Dynamics of Order Positions and Related Queues in a LOB

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Outline

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   - Second order approximation: Fluctuation analysis
     - Diffusion limits for the related queues
     - Fluctuation analysis of order positions and execution time around the fluid limits
   - Further analysis and applications
3. Discussions

Algorithmic trading

- In an electronic order-driven market, orders arrive to the exchange and wait in the LOB (Limit Order Book) to be executed
- An automatic and rapid trading of large quantities, with orders specified and implemented by computer algorithm
- In US, high-frequency trading firms represent 2% of the approximately 20,000 firms operating today, but account for over 70% of all equity orders volume
- Order flow is heavy: thousands of orders in seconds and tens of thousand of price changes in a day

Limit order book (LOB)

- The collection of outstanding buy and sell orders with both prices and quantities on the market
- 6 types of orders which may change the state of the limit order book
    - Limit bid/ask orders: orders to buy/sell at a specified price, added to the queue and executed in order of arrival until canceled
    - Market buy/sell orders: buy/sell orders executed immediately at the best available price
    - Cancelations: any unexecuted limit orders could be canceled by their owners (without penalty)
    - (Best) bid/ask price: the highest/lowest price in limit buy/sell orders
    - Regulatory guidelines generally require exchanges to honor price-time priority: best price + FIFO

A limit buy order
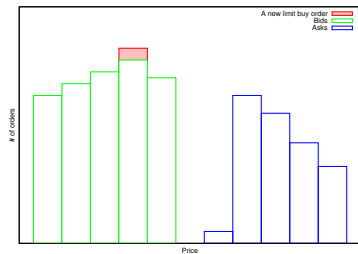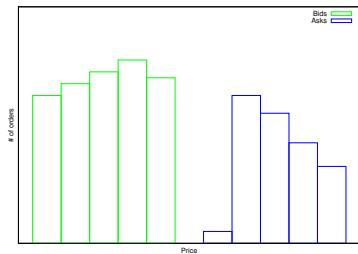
A market sell order
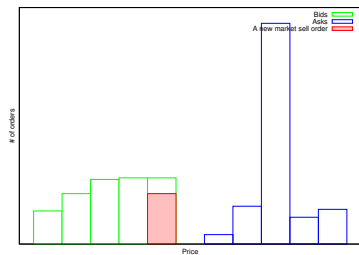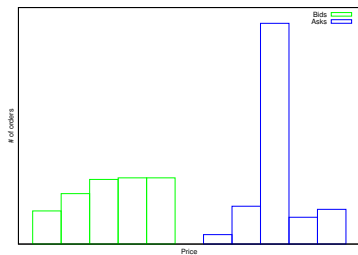
A cancelation of sell order
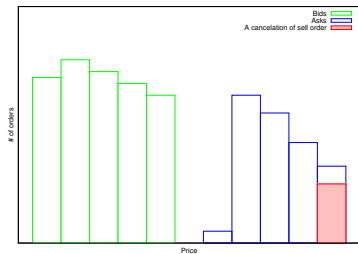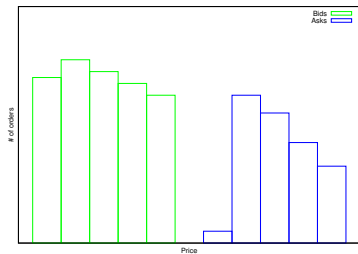
Cancelations account for over 80 percent of all orders

"A single mysterious computer program that placed orders—and then subsequently canceled them—made up 4 percent of all quote traffic in the U.S. stock market for the week (of October 5th, 2012).... The program placed orders in 25-millisecond bursts involving about 500 stocks and the algorithm never executed a single trade." (CNBC news)

Trade off between market and limit orders

When using limit orders:

- Earn a rebate or discount for providing liquidity
- However no guarantee of execution: *execution risk*

When using market orders

- Have to pay the spread and fees
- Yet immediate execution guaranteed

In essence: trade-off of paying the spread and fees vs execution/inventory risk.

Related optimization problems

- Market making Avelaneda and Stoikov (2008), Bayraktar and Ludkovski (2011), Cartea and Jaimungal (2013), Cartea, Jaimugal, and Ricci (2011), Veraarta (2010), Guilbaud and Pham (2011), and Guéant, Tapia, and Lehalla (2012), Horst, Lehalle and Li (2013)

- Optimal placement (across exchanges)
  Laruelle, Lehalle, and Pagès (2011), Cont and Kukanov (2012) and Guo et. al. (2014)

The value of order positions

- Key factors for solving market making or algorithmic trading problems.
- "Fighting" for good order positions is one of the motivations for huge expenditure for high frequency traders: a better order position means less waiting time and higher probability of execution
- Numerical study shows that for some stocks the "value of order positions" has the same order of magnitude of half spread in US equity market (Moellami and Yuan (2015))

Very little is known for order positions

- Probability of an order being executed is either assumed as constant (Guo et. al (2014)) or approximated by price changes (Cont and de Larrard (2012, 2013))
- The probability of an order at some level of LOB being executed by a given time depends on the queue length, its position in the LOB and frequency of price changes

$\rightarrow$ Order positions and related queues
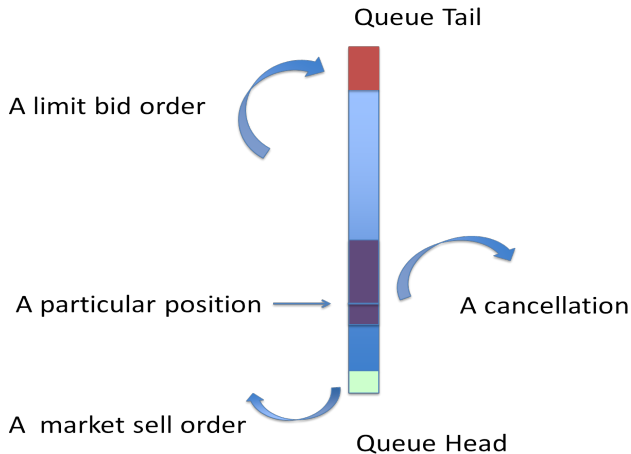
Figure : Orders happened in the best bid queue

Background
Main results
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Fluid limit for order positions and queues
Fluctuation analysis around fluid limits
Further analysis and applications

Model assumptions and notations

- Focus on the best bid and best ask as the rest are simpler.
- Six order types: best bid orders (1), market orders at the best bid (2), cancellation at the best bid(3), best ask orders (4), market orders at the best ask (5), and cancellation at the best ask (6).
- $\mathbf{N} = (N(t), t \geq 0)$ the order arrival process, with the inter-arrival times $\{D_i\}_{i \geq 1}$, and $N(t) = \max \left\{ m : \sum_{i=1}^{m} D_i \leq t \right\}$.
- $\{\overrightarrow{V}_i = (V_i^j, 1 \leq j \leq 6)\}_{i \geq 1}$: For each i, the component V_i^j represents the size of i-th order from the $j - th$ type. For example, V_i^2 is the size of the *market order* at the best bid for the $i-$th order.
- No simultaneous arrivals of different orders. For example, $\overrightarrow{V}_5 = (0, 0, 0, 4, 0, 0)$ means the fifth order is a best ask order of size 4.

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Assumptions

- (I). $\{D_i\}_{i \geq 1}$ is a stationary array of random variables with

$$\frac{D_1 + D_2 + ... + D_i}{i} \to \frac{1}{\lambda} \qquad \text{in probability.}$$

Here λ is a positive constant.

- (II). $\{\overrightarrow{V}_i\}_{i \geq 1}$ is a stationary array of random vectors with

$$\frac{\overrightarrow{V}_1 + \overrightarrow{V}_2 + ... + \overrightarrow{V}_i}{i} \to \overrightarrow{V} \qquad \text{in probability.}$$

Here $\overrightarrow{V} = (\bar{V}^j > 0, 1 \leq j \leq 6)$ is a constant vector.

- (III) Cancelations are uniformly distributed on every queue.

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Fluid limit net order flow

Define the scaled net order flow process

$$\overrightarrow{C}_n(t) = \frac{1}{n} \sum_{i=1}^{N(nt)} \overrightarrow{V}_i = \left(\frac{1}{n} \sum_{i=1}^{N(nt)} V_i^j, 1 \leq j \leq 6 \right),$$

Theorem

Under Assumptions (I) and (II),

$$\overrightarrow{\mathbf{C}}_n \Rightarrow \lambda \overrightarrow{V} \mathbf{e} \qquad \text{in } (D^6[0,\infty), J_1) \quad \text{as } n \to \infty.$$

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Fluid limit for order positions and related queues

Define scaled queue lengths $\mathbf{Q}_n^b$, $\mathbf{Q}_n^a$, and the scaled order position $\mathbf{Z}_n$ by

$$Q_n^b(t) = Q_n^b(0) + C_n^1(t) - C_n^2(t) - C_n^3(t)$$

$$Q_n^a(t) = Q_n^a(0) + C_n^4(t) - C_n^5(t) - C_n^6(t),$$

$$dZ_n(t) = -dC_n^2(t) - \frac{Z_n(t-)}{Q_n^b(t-)}dC_n^3(t).$$

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Theorem

Under Assumptions (I), (II), and (III) and assume
$(Q_n^b(0), Q_n^a(0), Z_n(0)) \Rightarrow (q^b, q^a, z)$. Then for $t < \tau$,

$$(\tilde{\mathbf{Q}}_n^b, \tilde{\mathbf{Q}}_n^a, \tilde{\mathbf{Z}}_n) \Rightarrow (\mathbf{Q}^b, \mathbf{Q}^a, \mathbf{Z}) \qquad \text{in} \quad (D^3[0,\infty), J_1),$$

where $(\mathbf{Q}^b, \mathbf{Q}^a, \mathbf{Z})$ is given by

$$Q^b(t) = q^b - \lambda v^b t,$$

$$Q^a(t) = q^a - \lambda v^a t.$$

$$\frac{dZ(t)}{dt} = -\lambda \left(\bar{V}^2 + \bar{V}^3 \frac{Z(t-)}{Q^b(t-)} \right), \qquad Z(0) = z.$$

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Here $\tau = \tau^a \wedge \tau^z$ with $\tau^a = \frac{q^a}{\lambda v^a}$

$$
\tau^z = \begin{cases}
\left(\dfrac{(1+c)z}{a} + b \right)^{c/(c+1)} b^{1/(c+1)} c^{-1} - b/c & c \notin \{-1, 0\}, \\
b(1 - e^{-\frac{z}{ab}}) & c = -1, \\
b \ln \left(\dfrac{z}{ab} + 1 \right) & c = 0.
\end{cases}
$$

Here

$$
a = \lambda \bar{V}^2, \quad b = q^b/(\lambda \bar{V}^3), \quad c = (\bar{V}^1 - \bar{V}^2 - \bar{V}^3)/\bar{V}^3,
$$

$$
v^b = -(\bar{V}^1 - \bar{V}^2 - \bar{V}^3), \quad v^a = -(\bar{V}^4 - \bar{V}^5 - \bar{V}^6).
$$

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Key ingredients for proof

- Fluid limit analysis of

$$(Q_n^b(t), Q_n^a(t)) \xrightarrow{n} (Q^b(t), Q^a(t))$$

- Convergence theorem for SDE's

Kurtz and Protter (1991)

Let $X_n(t) = U_n(t) + \int\limits_0^t F_n(X_n(s), s) dY_n(s)$, where U_n and Y_n are "nice".

Suppose $(U_n, Y_n) \to (U, Y)$, $(x_n, y_n) \to (x, y)$ implies
$(x_n, y_n, F_n(x_n)) \to (x, y, F(x))$. Then under some technical conditions
any limit point of the sequence X_n satisfy the SDE

$$X(t) = U(t) + \int\limits_0^t F(X, s) dY(s)$$

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The analysis could be delicate

Example: Let $\{X\}_{i \geq 1}$ is a sequence of identical binary random variables with values in $\{-1, 1\}$. And

$$\mathbb{P}(X_1 = 1) = \mathbb{P}(X_1 = -1) = \frac{1}{2}$$

$$\mathbb{P}(X_{i+1} = 1 | X_i = 1) = \mathbb{P}(X_{i+1} = -1 | X_i = -1) = \frac{1}{4} \text{ for } i = 1, 2, \ldots$$

Define

$$S_n(t) = \frac{1}{\sqrt{n}} \sum_{i=1}^{\lfloor nt \rfloor} X_i$$

Then $S_n(t)$ converges to $\sqrt{3}B(t)$.

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Now define a sequence of SDE's

$$dY_n = Y_n dS_n, \quad Y_n(0) = 1.$$

Clearly

$$Y_n(t) = \prod_{i=1}^{\lfloor nt \rfloor} (1 + \frac{X_i}{\sqrt{n}}),$$

and

$$Y_n(t) \to \exp\{\sqrt{3}B(t) - \frac{t}{2}\}, \quad n \to \infty.$$

However, the solution to

$$dY = Y d(\sqrt{3}B(t)), \quad Y(0) = 1$$

is

$$Y(t) = \exp\{\sqrt{3}B(t) - \frac{3t}{2}\}.$$

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Diffusion limit for order flow and related queues

- Define the centered and scaled net order flow $\overrightarrow{\boldsymbol{\Psi}}_n = (\overrightarrow{\Psi}_n(t), t \geq 0)$ by

$$\overrightarrow{\Psi}_n(t) = \frac{1}{\sqrt{n}} \left(\sum_{i=1}^{N(nt)} \overrightarrow{V}_i - \lambda \overrightarrow{V} nt \right)$$

with $\overrightarrow{V} = (\bar{V}^j, 1 \leq j \leq 6) = (\mathbb{E}[V_i^j], 1 \leq j \leq 6)$.

- Define the time rescaled queue length for the best bid and best ask

$$dR_n^b(t) = d(\Psi_n^1(t) + \lambda \bar{V}^1 t) - d(\Psi_n^2(t) + \lambda \bar{V}^2 t) - d(\Psi_n^3(t) + \lambda \bar{V}^3 t),$$
$$dR_n^b(t) = d(\Psi_n^4(t) + \lambda \bar{V}^4 t) - d(\Psi_n^5(t) + \lambda \bar{V}^5 t) - d(\Psi_n^6(t) + \lambda \bar{V}^6 t).$$

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Theorem

Assume that $\{D_i\}_{i\geq 1}$ is independent of $\{\overrightarrow{V}_i\}_{i\geq 1}$, and assume certain stationary and mixing conditions for $\{D_i\}_{i\geq 1}$, $\{\overrightarrow{V}_i\}_{i\geq 1}$. Then for $t \leq \tau$,

- $\overrightarrow{\Psi}_n \Rightarrow \overrightarrow{\Psi}$. Here $\overrightarrow{\Psi}$ is a 6-dimensional Brownian motion with drift $\lambda \overrightarrow{V}$ and variance-covariance matrix $(\lambda \Sigma^T \Sigma + \lambda^3 v_d^2 \overrightarrow{V} \cdot \overrightarrow{V}^T)$.

- The diffusion limit process for the best bid and ask queues is a two-dimensional Brownian motion with drift $\overrightarrow{\mu}$ and the variance-covariance matrix as $\overrightarrow{\mu} = \lambda A \cdot \overrightarrow{V}$ and $\sigma\sigma^T = A \cdot (\lambda \Sigma^T \Sigma + \lambda^3 v_d^2 \overrightarrow{V} \cdot \overrightarrow{V}^T) \cdot A^T$.

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Here Σ is given by $\Sigma\Sigma^T = (a_{jk})$ with

$$a_{jk} = \begin{cases} v_j^2 & \text{for} & j = k, \\ \rho_{j,k} v_j v_k & \text{for} & j \neq k, \end{cases}$$

$$v_j^2 = \text{Var}(V_1^j) + 2\sum_{i=2}^{\infty} \text{Cov}(V_1^j, V_i^j),$$

$$\rho_{j,k} = \frac{1}{v_j v_k} \left(\text{Cov}(V_1^j, V_1^k) + \sum_{i=2}^{\infty} \left(\text{Cov}(V_1^j, V_i^k) + \text{Cov}(V_1^k, V_i^j) \right) \right),$$

$$v_d^2 = \text{Var}(D_1) + 2\sum_{k=1}^{\infty} \text{Cov}(D_1, D_{k+1}).$$

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Fluctuation around the fluid limit

Under the same assumption as in those for diffusion limits,

$$\sqrt{n} \left[\begin{array}{c} \mathbf{Q}_n^b - \mathbf{Q}^b \\ \mathbf{Q}_n^a - \mathbf{Q}^a \\ \mathbf{Z}_n - \mathbf{Z} \end{array} \right] \Rightarrow \left[\begin{array}{c} \boldsymbol{\Psi}^1 - \boldsymbol{\Psi}^2 - \boldsymbol{\Psi}^3 \\ \boldsymbol{\Psi}^4 - \boldsymbol{\Psi}^5 - \boldsymbol{\Psi}^6 \\ \mathbf{Y} \end{array} \right],$$

where $\mathbf{Y}$ satisfies

$$dY(t) = (\frac{Z(t)(\Psi^1(t) - \Psi^2(t) - \Psi^3(t))}{Q^b(t)} - Y(t))\frac{\lambda \bar{V}^3}{Q_t^b} dt$$
$$- d\Psi^2(t) - \frac{Z_t}{Q_t^b} d\Psi^3(t)$$

and $Y(0) = 0$.

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Fluctuations around execution and hitting times

Theorem

Under the same assumptions as those in diffusion limits,

$$\lim_{n \to \infty} \mathbb{P}(\sqrt{n}(\tau_n^z - \tau^z) \geq x) = 1 - \Phi\left(\frac{ax}{\sigma_Y(\tau^z)}\right),$$

where $\Phi(x) := \int_{-\infty}^{x} \frac{e^{-y^2/2}}{\sqrt{2\pi}} dy$ is the cumulative probability distribution function of a standard Gaussian random variable.

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Further analysis

Analysis of order positions and related queues enables us to

- Applying large deviation principle to further analyze the probability that the queues and the execution time deviate from their fluid limits
- Derive semi-analytical expression for the probability of price decrease given the initial state of the queues
- Derive semi-analytical expression for the distribution of first hitting times for the queues
- When $\overrightarrow{\mu} = \overrightarrow{0}$, these semi-analytical expressions becomes analytic. For instance, the probability that the price decreases recovers that in Cont and de Larrard (2012).

Related to queuing theory

(Harrison, Rieman, Williams, Whitt ...)

- LOB vs. classical queues with reneging (Ward and Glynn (2003, 2005)): Cancelation rate in LOB is high and motivation and characteristics of cancelations are unclear. In fact, different assumptions of cancelation lead to different forms of diffusion approximation for $Z(t)$

- $Z(t)$ vs. the "workload process": classical queuing concerns on status/stability of the system, while algorithm trading focuses on the individual trade

- Our work vs. Cont and de Larrard (2012): Framework and assumptions are different. our results can recover theirs when adding their assumptions.

From here

- Data analysis is critical but tricky: understanding and identifying characteristics/motivation of cancelation in LOB, identifying hidden liquidity (dark pool and non-displayable orders)
- Diffusion limit for dynamics of order positions
- Related control problems for optimal execution and placement problems

This talk is based on

- X. Guo, T. L. Lai, and S. Wong (2015) Quantitative Trading: Algorithms, Analytics, Data, Models, Optimization.
- X. Guo and Z. Ruan, and L. J. Zhu (2015). Dynamics of order positions and related queues in a limit order book. *Preprint*.
- X. Guo, A. de Larrard and Z. Ruan (2013). Optimal order placement in a limit order book. *Under review*.
- X. Guo (2013). Optimal Placement in a Limit Order Book. *Tutorial for Operations Research, INFORMS*.