Optimal investment strategies and intergenerational risk sharing for target benefit pension plans

Yi Lu

Department of Statistics and Actuarial Science
Simon Fraser University

(Joint work with Suxin Wang and Barbara Sanders)

Mathematical Finance Colloquium
Department of Mathematics
University of Southern California

September 18, 2017

Outline

1. Introduction
2. TBP model, control problem and solutions
   - Model formulation
   - Optimal control problem
   - Solutions
3. Numerical illustrations
4. Conclusion

Sources of retirement income in Canada

- Canada Pension Plan (CPP)
- Old Age Security (OAS) pension
- Employer-sponsored retirement and pension plans
  - Defined benefit (DB) pension plans
  - Defined contribution (DC) pension plans
  - Group Registered Retirement Savings Plans (RRSP)
  - Pooled registered pension plans
- Converting your savings into income
  - Registered Retirement Income Fund (RRIF)
  - Annuities (term-certain or life)
  - Cash
- Getting money from your home

Source: https://www.canada.ca/en/financial-consumer-agency/services/retirement-planning/sources-retirement-income.html

DC and DB Plans

Defined Contribution (DC) pension plan

- Predefined contribution level (employee and/or employer)
- Sponsor liability limited to contributions
- Benefit levels depending on investment preference

Defined Benefit (DB) pension plan

- Predefined lifetime retirement benefits
- Contributions from both employer and employee
- Collective investment fund
- Mortality risk pooled among members

Registered Pension Plans and Members in Canada, by Type of Plan, 1992 and 2014

Type of Plan	Variable	1992	2014	Difference (%)
Defined Benefit	Plan	7,870	10,414	32.3
	Members	4,775,543	4,401,970	-7.8
Defined Contribution	Plan	9,901	6,511	-34.2
	Members	469,144	1,036,747	121.0
Others	Plan	257	832	223.7
	Members	73,403	746,442	916.9
Total	Plan	18,028	17,757	-1.5
	Members	5,318,090	6,185,159	16.3

Source: Raphalle Deraspe and Lindsay McGlashan (2016). The Target Benefit Plan: An Emerging Pension Regime. No. 2016-20-E, Library of Parliament. https://lop.parl.ca/Content/LOP/ResearchPublications/2016-20-e.pdf

Target Benefit Plans (TBPs)

- An emerging pension regime in Canada; TBP regimes in New Brunswick, Alberta, and British Columbia
- Collective Pension Scheme (CPS) with "fixed" contributions
- Target benefit amounts modified according to affordability and plan's investment performance
- Intergenerational Risk Sharing (IRS): investment and longevity risks

References:

1. Jana Steele (2016). Target Benefit Plans in Canada. Estates, Trusts & Pensions Journal, Vol. 36.
2. Raphalle Deraspe and Lindsay McGlashan (2016). The Target Benefit Plan: An Emerging Pension Regime. No. 2016-20-E, Library of Parliament.

Target Benefit Plans

Literature on CPS/IRS

- Cui et al. (2011) and Gollier (2008) estimated welfare gains from IRS within a funded CPS; welfare is improved comparing to DB/DC plans.
- Westerhout (2011) and Van Bommel (2007) pointed out that it is critical that IRS be implemented with a view to fairness.
- Boelaars (2016) compared welfare gains from IRS in funded collective pension schemes with individual retirement accounts.
- CIA (2015) provided a report of the task force on Canadian TBPs.

Target Benefit Plans

Practical objectives of a TBP

- Provide adequate benefits
- Maintain stability
- Respect intergenerational equity

Our work

- Considered a continuous-time stochastic optimal control problem for the TBP on asset allocation and benefit distribution
- Proposed an objective function which balances three practical objectives regarding benefit risks and discontinuity risk
- Obtained optimal asset allocation policy and benefit adjustment policy

Optimal Control problems

Literature review

- DC plans: focused on optimal investment allocation and income drawdown strategies (Gerrard et al., 2004; He and Liang, 2013, 2015)
- DB plans: concerned with optimal asset allocation and contribution policies (Boulier et al, 1995; Josa-Fombellida and Rincón-Zapatero, 2004, 2008; Ngwira and Gerrard, 2007)
- TBP-like plans: explored rules to reduce discontinuity risk (Gollier, 2008) and studied risk sharing between generations for a variety of realistic CPSs (Cui et al., 2011)
- Others: studied optimal portfolio problems (Haberman and Sung, 1994; Battocchio and Menoncin, 2004; Josa-Fombellida and Rincón-Zapatero, 2001)

Dynamics of financial market

- Risk-free asset $S_0(t)$

$$\mathrm{d}S_0(t) = r_0 S_0(t)\mathrm{d}t, \quad t \geq 0,$$

where r_0 represents the risk-free interest rate.

- Risky asset $S_1(t)$

$$\mathrm{d}S_1(t) = S_1(t)[\mu \mathrm{d}t + \sigma \mathrm{d}W(t)], \quad t \geq 0,$$

where μ is the appreciation rate of the stock, σ is the volatility rate, and $W(t)$ is a standard Brownian motion.

Membership provision

- Fundamental elements in a TBP model:

  $n(t)$: density of new entrants aged a at time t,
  $s(x)$: survival function with $s(a) = 1$ and $a \leq x \leq \omega$.

- Density of those who attain age x at time t is

$$n(t - (x - a))s(x), \quad x > a.$$

Dynamics of salary rates

- We assume that the annual salary rate for a member who retires at time t satisfies

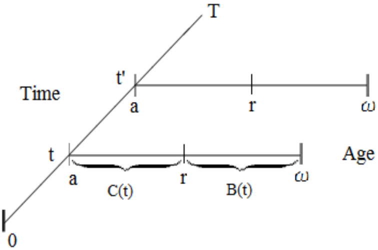
$$\mathrm{d}L(t) = L(t)\left(\alpha \mathrm{d}t + \eta \mathrm{d}\overline{W}(t)\right), \quad t \geq 0,$$

where $\alpha \in \mathbb{R}^+$ and $\eta \in \mathbb{R}$. $\overline{W}$ is a standard Brownian motion correlated with W, such that $E[W(t)\overline{W}(t)] = \rho t$.

- For a retiree age x at time t $(x \geq r)$, we define his assumed salary at retirement $(x - r$ years ago) as

$$\widetilde{L}(x,t) = L(t)e^{-\alpha(x-r)}, \quad t \geq 0,\ x \geq r.$$

Plan Provision: time-age structure



Plan Provision: benefit payments

- Individual pension payment rate at time t for those aged x:

$$B(x,t) = f(t)\widetilde{L}(x,t)e^{\zeta(x-r)} = f(t)L(t)e^{-(\alpha-\zeta)(x-r)}, \quad x \geq r.$$

where $e^{\zeta(x-r)}$ represents the cost-of-living adjustments, and $f(t)$ is the benefit adjustment variable at time t.

- Aggregate pension benefit rate for all the retirees at time t:

$$B(t) = \int_r^{\omega} n(t-x+a)s(x)B(x,t)\mathrm{d}x = I(t)f(t)L(t), \quad t \geq 0.$$

- B^* is a pre-set aggregate retirement benefit target at time 0 and updated aggregate benefit target at time t is $B^*e^{\beta t}$, where β can be viewed as a inflation related growth rate.

Plan Provision: contributions

- Individual contribution rate for an active member aged x at time t:

$$C(x,t) = c_0 e^{\alpha t}, \quad a \le x < r,$$

where c_0 is the instantaneous contribution rate at time 0 in respect of each active member, expressed as a dollar amount per year.

- Aggregate contribution rate in respect of all active members at time t:

$$C(t) = \int_a^r n(t-x+a)s(x)C(x,t)\mathrm{d}x = C_1(t) \cdot e^{\alpha t}, \quad t \ge 0.$$

Pension Fund Dynamic

Let $X(t)$ be the wealth of the pension fund at time t after adopting the investment strategy $\pi(t)$.

The pension fund dynamic can be described as

$$\begin{cases} \mathrm{d}X(t) = \pi(t)\frac{\mathrm{d}S_1(t)}{S_1(t)} + (X(t) - \pi(t))\frac{\mathrm{d}S_0(t)}{S_0(t)} + (C(t) - B(t))\mathrm{d}t, \\ X(0) = x_0, \end{cases}$$

where $\pi(t)$ denotes the amount to be invested in the risky asset at time t.

The objective function

- Let $J(t,x,l)$ be the objective function at time t with the fund value and the salary level being x and l. It is defined as

$$\begin{cases} J(t,x,l) = E_{\pi,f}\Big\{ \int_t^T \Big[\left(B(s) - B^* e^{\beta s}\right)^2 - \lambda_1 \left(B(s) - B^* e^{\beta s}\right) \Big] e^{-r_0 s} \mathrm{d}s \\ \qquad\qquad + \lambda_2 \left(X(T) - x_0 e^{r_0 T}\right)^2 e^{-r_0 T} \Big\}, \\ J(T,x,l) = \lambda_2 \left(X(T) - x_0 e^{r_0 T}\right)^2 e^{-r_0 T}, \end{cases}$$

  where $\lambda_1, \lambda_2 \geq 0$.

- The value function is defined as

$$\phi(t,x,l) := \min_{(\pi,f)\in\Pi} J(t,x,l), \qquad t,x,l > 0,$$

  where Π is a set of all the admissible strategies of (π, f).

HJB EQUATION

Using variational methods and Itô's formula, we get the following HJB equation satisfied by the value function $\phi(t, x, l)$:

$$\min_{\pi, f} \Big\{ \phi_t + \left[r_0 x + (\mu - r_0)\pi + C_1(t) e^{\alpha t} - fl \cdot I(t) \right] \phi_x + \alpha l \phi_l + \frac{1}{2}\pi^2\sigma^2\phi_{xx} + \frac{1}{2}\eta^2 l^2 \phi_{ll} + \rho\sigma\eta l \pi \phi_{xl} + \left[\left(fl \cdot I(t) - B^* e^{\beta t} \right)^2 - \lambda_1 \left(fl \cdot I(t) - B^* e^{\beta t} \right) \right] e^{-r_0 t} \Big\} = 0.$$

Solutions

$$\min_{\pi}\left\{\phi_t + \left[r_0 x + (\mu - r_0)\pi + C_1(t)e^{\alpha t}\right]\phi_x + \alpha l \phi_l + \rho\sigma\eta l \pi \phi_{xl} + \frac{1}{2}\pi^2\sigma^2\phi_{xx}\right\} = 0$$

$$\min_{f}\left\{-fl \cdot I(t)\phi_x + \frac{1}{2}\eta^2 l^2 \phi_{ll} + \left[\left(fl \cdot I(t) - B^* e^{\beta t}\right)^2 - \lambda_1\left(B(t) - B^* e^{\beta t}\right)\right] e^{-r_0 t}\right\} = 0$$

Then the optimal solutions are given by

$$\pi^*(t,x,l) = -\frac{\delta\phi_x + \rho\eta l \phi_{xl}}{\sigma\phi_{xx}},$$

$$f^*(t,x,l) = \frac{1}{l \cdot I(t)}\left[\frac{\phi_x e^{r_0 t} + \lambda_1}{2} + B^* e^{\beta t}\right],$$

where $\delta = (\mu - r_0)/\sigma$ is the Sharp Ratio.

Solutions

By the terminal condition, we postulate that $\phi(t, x, l)$ is of the form

$$\phi(t, x, l) = \lambda_2 e^{-r_0 t} P(t)[x^2 + Q(t)x] + R(t)xl + U(t)l^2 + V(t)l + K(t).$$

The boundary condition implies that $R(T) = U(T) = V(T) = 0$ and

$$P(T) = 1, \qquad Q(T) = -2x_0 e^{r_0 T}, \qquad K(T) = x_0^2 e^{2r_0 T}.$$

Solutions

By comparing the coefficients, we get the following system of differential equations:

$$P_t + \left(r_0 - \delta^2 - \lambda_2 P(t)\right) P(t) = 0$$

$$U_t + (2\alpha + \eta^2)U(t) - \left(P(t) + \frac{(\delta + \rho\eta)^2}{\lambda_2}\right) \frac{e^{r_0 t}[R(t)]^2}{4P(t)} = 0$$

$$R_t + \left(r_0 - \delta^2 + \alpha - \delta\rho\eta - \lambda_2 P(t)\right) R(t) = 0$$

$$Q_t + \left[\frac{P_t}{P(t)} - \delta^2 - \lambda_2 P(t)\right] Q(t) + 2\left(C_1(t)e^{\alpha t} - B^* e^{\beta t}\right) = 0$$

$$V_t + \alpha V(t) + \left(C_1(t)e^{\alpha t} - B^* e^{\beta t} - \frac{1}{2}\left(\delta^2 + \delta\rho\eta + \lambda_2 P(t)\right) Q(t)\right) R(t) = 0$$

$$K_t + \lambda_2 e^{-r_0 t} P(t)Q(t)\left[C_1(t)e^{\alpha t} - B^* e^{\beta t} - \frac{1}{4}\left(\delta^2 + \lambda_2 P(t)\right) Q(t)\right] - \frac{\lambda_1^2 e^{-r_0 t}}{4} = 0$$

Solution to the optimization problem

$$P(t)=\begin{cases}\frac{1}{\lambda_2(T-t)+1}, & r_0=\delta^2,\\ \frac{r_0-\delta^2}{\lambda_2+(r_0-\delta^2-\lambda)e^{-(r_0-\delta^2)(T-t)}}, & r_0\neq\delta^2,\end{cases}$$

$$Q(t)=\begin{cases}2e^{r_0 t}\left[\int_t^T C_1(s)e^{(\alpha-r_0)s}\mathrm{d}s-B^*(T-t)-x_0\right], & \beta=r_0,\\ 2e^{r_0 t}\left[\int_t^T C_1(s)e^{(\alpha-r_0)s}\mathrm{d}s-B^*\frac{\left(e^{(\beta-r_0)T}-e^{(\beta-r_0)t}\right)}{\beta-r_0}-x_0\right], & \beta\neq r_0,\end{cases}$$

$$K(t)=\lambda_2\int_t^T e^{-r_0 t}\Big\{P(s)Q(s)\Big[C_1(s)e^{\alpha s}-B^*e^{\beta s} -\frac{1}{4}\left(\delta^2+\lambda_2P(s)\right)Q(s)\Big]-\frac{\lambda_1^2}{4}\Big\}\mathrm{d}s.$$

$$R(t)=U(t)=V(t)=0$$

Solution to the optimization problem

- Optimal strategies are

$$\pi^*(t,x,l) = -\frac{\delta}{2\sigma}\left[2x + Q(t)\right],$$
$$f^*(t,x,l) = \frac{1}{l \cdot I(t)}\left[\frac{\lambda_1}{2} + \frac{\lambda_2}{2}\left(2x + Q(t)\right)P(t) + B^* e^{\beta t}\right].$$

- Corresponding value function is given by

$$\phi(t,x,l) = \lambda_2 e^{-r_0 t} P(t)[x^2 + xQ(t)] + K(t).$$

Assumptions for numerical illustrations

- $a = 30$, $r = 65$, $\omega = 100$
- Force of mortality follows Makeham's Law (Dickson et al., 2013)
- $n(t) = 10$ for all $t \geq 0$, implying a stationary population
- $B^* = 100$, $\beta = 0.025$
- Cost-of-living adjustment rate $\zeta = 0.02$
- $r_0 = 0.01$, $\mu = 0.1$, $\sigma = 0.3 \Rightarrow \delta = 0.3$
- $\alpha = 0.03$, $\eta = 0.01$; initial salary rate $L(0) = 1$
- Correlation coefficient $\rho = 0.1$; $\lambda_1 = 15$, $\lambda_2 = 0.2$
- $X(0) = 2500$; $c_0 = 0.1$

Numerical analysis

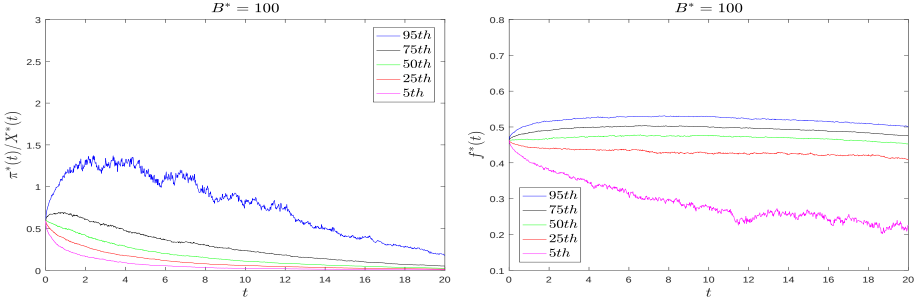


FIGURE: Percentiles of $\pi^*(t)/X^*(t)$ and $f^*(t)$

Numerical analysis

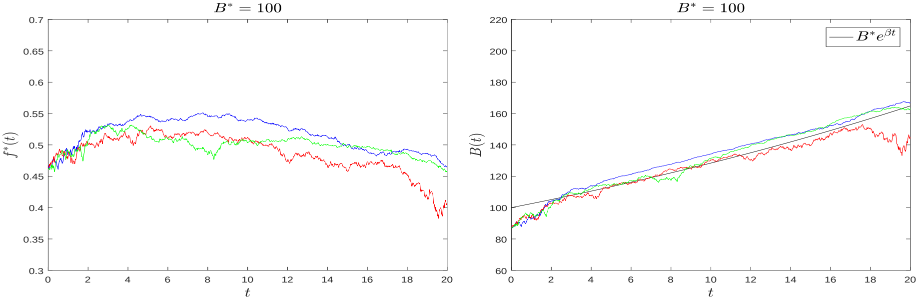


Figure: Sample paths of $f^*(t)$ and $B(t)$

Numerical analysis

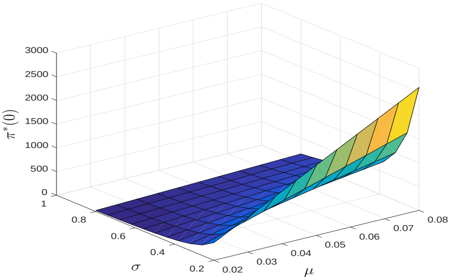


Figure: Effects of risky asset model parameters

Numerical analysis

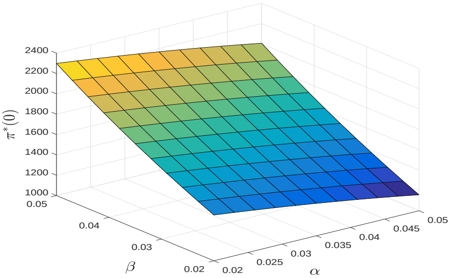


Figure: Effects of salary and target benefit growth rates

Numerical analysis

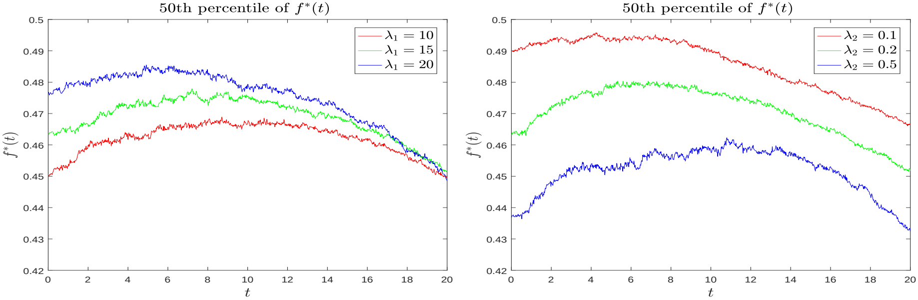


Figure: Medians of $f^*(t)$ for different values of λ_1 and λ_2

Concluding remarks

- Assumed non-stationary population and applied the Black-Scholes framework for plan assets with one risk-free and one risky asset
- Considered three key objectives for the plan trustees (benefit adequacy, stability and intergenerational equity)
- Solved optimal control problem for TBPs in continuous time and found optimal investment and benefit adjustment strategies
- Analyzed properties of the optimal strategies and sensitivities to the model parameters using Monte Carlo simulations
- Observed that intergenerational risk sharing are effective under our model settings

References I

- Boelaars, I.A. (2016). Intergenerational risk-sharing in funded pension schemes under time-varying interest rates. Draft.
- Boulier, J.-F., Trussant, E., and Florens, D. (1995). A dynamic model for pension funds management. In Proceedings of the 5th AFIR International Colloquium, volume 1, pages 361C384.
- CIA (2015). Report of the task force on target benefit plans.
- Cui, J., De Jong, F., and Ponds, E. (2011). Intergenerational risk sharing within funded pension schemes. Journal of Pension Economics and Finance, 10(01):1-29.
- Gerrard, R., Haberman, S., and Vigna, E. (2004). Optimal investment choices post-retirement in a defined contribution pension scheme. Insurance: Mathematics and Economics, 35(2):321C342.
- Gollier, C. (2008). Intergenerational risk-sharing and risk-taking of a pension fund. Journal of Public Economics, 92(5):1463-1485.
- Haberman, S. and Sung, J.-H. (1994). Dynamic approaches to pension funding. Insurance: Mathematics and Economics, 15(2):151C162.
- He, L. and Liang, Z. (2013). Optimal dynamic asset allocation strategy for ELA scheme of DC pension plan during the distribution phase. Insurance: Mathematics and Economics, 52(2):404C410.

References II

- He, L. and Liang, Z. (2015). Optimal assets allocation and benefit outgo policies of DC pension plan with compulsory conversion claims. Insurance: Mathematics and Economics, 61:227-234.
- Josa-Fombellida, R. and Rincon-Zapatero, J. P. (2001). Minimization of risks in pension funding by means of contributions and portfolio selection. Insurance: Mathematics and Economics, 29(1):35C45.
- Josa-Fombellida, R. and Rincon-Zapatero, J. P. (2004). Optimal risk management in defined benefit stochastic pension funds. Insurance: Mathematics and Economics, 34(3):489C503.
- Josa-Fombellida, R. and Rincon-Zapatero, J. P. (2008). Funding and investment decisions in a stochastic defined benefit pension plan with several levels of labor-income earnings. Computers & Operations Research, 35(1):47C63.
- Ngwira, B. and Gerrard, R. (2007). Stochastic pension fund control in the presence of Poisson jumps. Insurance: Mathematics and Economics, 40(2):283-292.
- Van Bommel, J. (2007). Intergenerational risk sharing and bank raids. Working Paper, University of Oxford.
- Westerhout, E. (2011). Intergenerational risk sharing in time-consistent funded pension schemes. Discussion Paper 03/2011-028, Netspar.