_{t,s}(\mathbb{Q}, w) + \mathbb{E}^{\mathbb{Q}}\left[\beta_s(\mathbb{Q}, w_t^s(\mathbb{Q}, w)) \middle| \mathcal{F}_t\right]\right)$;
- $V_t^{(\mathbb{Q},w)}(X) \subseteq \mathbb{E}^{\mathbb{Q}}\left[V_s^{(\mathbb{Q},w_t^s(\mathbb{Q},w))}(X) \middle| \mathcal{F}_t\right]$ for every $X \in L^p$ where

$$V_t^{(\mathbb{Q},w)}(X) := \mathrm{cl}\left[R_t(X) + \beta_t(\mathbb{Q}, w)\right].$$

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$$V_t^{(\mathbb{Q},w)}(X) := \operatorname{cl}\left[R_t(X) + \beta_t(\mathbb{Q}, w)\right].$$

When $(R_t)_{t=0}^T$ is coherent then

$$\beta_t(\mathbb{Q}, w) = \begin{cases} G_t(w) \cap M_t & \text{if } (\mathbb{Q}, w) \in \mathcal{W}_t^{\max} \\ \emptyset & \text{else} \end{cases}.$$

Multiportfolio Time Consistency

If $(R_t)_{t=0}^T$ is normalized, c.u.c., and conditionally convex with dual representation defined on $\mathcal{W}_t^e := \{(\mathbb{Q}, w) \in \mathcal{W}_t \mid \mathbb{Q} \sim \mathbb{P}\}$ then the following are equivalent:

- $(R_t)_{t=0}^T$ is multiportfolio time consistent;
- $\alpha_t(\mathbb{Q}, w) = \operatorname{cl}\left(\alpha_{t,s}(\mathbb{Q}, w) + \mathbb{E}^{\mathbb{Q}}\left[\alpha_s(\mathbb{Q}, w_t^s(\mathbb{Q}, w)) \middle| \mathcal{F}_t\right]\right)$;
- $\mathbb{V}_t^{(\mathbb{Q},w)}(X) \subseteq \operatorname{cl} \mathbb{E}^{\mathbb{Q}}\left[\mathbb{V}_s^{(\mathbb{Q},w_t^s(\mathbb{Q},w))}(X) \middle| \mathcal{F}_t\right]$ for every $X \in L^p$ where

$$\mathbb{V}_t^{(\mathbb{Q},w)}(X) := \operatorname{cl}\left[R_t(X) + \alpha_t(\mathbb{Q}, w)\right].$$

When $(R_t)_{t=0}^T$ is conditionally coherent then

$$\alpha_t(\mathbb{Q}, w) = \begin{cases} \Gamma_t(w) \cap M_t & \text{if } (\mathbb{Q}, w) \in \mathcal{W}_t^{\max} \\ \emptyset & \text{else} \end{cases}.$$

- Let $\mathbb{Q}, \mathbb{R} \in \mathcal{M}^d$ then $\mathbb{S} = \mathbb{Q} \oplus^s \mathbb{R}$ if

$$\frac{d\mathbb{S}_i}{d\mathbb{P}} := \begin{cases} \mathbb{E}\left[\frac{d\mathbb{Q}_i}{d\mathbb{P}}\middle|\mathcal{F}_s\right] \cdot \frac{d\mathbb{R}_i}{d\mathbb{P}} \Big/ \mathbb{E}\left[\frac{d\mathbb{R}_i}{d\mathbb{P}}\middle|\mathcal{F}_s\right] & \text{on } \left\{\mathbb{E}\left[\frac{d\mathbb{R}_i}{d\mathbb{P}}\middle|\mathcal{F}_s\right] > 0\right\}, \\ \mathbb{E}\left[\frac{d\mathbb{Q}_i}{d\mathbb{P}}\middle|\mathcal{F}_s\right] & \text{else} \end{cases}$$

Stability

A set $W_t \subseteq \mathcal{W}_t$ is ***stable*** at time t with respect to $W_{t,s}$ and W_s if

- $(\mathbb{Q}, w) \in W_t$ implies $(\mathbb{Q}, w_t^s(\mathbb{Q}, w)) \in W_s$ and
- $(\mathbb{Q}, w) \in W_{t,s}$ and $\mathbb{R} \in \mathcal{M}^d$ such that $(\mathbb{R}, w_t^s(\mathbb{Q}, w)) \in W_s$ implies $(\mathbb{Q} \oplus^s \mathbb{R}, w) \in W_t$

Multiportfolio Time Consistency

If $(R_t)_{t=0}^T$ is normalized, c.u.c., and coherent then the following are equivalent then $(R_t)_{t=0}^T$ is multiportfolio time consistent if and only if any of the following equivalent properties holds for all times $t < s$

- $\mathcal{W}_t^{\max}$ is stable with respect to $\mathcal{W}_{t,s}^{\max}$ and $\mathcal{W}_s^{\max}$;
- $\mathcal{W}_t^{\max} = \left\{(\mathbb{Q} \oplus^s \mathbb{R}, w) \mid (\mathbb{Q}, w) \in \mathcal{W}_{t,s}^{\max}, (\mathbb{R}, w_t^s(\mathbb{Q}, w)) \in \mathcal{W}_s^{\max}\right\}$;
- $\mathcal{W}_t^{\max} = \mathcal{W}_{t,s}^{\max} \cap H_t^s(\mathcal{W}_s^{\max})$ where $H_t^s(W) := \{(\mathbb{Q}, w) \in \mathcal{W}_t : (\mathbb{Q}, w_t^s(\mathbb{Q}, w)) \in W\}$.

Time Consistency:
Composition of Risk Measures

3.2 Time consistency: Composition of risk measures

Composition of One-Step Risk Measures

Let $(R_t)_{t=0}^T$ be a risk measure then $\left(\tilde{R}_t\right)_{t=0}^T$ is the multiportfolio time consistent version if

$$\tilde{R}_T(X) := R_T(X); \qquad \tilde{R}_t(X) := \bigcup_{Z \in \tilde{R}_{t+1}(X)} R_t(-Z)$$

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$$\tilde{R}_T(X) := R_T(X); \qquad \tilde{R}_t(X) := \bigcup_{Z \in \tilde{R}_{t+1}(X)} R_t(-Z)$$

- Also given by:

$$\begin{aligned}
\tilde{A}_t &:= A_{t,t+1} + \tilde{A}_{t+1};\\
\tilde{\beta}_t(\mathbb{Q}, w) &:= \operatorname{cl}\left(\beta^{\min}_{t,t+1}(\mathbb{Q}, w) + \mathbb{E}^{\mathbb{Q}}\left[\tilde{\beta}_{t+1}(\mathbb{Q}, w_t^{t+1}(\mathbb{Q}, w)) \middle| \mathcal{F}_t\right]\right);\\
\tilde{\alpha}_t(\mathbb{Q}, w) &:= \operatorname{cl}\left(\alpha^{\min}_{t,t+1}(\mathbb{Q}, w) + \mathbb{E}^{\mathbb{Q}}\left[\tilde{\alpha}_{t+1}(\mathbb{Q}, w_t^{t+1}(\mathbb{Q}, w)) \middle| \mathcal{F}_t\right]\right);\\
\widetilde{\mathcal{W}}_t &:= \mathcal{W}^{\max}_{t,t+1} \cap H_t^{t+1}(\widetilde{\mathcal{W}}_{t+1}).
\end{aligned}$$

Recursive Algorithm

4. Recursive algorithm: Setting

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- $R_t(X)[\omega_t] = \{u(\omega_t) : u \in R_t(X)\}$

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Pointwise Representation

If R_t has closed and conditionally convex images then $u \in R_t(X)$ if and only if $u(\omega_t) \in R_t(X)[\omega_t]$ for every $\omega_t \in \Omega_t$.

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If R_t has closed and conditionally convex images then $u \in R_t(X)$ if and only if $u(\omega_t) \in R_t(X)[\omega_t]$ for every $\omega_t \in \Omega_t$.

- If R_t is closed and conditionally convex, then R_t has closed and conditionally convex images.
- R_t is ***local*** if $1_D R_t(X) = 1_D R_t(1_D X)$ for every $D \in \mathcal{F}_t$

- $R_t(X)[\omega_t] = \{u(\omega_t) : u \in R_t(X)\}$
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Pointwise Representation

If R_t has closed and conditionally convex images then $u \in R_t(X)$ if and only if $u(\omega_t) \in R_t(X)[\omega_t]$ for every $\omega_t \in \Omega_t$.

- If R_t is closed and conditionally convex, then R_t has closed and conditionally convex images.
- R_t is ***local*** if $1_D R_t(X) = 1_D R_t(1_D X)$ for every $D \in \mathcal{F}_t$
- If R_t is local then $R_t(X)[\omega_t] = R_t(1_{\omega_t} X)[\omega_t]$

Recursive Algorithm: Bellman's Principle

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Want: The **closed multi-portfolio time consistent** version of R_t

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Pointwise Representation

$$\bar{R}_T(X)[\omega_T] = \mathrm{cl}(R_T(X)[\omega_T])$$

$$\bar{R}_t(X)[\omega_t] = \mathrm{cl}\bigcup\{R_{t,t+1}(-Z)[\omega_t] : \forall \omega_{t+1} \in \mathrm{succ}(\omega_t) : Z(\omega_{t+1}) \in \bar{R}_{t+1}(X)[\omega_{t+1}]\Big\}$$

Dynamic set optimization:

$\bar{R}_t(X)[\omega_t] = \inf_{Z \in \bar{\mathcal{Z}}_{t+1}[\omega_t]} R_{t,t+1}(-Z)[\omega_t]$ with

$$\bar{\mathcal{Z}}_{t+1}[\omega_t] = \big\{ Z \in L^p_{t+1} : \forall \omega_{t+1} \in \operatorname{succ}(\omega_t) : Z(\omega_{t+1}) \in \bar{R}_{t+1}(X)[\omega_{t+1}] \big\}$$

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Equivalent to **vector optimization** problem:

$$\bar{R}_t(X)[\omega_t] = \inf_{(Z,Y) \in \mathcal{Z}_{t+1}[\omega_t]} \Gamma(Z,Y)$$

for $\Gamma(Z,Y) = Y$ and
$\mathcal{Z}_{t+1}[\omega_t] = \left\{(Z,Y) \in \bar{\mathcal{Z}}_{t+1}[\omega_t] \times M : Y \in R_{t,t+1}(-Z)[\omega_t]\right\}$

Interpretation as Bellman's Principle:

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The at t truncated optimal solution $(Z_s)_{s=t}^T$ obtained at time 0 from a given $Z_0 \in \tilde{R}_0(X)$ is still optimal at any later time point $t \in \{0, ..., T\}$.

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Meaning: For the risk compensating portfolio holding $Z_t \in \tilde{R}_t(X)$, $(Z_s)_{s=t}^T$ satisfies the conditions $Z_s \in \tilde{R}_s(X)$ and $Z_{s-1} \in R_{s-1}(-Z_s)$, $s \in \{t, ..., T\}$.

Computation

Approximation: Not always possible or feasible to find the risk measure exactly

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R_t^δ is a δ***-approximation*** of R_t if for every $X \in L^p$

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for a fixed $m \in \operatorname{int}(M_+)$

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Question: How do errors grow with time steps?

Propogation of Errors: Errors propogate linearly

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If $\bar{R}^{\epsilon}_{t+1}(X)$ is an ϵ-approximation of $\bar{R}_{t+1}(X)$ then composed backwards $\bar{R}^{\epsilon}_{t}(X)$ is an ϵ-approximation of $\bar{R}_{t}(X)$

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If $\bar{R}^{\epsilon,\gamma}_{t}(X)$ is a γ-approximation of ϵ-approximation $\bar{R}^{\epsilon}_{t}(X)$, then $\bar{R}^{\epsilon,\gamma}_{t}(X)$ is an $(\epsilon+\gamma)$-approximation of $\bar{R}_{t}(X)$

Polyhedral risk measures:

- Linear vector optimization
- $R_{t,t+1}(-Z)[\omega_t] = \{P_t z + M_{t,+} : A_t Z + B_t z \leq b_t\}$
- $\dim(M)$-dimensional problem with
  $d \times |\operatorname{succ}(\omega_t)| + |z|$-dimensional pre-image space
- Benson's algorithm can be applied directly

Convex risk measures:

- Convex vector optimization
- $R_{t,t+1}(-Z)[\omega_t] = \{P_t(z) + M_{t,+} : g(Z,z) \leq 0\}$
- To calculate: a polyhedral δ-approximation of $\bar{R}_t$ by recursively calculating backwards in time
- $\dim(M)$-dimensional problem with $d \times |\operatorname{succ}(\omega_t)| + |z|$-dimensional pre-image space
- Convex extension of Benson's algorithm can be applied directly with a given approximation error desired

Examples:
Superhedging

- Convex transaction costs at time t: closed convex set $\mathbb{R}^d_+ \subseteq K_t[\omega] \subseteq \mathbb{R}^d$ (solvency region), positions transferable into non-negative portfolios.

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Self-Financing Portfolio Process

A **self-financing portfolio process** $(V_t)_{t=0}^T$ is a stochastic process of portfolio vectors (of "physical units") if starting with no assets you can trade for V_t from the portfolio V_{t-1}.

$$\forall t = 0, ..., T : \; V_t - V_{t-1} \in -K_t$$

with $V_{-1} = 0$ and K_t is a solvency region.

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- Let $K_{t,s} := -\sum_{r=t}^{s} L^p_r(K_r)$ denote the portfolios reachable from time t at time s.

6.1 Examples: Superhedging

- $SHP_t(X) := \{u \in L_t^p \mid -X + u \in -K_{t,T}\}$.

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- If the market model $(K_t)_{t=0}^T$ satisfies proper no-arbitrage argument (robust no scalable arbitrage) then the superhedging portfolios can be found via the dual representation with penalty function:

$$\alpha_t^{SHP}(\mathbb{Q}, w) = \sum_{s=t}^{T} \{u \in L_t^p \mid \\ \operatorname*{ess\,sup}_{k \in L_s^p(K_s)} -w^\mathsf{T} \mathbb{E}^{\mathbb{Q}} \left[k \middle| \mathcal{F}_t \right] \leq w^\mathsf{T} u \right\}$$

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- This is multiportfolio time consistent, but not necessarily self-recursive.

6.1 Examples: Superhedging

- If solvency cones (K_t is a.s. a cone) then the superhedging portfolios are conditionally coherent with dual variables defined by

$$\mathcal{W}_{\{t,...,T\}} = \{(\mathbb{Q}, w) \in \mathcal{W}_t \mid \forall s \in \{t, ..., T\} : w_t^s(\mathbb{Q}, w) \in L_s^q(K_s^+)\}$$

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Examples:
Average Value-at-Risk

- Level $\lambda^t \in L_t^\infty$ bounded away from 0.
- Average Value-at-Risk at time t is closed and conditionally coherent risk measure defined by the dual variables:

$$\mathcal{W}_t^\lambda := \left\{ (\mathbb{Q}, w) \in \mathcal{W}_t \mid 0 \preceq w \cdot \frac{d\mathbb{Q}}{d\mathbb{P}} \preceq w/\lambda^t \right\}$$

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- Average Value-at-Risk is not time consistent.

- Consider $M := \mathbb{R}^d$ and $p = +\infty$
- Multiportfolio time consistent version has dual variables:

$$\widetilde{\mathcal{W}}_t^{\lambda} := \{(\mathbb{Q}, w) \in \mathcal{W}_t \mid \forall s \in \{t, t+1, ..., T-1\} : \\ w_t^s(\mathbb{Q}, w)/\lambda^s \succeq w_t^{s+1}(\mathbb{Q}, w)\} \\ = \{(\mathbb{Q}, w) \in \mathcal{W}_t \mid \forall s \in \{t, t+1, ..., T-1\} \forall i : \\ \mathbb{P}\left(\mathbb{E}\left[\frac{d\mathbb{Q}_i}{d\mathbb{P}}\middle|\mathcal{F}_{s+1}\right] \leq \frac{1}{\lambda_i^s}\mathbb{E}\left[\frac{d\mathbb{Q}_i}{d\mathbb{P}}\middle|\mathcal{F}_s\right] \text{ or } w_i = 0\right) = 1 \; \forall i\right\}$$

Examples:
Entropic Risk Measure

- Utility based shortfall risk measures:

$$R_t^u(X) := \{m \in M_t \mid \mathbb{E}\left[u(X+m)\middle|\, \mathcal{F}_t\right] \in C_t\}$$

for some vector utility function u and $C_t \in \mathcal{G}(L_t^p; L_{t,+}^p)$

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- Restrictive entropic risk measure: $M = \mathbb{R}^d$, $u_i(x) = \frac{1-\exp(-\lambda_i x)}{\lambda_i}$ and $C_t = L_{t,+}^p$

6.3 Examples: Entropic risk measure

- Utility based shortfall risk measures:

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- Restrictive entropic risk measure: $M = \mathbb{R}^d$, $u_i(x) = \frac{1-\exp(-\lambda_i x)}{\lambda_i}$ and $C_t = L_{t,+}^p$
- Closed convex risk measure with penalty function

$$-\alpha_t^{ent}(\mathbb{Q}, w) = H_t(\mathbb{Q}|\mathbb{P})/\lambda + \Gamma_t(w)$$
$$-\beta_t^{ent}(\mathbb{Q}, w) = H_t(\mathbb{Q}|\mathbb{P})/\lambda + G_t(w)$$

where $\hat{H}_t(\mathbb{Q}|\mathbb{P}) = \mathbb{E}^{\mathbb{Q}}\left[\log(\frac{d\mathbb{Q}}{d\mathbb{P}})\middle|\, \mathcal{F}_t\right]$

- c.u.c., normalized, and multiportfolio time consistent.

Examples:
2 assets, proportional transaction costs

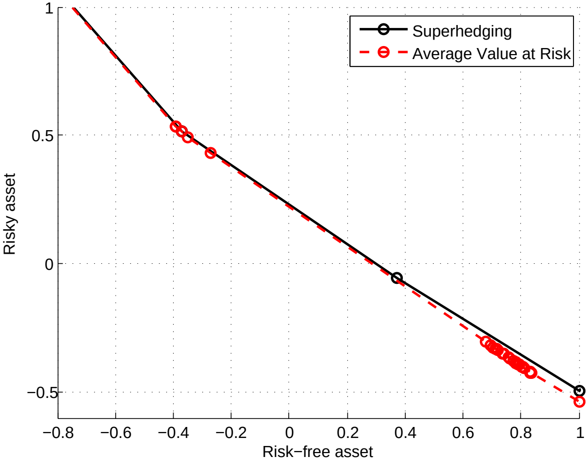

Figure: The superhedging risk measure and composed average value-at-risk of an at-the-money European put option

Examples:
3 assets, proportional transaction costs

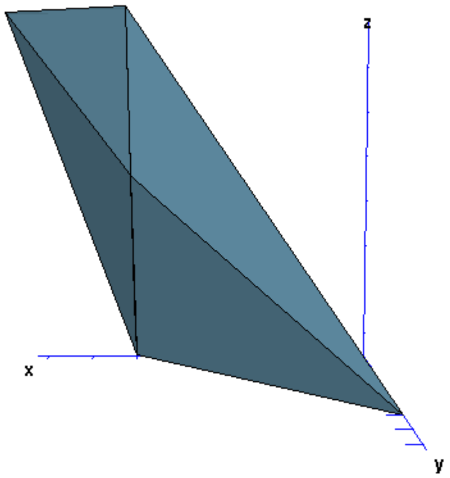

Figure: The composed relaxed worst case risk measure of an outperformance option

6. Examples: 3 assets, proportional transaction costs

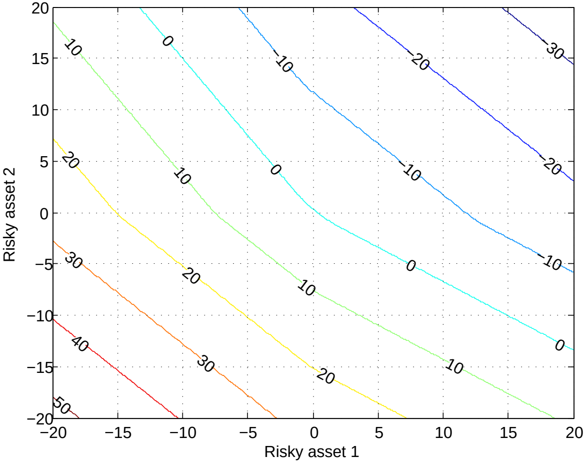

Figure: Contour plot of the efficient frontier of the at differing levels of captial in the risk-less asset

Examples:
2 assets, convex transaction costs

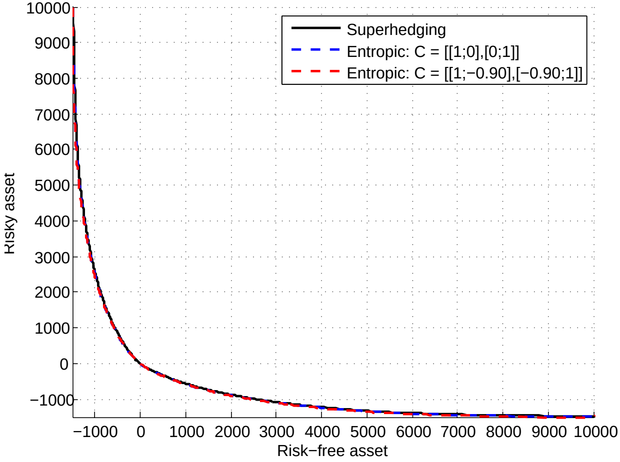

Figure: The superhedging risk measure and composed entropic risk measures of a binary option

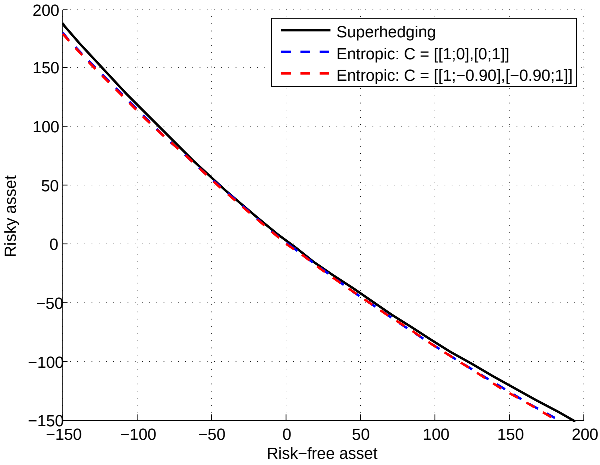

Figure: The superhedging risk measure and composed entropic risk measures of a binary option; zoomed-in

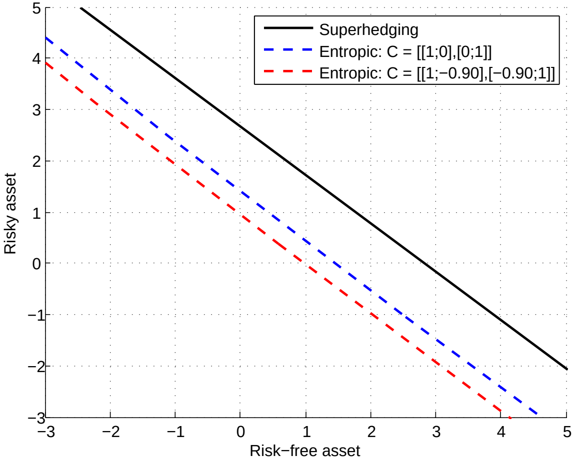

Figure: The superhedging risk measure and composed entropic risk measures of a binary option; near 0

Thank you

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