Time-Inconsistent Stopping Problems

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Joint work with
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Outline

Motivation
- What is time inconsistency? Why do we have it?

Methodology
- Game-theoretic approach

Application
- Probability Distortion

Extension
- Non-exponential Discounting

Classical Optimal Stopping

Consider

- a continuous Markovian process $X : [0, \infty) \times \Omega \mapsto \mathbb{R}^d$.
- a payoff function $u : \mathbb{R}^d \mapsto \mathbb{R}_+$.

Optimal Stopping

$$\sup_{\tau \in \mathcal{T}} \mathbb{E}_x[u(X_\tau)] = \underset{\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim}{\mathbb{E}_x[u(X_{\tilde{\tau}_x})]}$$

- $\mathcal{T}$: set of stopping times.

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- $\mathcal{T}$: set of stopping times.
- Does $\tilde{\tau}_x \in \mathcal{T}$ exist?
  - Dynamic programming (free boundary problems)
  - martingale method (Snell envelope)

Probability Distortion

- Standard formulation:

$$\mathbb{E}_x[u(X_\tau)] = \int_0^\infty \mathbb{P}[u(X_\tau^x) > y]dy$$

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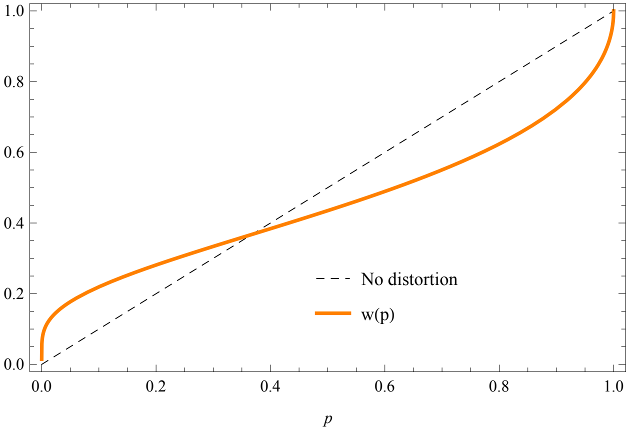
- $w : [0,1] \mapsto [0,1]$ is called a *probability weighting function*
  - w is continuous, increasing;
  - $w(0) = 0$ and $w(1) = 1$.

Reverse S-shaped w



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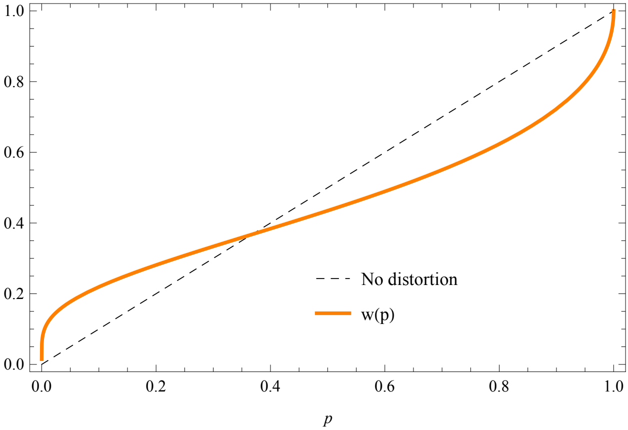
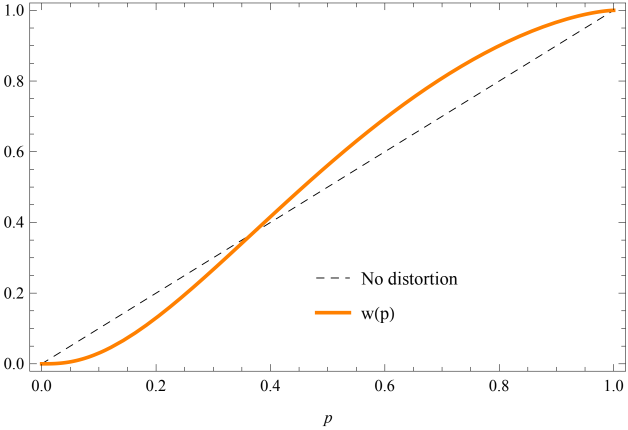


- *exaggerate* prob. of **"very good state"** *(Hope, Greed)*
- *exaggerate* prob. of **"very bad state"** *(Fear)*

S-SHAPED w



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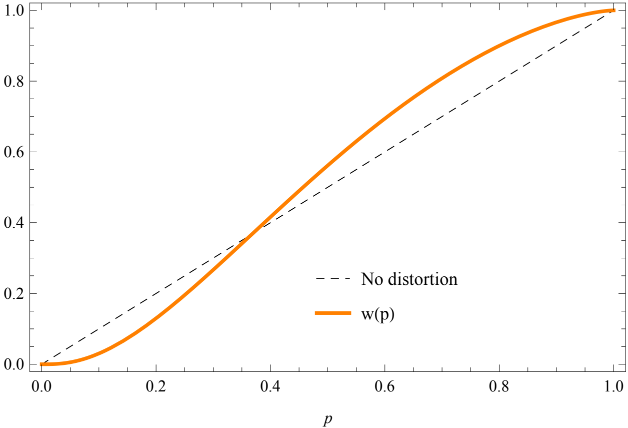

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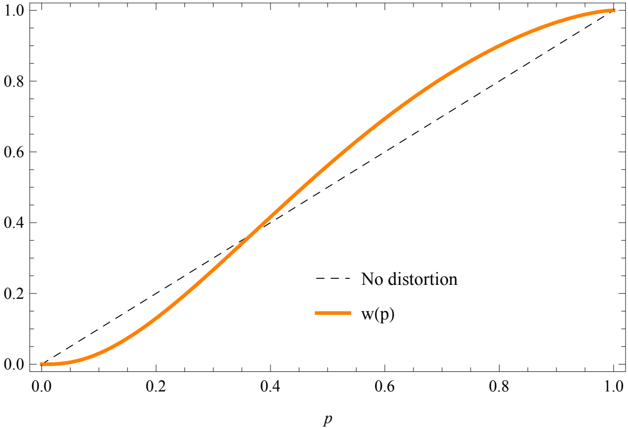

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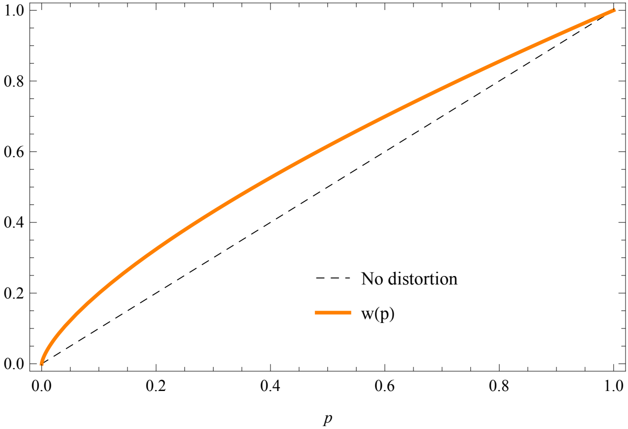
$\Longrightarrow$ *believes "the asset is stable"*

Concave w



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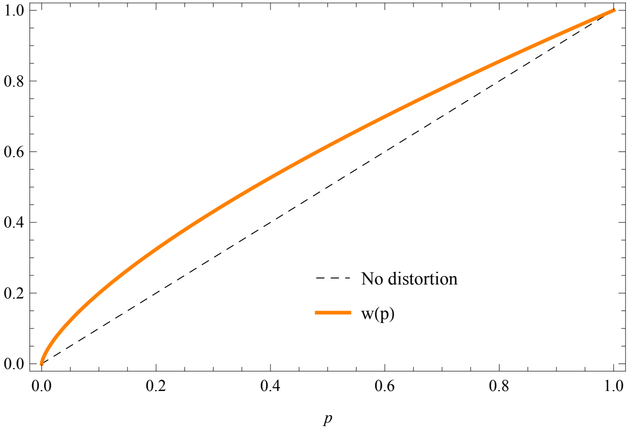

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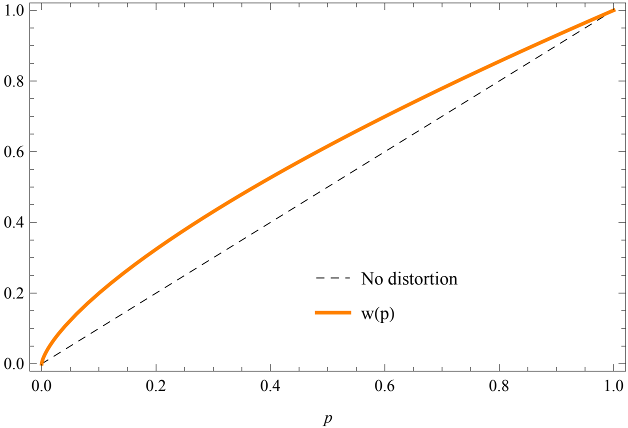

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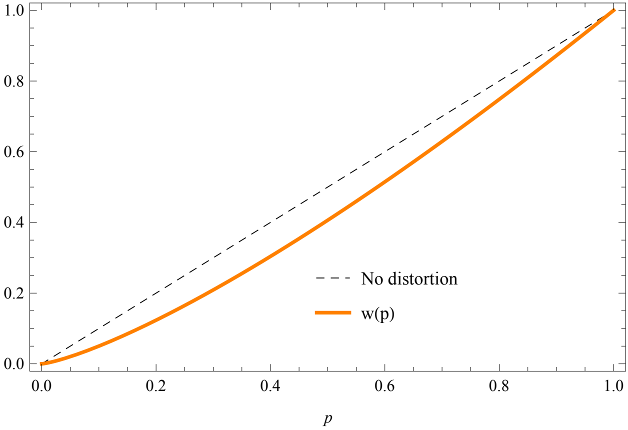

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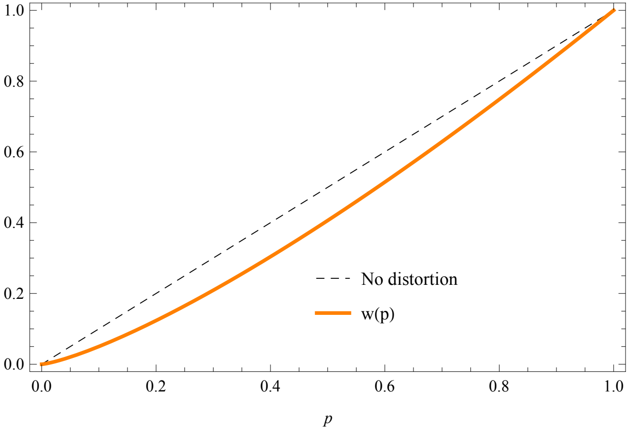

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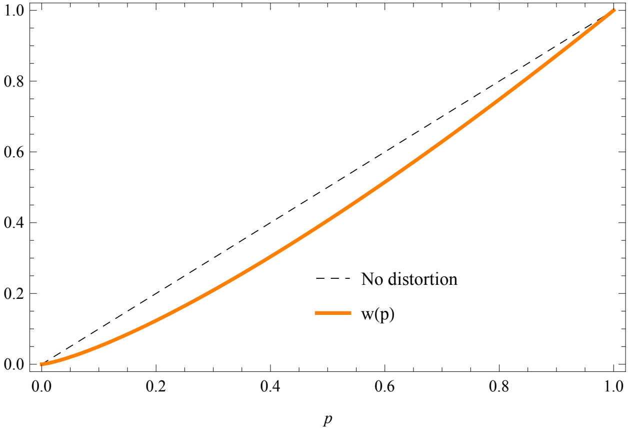

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  - **Xu & Zhou (2013)** characterized $\widetilde{\tau}_x \in \mathcal{T}$ using distribution/quantile formulation.

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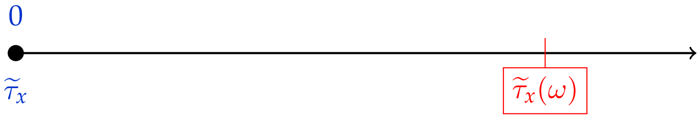

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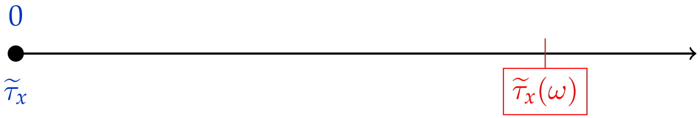

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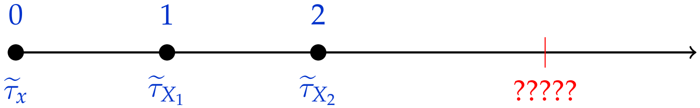

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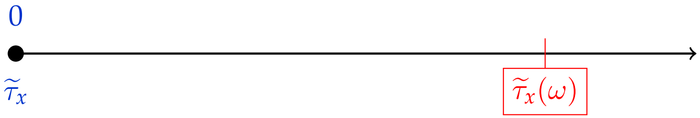

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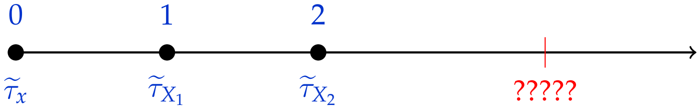

- **Time Inconsistency:**
  - $\widetilde{\tau}_x$, $\widetilde{\tau}_{X_1}$, $\widetilde{\tau}_{X_2}$ may all be different.
  - Is it reasonable to apply $\widetilde{\tau}_x$ at time t?

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Under the condition

$$\frac{\mu}{\sigma^2} \in \left(\frac{\gamma}{2}, \frac{1}{2}\right),$$

- optimal stopping time:

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- time inconsistency $\Longrightarrow$ procrastination ("never stop"!!)

General Optimal Stopping

$$\sup_{\tau \in \mathcal{T}} J(x;\tau),$$

Assumption: $J : \mathbb{R}^d \times \mathcal{T} \mapsto \mathbb{R}$ satisfies

1) $J(x;0) = u(x)$;
2) $J(x;\tau_n) \to J(x;\tau)$ if $\tau_n \downarrow \tau$ a.s.;
3) With $D \in \mathcal{B}(\mathbb{R}^d)$ and T_D the first hitting time of D,

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- Expected payoff: $J(x; \tau) := \mathbb{E}_x[u(X_\tau)]$.
- Probability Distortion:

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Stopping Policies

Definition

A *Borel* function $\tau : \mathbb{R}^d \mapsto \{0, 1\}$ is called a **stopping policy**.

$$\tau(x) = 0 \implies \text{stop}$$
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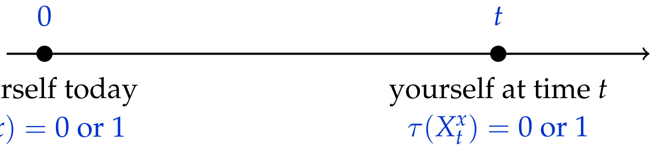

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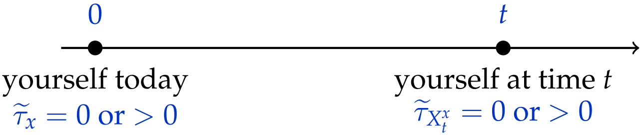
- When eventually will we stop?

$$\mathcal{L}\tau(x) := \inf\{t \geq 0 \ : \ \tau(X_t^x) = 0\} \in \mathcal{T}$$

A *naive agent* simply follows $\widetilde{\tau}_x$ over time:



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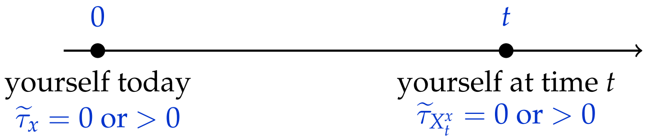

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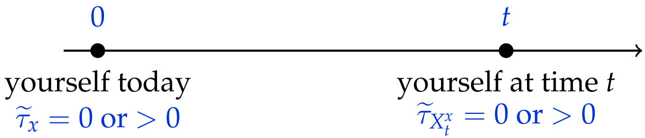

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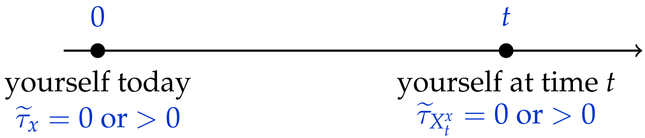

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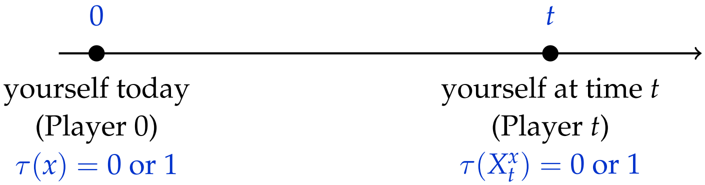

Game-Theoretic Approach

- Given a stopping policy $\tau : \mathbb{R}^d \mapsto \{0, 1\}$,

yourself today (Player 0) $\tau(x) = 0$ or 1

yourself at time t (Player t) $\tau(X_t^x) = 0$ or 1

- **Game-theoretic** thinking of Player 0:
  Given that each Player t will follow τ,
  - what is the best stopping strategy at time 0?
  - can it just be $\tau(x)$?

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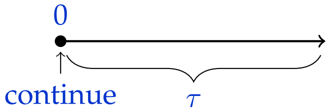

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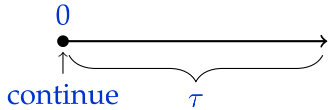

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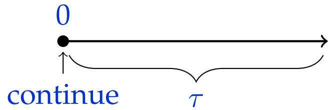

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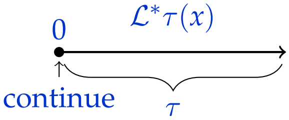

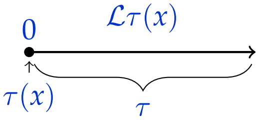

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III. $u(x) = J(x; \mathcal{L}^*\tau(x)) \Rightarrow$

- **indifferent** between to stop and to continue at time 0.
- no incentive to deviate from $\tau(x)$

- Summarize the best stopping strategy for Player 0 as

$$\Theta\tau(x) := \begin{cases} 0, & \text{if } x \in S_\tau; \\ 1, & \text{if } x \in C_\tau; \\ \tau(x), & \text{if } x \in I_\tau; \end{cases}$$

where

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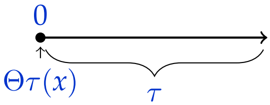

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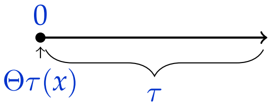

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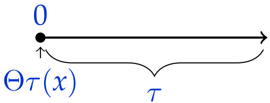

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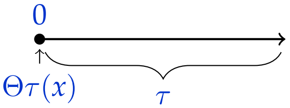

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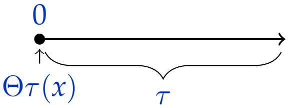

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2. Now, one follows $\Theta\tau$.

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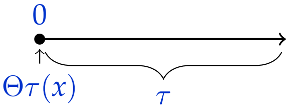

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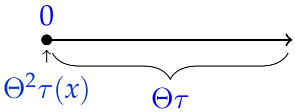

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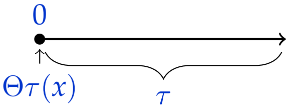

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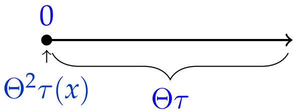

$\Longrightarrow$ switch from $\Theta\tau$ to $\Theta^2\tau$

3. Continue this procedure *until* we reach

$$\tau_*(x) := \lim_{n\to\infty} \Theta^n \tau(x)$$

Expect: $\Theta\tau_*(x) = \tau_*(x)$, i.e. cannot improve anymore.

Equilibrium Policies

Definition

A stopping policy τ is called an **equilibrium policy** if

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- **In general**, given a stopping policy τ, carry out iteration:

$$\tau \longrightarrow \Theta\tau \longrightarrow \Theta^2\tau \longrightarrow \cdots \longrightarrow \text{"equilibrium"??}$$

- **To show:** (i) $\tau_* := \lim_{n\to\infty} \Theta^n\tau$ converges (ii) $\Theta\tau_* = \tau_*$.

One-Dimensional Case

Consider a 1-D diffusion

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Main Result

Suppose Z is a martingale. Then, for any stopping policy τ,

$$\tau_*(x) := \lim_{n\to\infty} \Theta^n \tau(x) \quad \text{converges}, \quad \forall x \in \mathbb{R}.$$

Moreover, τ_* is an *equilibrium policy*, i.e.

$$\Theta\tau_*(x) = \tau_*(x), \quad \forall x \in \mathbb{R}.$$

Literature

Strotz (1955): 3 different reactions to time inconsistency

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Question: How to formulate sophisticated strategies in continuous time ?

Unclear in the literature...

Literature

- Ekeland & Lazrak (2006): **Subgame perfect Nash equilibriums** emerge as the proper formulation for sophisticated strategies, for **control problems**.

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- Extending the equilibrium idea to **stopping problems**:

  difficult, **unresolved**.

  Xu & Zhou (2013), Barberis (2002), Grenadier & Wang (2007).

From "Naive" to "Sophosticated"

$$\widetilde{\tau}_* = \lim_{n \to \infty} \Theta^n \widetilde{\tau}$$

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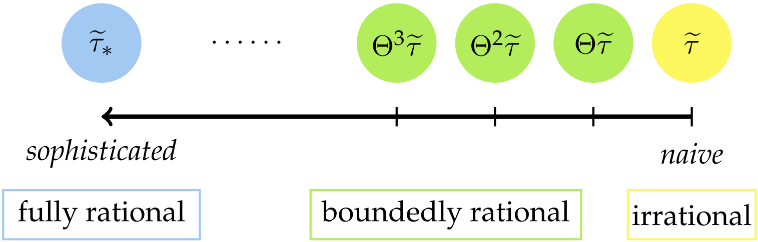

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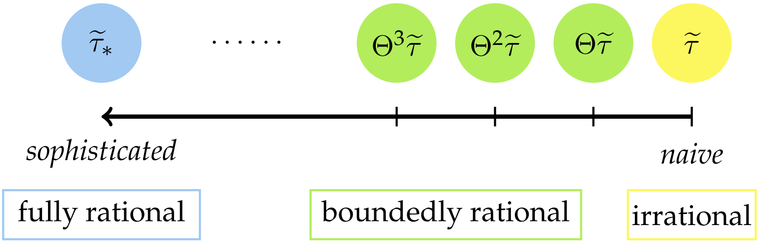

- **Bounded Rationality** proposed by H. Simon (1982).
- This connection is new in the literature.

Probability Distortion

Follow the setup in **Xu & Zhou (2013)**:

- Price process:

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  - Utility (Payoff) function $U : \mathbb{R}_+ \mapsto \mathbb{R}_+$:

$$\text{nondecreasing, continuous, } U(0) = 0.$$

  - Prob. weighting function $w : [0,1] \mapsto [0,1]$:

$$\text{increasing, continuous, } w(0) = 0, w(1) = 1.$$

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- $\beta > 0$
  - The asset is just "average", or even "bad".
  - Time inconsistency arises!
  - The naive policy $\tilde{\tau}$ may not be an equilibrium.

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For the case $\beta > 0$,

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- Define $u(x) := U(x^{1/\beta})$
  - u is nondecreasing, and $u(0) = 0$
  - $$J(x;\tau) = \int_0^{\infty} w\left(\mathbb{P}_x[u(X_\tau) > y]\right) dy.$$

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For a *completely rational* agent,

- $w(p) = p$ (no prob. distortion), i.e. $J(x;\tau) = \mathbb{E}[u(X_\tau^x)]$.

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For an *partially optimistic* agent,

- w is reverse S-shaped
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Example (concave u, concave w)

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- Want to find

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- First iteration $\Theta\widetilde{\tau}$:

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$$= \int_0^\infty w\left(\mathbb{P}[0 > y]\right) = 0 < u(x), \quad \forall x \in \mathbb{R}_+.$$

$$\implies S_{\widetilde{\tau}} = \mathbb{R}_+ \implies \Theta\widetilde{\tau}(x) = 0 \text{ for all } x.$$

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- **Conclude:**

$$\tilde{\tau}_*(x) \;=\; \lim_{n\to\infty} \Theta^n\tilde{\tau}(x) = \Theta\tilde{\tau}(x) \equiv 0.$$

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- **Conclude:**

$$\widetilde{\tau}_*(x) = \lim_{n\to\infty} \Theta^n\widetilde{\tau}(x) = \Theta\widetilde{\tau}(x) \equiv 0.$$

- This coincides with completely rational behavior!

Proposition

Suppose u is strictly concave, and w satisfies either (i) or (ii):

(i) w is concave;

(ii) w is reverse S-shaped and $w'(0+) = \infty$.

Then,

$$\widetilde{\tau}(x) = 1, \quad \forall x \in \mathbb{R}_+$$
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Implications:
A *sophisticated* agent may behave as a *completely rational* one.

Reverse S-shaped w

Three main forms:

- **Tversky & Kahneman (1992):**

$$w(x) = \frac{x^\gamma}{(x^\gamma + (1-x)^\gamma)^{1/\gamma}}, \quad 0.279... \leq \gamma < 1$$

- **Goldstein & Einhorn (1987):**

$$w(x) = \frac{\alpha x^\gamma}{\alpha x^\gamma + (1-x)^\gamma}, \quad \alpha > 0, \gamma \in (0,1).$$

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Common property: $w'(0+) = \infty.$

Non-Exponential Discounting

Optimal Stopping

$$\sup_{\tau \in \mathcal{T}} \mathbb{E}_{t,x}[\delta(\tau - t)u(X_\tau)]$$

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- $\delta : \mathbb{R}_+ \mapsto [0, 1]$ is decreasing with $\delta(0) = 1$
  - If $\delta(t, s) := e^{-\alpha(s-t)}$, *time-consistent*
  - If $\delta(t, s)$ is of non-exponential form, *time-inconsistent*

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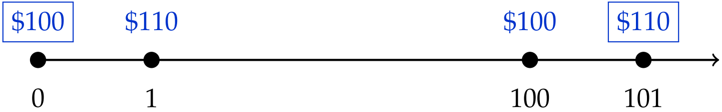

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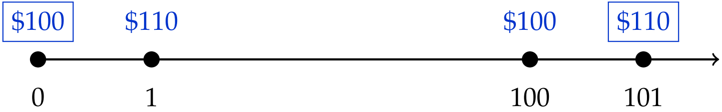

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$\Rightarrow$ Does not capture "decreasing impatience".

- A *Borel* map $\tau : \mathbb{R}_+ \times \mathbb{R}^d \mapsto \{0, 1\}$ is a **stopping policy**.

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- If future selves will follow τ, the best stopping strategy for Player t is

$$\Theta\tau(t,x) := \begin{cases} 0, & \text{if } x \in S_\tau; \\ 1, & \text{if } x \in C_\tau; \\ \tau(t,x), & \text{if } x \in I_\tau; \end{cases}$$

where

$$S_\tau := \{(t,x) : u(x) > \mathbb{E}_{t,x}\left[\delta(\mathcal{L}^*\tau(t,x) - t)u\left(X_{\mathcal{L}^*\tau(t,x)}\right)\right]\},$$
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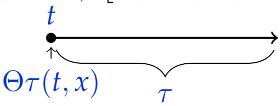

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$\Longrightarrow \ C_\tau = [0,\infty) \times \mathbb{R}^d$, and thus $\Theta\tau(t,x) = \tau(t,x), \ \forall (t,x)$.

Equilibrium Policies

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- **In general**, given a stopping policy τ, carry out iteration:

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- **To show:** (i) $\tau_* := \lim_{n\to\infty} \Theta^n\tau$ converges (ii) $\Theta\tau_* = \tau_*$.

Decreasing Impatience

- **Assumption:** the discount function $\delta : \mathbb{R}_+ \mapsto [0,1]$ satisfies

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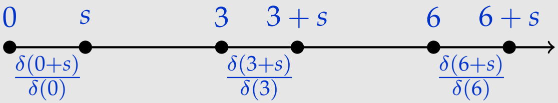

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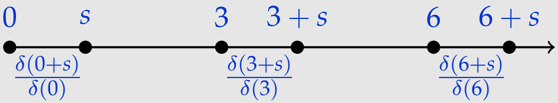

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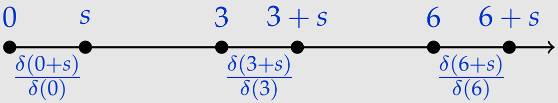

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- Once we consider **DI**, (1) is automatically satisfied.

Main Result

Lemma

Assume (1). Let τ be a stopping policy. Then,

$$\begin{aligned} \textit{if} \quad & \boxed{\Theta\tau(t,x) \le \tau(t,x) \;\; a.s. \quad \forall (t,x)}, \\ \textit{then} \quad & \Theta^{n+1}\tau(t,x) \le \Theta^{n}\tau(t,x) \;\; a.s. \quad \forall (t,x) \text{ and } n. \end{aligned} \tag{2}$$

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Theorem

Assume (1) *and* (2). *Then, for any* (t,x),

$$\tau_*(t,x) := \downarrow \lim_{n\to\infty} \Theta^n\tau(t,x) \quad \text{converges}.$$

Moreover, τ_* *is an equilibrium policy, i.e.*

$$\Theta\tau_*(t,x) = \tau_*(t,x) \quad \forall (t,x).$$

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  - Hyperbolic discounting:

$$\delta(s) = \frac{1}{1+s} \quad \forall s \geq 0.$$

- **Classical Theory:** For each $t \in [0, T]$,

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  - **time inconsistency $\Longrightarrow$ procrastination**

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  - Since $e^{\frac{1}{2}s} = 1 + s$ at $s = 0$ and $s^* \approx 2.513$,

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- **Conclude:**

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- **τ_0 says "Stop Smoking Immediately!!"**
(unless you're too old...)

Example (BES(1))

- X_t : one-dimensional Brownian motion
- Hyperbolic discount function

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We can characterize the whole set $\mathcal{E}$ of equilibrium policies.

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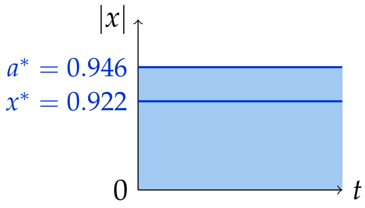

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- **Pareto efficiency:**
  How to formulate this under current setting?

THANK YOU!!

Preprint available @ arXiv:1502.03998
"Time-consistent stopping under decreasing impatience"
(H. and Nguyen-Huu)

First draft in preparation
"Time-consistent stopping under probability distortion"
(H., Nguyen-Huu, and Zhou)