Recent Developments on Functional Itô Calculus: Lie Bracket and Tanaka Formula

Yuri F. Saporito

FGV EMAp
ESCOLA DE MATEMÁTICA APLICADA

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Itô Formula

Theorem (Itô Formula)

Consider $f \in C^{1,2}([0,T] \times \mathbb{R})$. *Then*

$$f(t, w_t) = f(0,0) + \int_0^t \frac{\partial f}{\partial t}(s, w_s)ds + \int_0^t \frac{\partial f}{\partial x}(s, w_s)dw_s + \\ + \frac{1}{2}\int_0^t \frac{\partial^2 f}{\partial x^2}(s, w_s)ds.$$

- Only functions of the current value (current source of randomness).
- This is a Fundamental Theorem of Calculus for functions with unbounded variation and finite quadratic variation.
- The term with the second derivative materializes the bounded *quadratic* variation feature.

Motivation

Various examples in real life show that "often the impact of randomness is cumulative and depends on the history of the process":

- The price of a path-dependent options may depend on the whole history of the underlying asset, e.g. option on the average (Asian), on the maximum/minimum (Lookback, Barrier), etc.
- Blood glucose levels depends on the history of food composition, portion sizes and timing of meals.
- The temperature at a certain location depends on the temperature history and the history of other factors like humidity, ocean currents, winds, clouds, solar radiation, etc.
- The credit score of a person depends on payment history, amounts owed, length of credit history, new credit accounts, etc.

Bruno Dupire, "Functional Itô Calculus", (July 17, 2009), Available at SSRN: http://ssrn.com/abstract=1435551

Functional Itô Calculus - Space of Paths

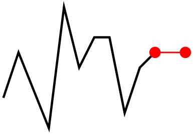
- $\Lambda_t = \{\text{bounded càdlàg paths from } [0,t] \to \mathbb{R}\}$.
- Space of paths:

$$\Lambda = \bigcup_{t \in [0,T]} \Lambda_t.$$

- Upper case - Paths: When $X \in \Lambda$ belongs to the specific Λ_t, we denote it by X_t.
  - Since the Λ_t are disjoints, this is a consistent notation.
  - If X_t is fixed, we denote the restriction of X_t to $[0,s]$ by X_s.
- Lower case - Process: $X \in \Lambda_t \Rightarrow x_s = X_t(s),\ s \in [0,t]$.

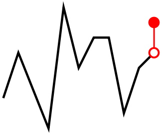
Flat Extension:

$$X_{t,s-t}(u) = \begin{cases} x_u \ , & \text{if } 0 \leq u \leq t \\ x_t \ , & \text{if } t \leq u \leq s. \end{cases}$$



Bump:

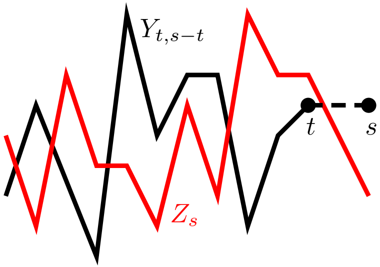
$$X_t^h(u) = \begin{cases} x_u \ , & \text{if } 0 \leq u < t \\ x_t + h \ , & \text{if } u = t. \end{cases}$$



Topology

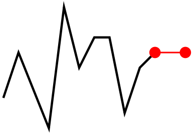
For any $X_t, Y_s \in \Lambda$, where we assume without loss of generality that $s \geq t$, define

$$d_\Lambda(X_t, Y_s) = \|X_{t,s-t} - Y_s\|_\infty + s - t.$$



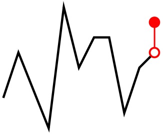
Time Derivative:

$$\Delta_t f(X_t) = \lim_{\Delta t \to 0^+} \frac{f(X_{t,\Delta t}) - f(X_t)}{\Delta t}$$



Space Derivative:

$$\Delta_x f(X_t) = \lim_{h \to 0} \frac{f(X_t^h) - f(X_t)}{h}.$$



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Definition (Λ-Continuity)

As usual, a functional $f : \Lambda \longrightarrow \mathbb{R}$ is said Λ-*continuous* if it continuous with respect to the metric d_Λ.

Whenever it is defined, we will write $\Delta_{xx} f = \Delta_x(\Delta_x f)$.

Definition (Smooth Functional)

We will call a functional $f : \Lambda \longrightarrow \mathbb{R}$ *smooth* if it is Λ-continuous and it has Λ-continuous functional derivatives $\Delta_t f$, $\Delta_x f$ and $\Delta_{xx} f$.

$f(X_t) = h(t, x_t)$.

- f is Λ-continuous if and only if h is continuous.
- Time Derivative:

$$\begin{aligned}\Delta_t f(X_t) &= \lim_{\Delta t \to 0^+} \frac{h(t+\Delta t, X_{t,\Delta t}(t+\Delta t)) - h(t, x_t)}{\Delta t} = \\ &= \lim_{\Delta t \to 0^+} \frac{h(t+\Delta t, x_t) - h(t, x_t)}{\Delta t} = \\ &= \frac{\partial h}{\partial t}(t, x_t).\end{aligned}$$

- Space Derivative:

$$\Delta_x f(X_t) = \lim_{h \to 0} \frac{h(t, x_t + h) - h(t, x_t)}{h} = \frac{\partial h}{\partial x}(t, x_t).$$

Examples

$$f(X_t) = \int_0^t x_u du.$$

- Time Derivative:

$$\Delta_t f(X_t) = \lim_{\Delta t \to 0^+} \frac{\int_0^{t+\Delta t} X_{t,\Delta t}(u)du - \int_0^t x_u du}{\Delta t}$$
$$= \lim_{\Delta t \to 0^+} \frac{\int_t^{t+\Delta t} x_t du}{\Delta t} = x_t.$$

- Space Derivative:

$$\Delta_x f(X_t) = \lim_{h \to 0} \frac{\int_0^t X_t^h(u)du - \int_0^t x_u du}{h} = 0.$$

Functional Itô Formula

Theorem

Let x be a semi-martingale and f be a smooth functional. Then, for any $t \in [0, T]$, we have

$$f(X_t) = f(X_0) + \int_0^t \Delta_t f(X_s) ds + \int_0^t \Delta_x f(X_s) dx_s + \frac{1}{2} \int_0^t \Delta_{xx} f(X_s) d\langle x \rangle_s.$$

Pricing PPDE

- Local vol model: $dx_t = rx_t dt + \sigma(X_t) dw_t$.

- Derivative: maturity T and payoff functional $g : \Lambda_T \longrightarrow \mathbb{R}$.

- The no-arbitrage price of this derivative is given by

$$f(Y_t) = e^{-r(T-t)} \mathbb{E}\left[g(X_T) \mid Y_t \right],$$

for any path $Y_t \in \Lambda$.

Pricing PPDE

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Theorem (Pricing PPDE)

If the f is a smooth functional, then, for any Y_t in the topological support of the process x,

$$\Delta_t f(Y_t) + \frac{1}{2}\sigma^2(Y_t)\Delta_{xx} f(Y_t) + r(y_t \Delta_x f(Y_t) - f(Y_t)) = 0,$$

with final condition $f(Y_T) = g(Y_T)$. Equations of this type are called Path-dependent PDE, or PPDE.

Under suitable assumptions on σ, the PPDE above will hold for any continuous path.

Adjoint of Δ_x

R. Cont & D.-A. Fournié (2013) "Functional Itô Calculus and Stochastic Integral Representation of Martingales", *The Annals of Probability* **41** (1), 109–133.

From now on, we consider the local volatility model $dx_t = \sigma(x_t)dw_t$ (so x is a martingale) and define

$$\mathcal{H}_x^2 = \left\{ f : \Lambda \to \mathbb{R} \; ; \; \mathbb{E}\left[\int_0^T f^2(X_t) d\langle x \rangle_t\right] < +\infty \right\},$$

$$\mathcal{M}_x^2 = \{ f : \Lambda \to \mathbb{R} \; ; \; f(X_t) \text{ is also a martingale}\},$$

$$\langle f, g \rangle_{\mathcal{H}_x^2} = \mathbb{E}\left[\int_0^T f(X_t) g(X_t) d\langle x \rangle_t\right]$$

$$\langle f, g \rangle_{\mathcal{M}_x^2} = \mathbb{E}\left[f(X_T) g(X_T)\right]$$

Adjoint of Δ_x

We write

$$\mathcal{I}_x(f)(t) = \int_0^t f(X_s)dx_s,$$

for the classical Itô integral of the process $(f(X_t))_{t\in[0,T]}$ with respect to the martingale $(x_t)_{t\in[0,T]}$.

First notice we can write Itô Isometry as

$$\langle f, g\rangle_{\mathcal{H}_x^2} = \langle \mathcal{I}_x(f), \mathcal{I}_x(g)\rangle_{\mathcal{M}_x^2}.$$

We then have the following result:

Proposition

The adjoint of $\Delta_x : \mathcal{M}_x^2 \longrightarrow \mathcal{H}_x^2$ is the Itô integral:

$$\langle \Delta_x f, g\rangle_{\mathcal{H}_x^2} = \langle f, \mathcal{I}_x(g)\rangle_{\mathcal{M}_x^2}.$$

Path-dependence and the Computation of Greeks

Samy Jazaerli and Yuri F. Saporito, "Functional Itô Calculus, Path-Dependence and the Computation of Greeks", *Submitted* (2013) Available at arXiv: http://arxiv.org/abs/1311.3881

- We will show that for a class of (path-dependent) payoffs, there exists a "weight" π such

$$\text{Greek} = \mathbb{E}[\text{ payoff } \times \pi].$$

- In this talk we will show the details only for the Delta (the sensitivity to the initial value of the asset).

- Similar computations can be performed for the Gamma and Vega.

- An application of Malliavin Calculus to this problem can be found in the important paper:

  E. Fournié *et al*, "Applications of Malliavin calculus to Monte Carlo methods in finance", *Finance and Stochastics* **5** (1999) 201–236.

- Suppose the underlying asset is described by the local volatility model

$$dx_t = \sigma(x_t) dw_t.$$

- For fixed $0 < t_1 < \cdots < t_m = T$, consider the price

$$u(x) = \mathbb{E}[\phi(x_{t_1}, \ldots, x_{t_m}) \mid x_0 = x].$$

- Under mild conditions,

$$u'(x) = \mathbb{E}\left[\phi(x_{t_1}, \ldots, x_{t_m}) \int_0^T \frac{a(t) z_t}{\sigma(x_t)} dw_t \;\middle|\; x_0 = x \right],$$

where y is the tangent process $dz_t = \sigma'(x_t) z_t dw_t$ and

$$a \in \Gamma_m = \left\{ a \in L^2([0, T]) \; ; \; \int_0^{t_i} a(t) dt = 1, \; \forall \; i = 1, \ldots, m \right\}.$$

Malliavin Calculus Approach

- Regarding the Malliavin Calculus, the payoffs of the form $\phi(x_{t_1}, \ldots, x_{t_m})$ seem arbitrary (but it is what is used in practice).

- Similar formulas for other path-dependent derivatives can be found.

- The set Γ_m degenerates if $t_1 = 0$.

Goals

- Introduce a measure of path-dependence for functionals.
- Understand the assumption that the payoff depends only on $x_{t_1}, \ldots, x_{t_m}$.
- Derive these weights using the framework of functional Itô calculus.

Delta

- The Delta of a derivative contract is the sensitivity of its price with respect to the current value of the underlying asset: $\Delta_x f(X_0)$.

- Define the tangent process: $dz_t = \sigma'(x_t) z_t dw_t$, with $z_0 = 1$.

- Applying Δ_x to the functional PDE gives us

$$\Delta_{tx} f(X_t) + \sigma(x_t)\sigma'(x_t)\Delta_{xx} f(X_t) + \frac{1}{2}\sigma^2(x_t)\Delta_{xxx} f(X_t) = 0,$$

where $\Delta_{tx} = \Delta_x \Delta_t$.

- Hence

$$d(\Delta_x f(X_t) z_t) = (\Delta_{xt} f(X_t) - \Delta_{tx} f(X_t)) z_t dt + dm_t.$$

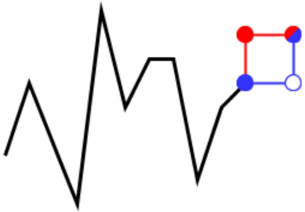
Lie Bracket

Definition (Lie Bracket)

The *Lie bracket* of the operators Δ_t and Δ_x will play a fundamental role in what follows. It is defined as

$$[\Delta_x, \Delta_t]f(Y_t) = \Delta_{tx}f(Y_t) - \Delta_{xt}f(Y_t),$$

where $\Delta_{tx} = \Delta_x\Delta_t$.



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Definition (Weak Path-Dependence)

A functional $f : \Lambda \longrightarrow \mathbb{R}$ is called *weakly path-dependent* if

$$[\Delta_x, \Delta_t]f = 0.$$

Lie Bracket

- $f(X_t) = \int_0^t x_s ds$ is not weakly path-dependent: $[\Delta_x, \Delta_t]f = 1$.

- $f(X_t) = h(t, x_t)$ is weakly path-dependent.

- $f(X_t) = \int_0^t \int_0^s x_u duds$ is weakly path-dependent.

- $f(X_t) = \langle x \rangle_t$ is weakly path-dependent at continuous paths.

- $f(X_t) = \mathbb{E}[\phi(x_{t_1}, \ldots, x_{t_m}) \mid X_t]$ has zero Lie bracket but for $t_1, \ldots, t_m$.

Delta

Remember that

$$d(\Delta_x f(X_t) z_t) = -[\Delta_x, \Delta_t] f(X_t) z_t dt + dm_t.$$

Hence, $[\Delta_x, \Delta_t] f = 0$ implies

$$\begin{aligned}
\Delta_x f(X_t) z_t = \Delta_x f(X_0) + m_t \Rightarrow \int_0^T \Delta_x f(X_t) z_t dt &= \Delta_x f(X_0) T + \int_0^T m_t dt \\
\Rightarrow \Delta_x f(X_0) T &= \mathbb{E}\left[\int_0^T \Delta_x f(X_t) z_t dt\right] \\
\Rightarrow \Delta_x f(X_0) &= \left\langle \Delta_x f(X), \frac{1}{T} \frac{z}{\sigma^2(x)} \right\rangle_{\mathcal{H}_x^2} \\
\Rightarrow \Delta_x f(X_0) &= \left\langle f(X_t), \mathcal{I}_x \left(\frac{1}{T} \frac{z_t}{\sigma^2(x)} \right) \right\rangle_{\mathcal{M}_x^2} \\
\Rightarrow \Delta_x f(X_0) &= \mathbb{E}\left[g(X_T) \frac{1}{T} \int_0^T \frac{z_t}{\sigma(x_t)} dw_t \right]
\end{aligned}$$

Delta

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Theorem

Consider a path-dependent derivative with maturity T and contract $g : \Lambda_T \longrightarrow \mathbb{R}$. So, if the price of this derivative satisfy certain smoothness conditions and it is weakly path-dependent, then $\Delta_x f(X_t) z_t$ is a martingale and the following formula for the Delta is valid:

$$\Delta_x f(X_0) = \mathbb{E}\left[g(X_T)\frac{1}{T}\int_0^T \frac{z_t}{\sigma(x_t)} dw_t\right].$$

- This is the same formula found by Fournié *et al* using Malliavin calculus when $g(X_T) = \phi(x_T)$.
- Our proof can be adapted to deal with payoffs $\phi(x_{t_1}, \ldots, x_{t_m})$ and Fournié *et al*'s formula is derived.

Strong Path-Dependence

How would these formulas change if f is strongly path-dependent?

$$\Delta_x f(X_0) = \Delta_x f(X_t) z_t + \int_0^t [\Delta_x, \Delta_t] f(X_s) z_s ds - m_t.$$

One can show then

$$\Delta_x f(X_0) = \mathbb{E}\left[g(X_T) \frac{1}{T} \int_0^T \frac{z_t}{\sigma(x_t)} dw_t \right]$$

$$+ \mathbb{E}\left[\frac{1}{T} \int_0^T \int_0^t [\Delta_x, \Delta_t] f(X_s) z_s ds dt \right].$$

- For the second term, one should study the adjoint and/or an integration by parts for Δ_t and Δ_x in $\mathcal{H}^2_x$.

The Functional Meyer-Tanaka Formula

The Meyer-Tanaka Formula

Classical versions of *the Meyer-Tanaka Formula*:

- $$f(x_t) = f(x_0) + \int_0^t f_x(x_s)dx_s + \int_{\mathbb{R}} L^x(t,y)d_y f_x(y),$$

- $$f(t,x_t) = f(0,x_0) + \int_0^t f_t(s,x_s)ds + \int_0^t f_x(s,x_s)dx_s$$
$$+ \int_{\mathbb{R}} L^x(t,y)d_y f_x(t,y) - \int_{\mathbb{R}}\int_0^t L^x(s,y)d_{s,y} f_x(s,y),$$

where $L^x(t,y)$ is the local time of the process x at y:

$$L^x(t,y) = \lim_{\varepsilon \to 0^+} \frac{1}{4\varepsilon} \int_0^t 1_{[y-\varepsilon, y+\varepsilon]}(x_s)d\langle x\rangle_s.$$

Convex Functionals

For a functional $f : \Lambda \longrightarrow \mathbb{R}$, we define $F : \Lambda \times \mathbb{R} \longrightarrow \mathbb{R}$ as

$$F(Y_t, h) = f(Y_t^h).$$

Definition (Convex Functionals)

We say f is a *convex functional* if $F(Y_t, \cdot)$ is a convex real function for any $Y_t \in \Lambda$.

Example: The *running maximum* is a simple example of a (non-smooth) convex functional

$$m(Y_t) = \sup_{0 \leq s \leq t} y_s.$$

Definition (Mollified Functionals)

For a given functional f, we define the sequence of *mollified functionals* as

$$F_n(Y_t, h) = \int_{\mathbb{R}} \rho_n(h-\xi)F(Y_t,\xi)d\xi = \int_{\mathbb{R}} \rho_n(\xi)F(Y_t, h-\xi)d\xi.$$

Remark: The mollifier can be taken as

$$\rho(z) = c \ \exp\left\{\frac{1}{(z-1)^2-1}\right\} 1_{[0,2]}(z),$$

where c is chosen in order to have $\int_{\mathbb{R}} \rho(z)dz = 1$.

The Functional Meyer-Tanaka Formula

Theorem (Functional Meyer-Tanaka Formula)

Consider a convex functional $f : \Lambda \longrightarrow \mathbb{R}$ *satisfying some regularity assumptions. Then*

$$f(X_t) = f(X_0) + \int_0^t \Delta_t f(X_s) ds + \int_0^t \Delta_x^- f(X_s) dx_s + \int_{\mathbb{R}} L^x(t,y) d_y \partial_y^- \mathcal{F}(X_t, y) - \int_0^t \int_{\mathbb{R}} L^x(s,y) d_{s,y} \partial_y^- \mathcal{F}(X_s, y).$$

Yuri F. Saporito (2014) "Functional Meyer-Tanaka Formula", *Submitted*.

Thank you!