Optimal consumption-investment problems under time-varying incomplete preferences

Weixuan Xia

(Based on PhD Thesis Chapter I (Boston University))

Mathematical Finance Colloquium
University of Southern California

January 9th, 2023

Outline

- Motivation and contribution
- Problem formulation
- Solution by way of scalarization
- Optimal investment
- Simulation and numerical experiments
- Conclusions

Motivation and contribution

Major considerations

- A study of optimal consumption-investment problems ([Merton, 1969], [Merton, 1971]) under incomplete preferences over multiple goods that fluctuate in continuous time
- Significant relaxation of completeness axiom – conflicts in pairwise comparison of consumption bundles; analogous to different currency-valued assets: [Campi and Owen, 2011], [Hamel and Wang, 2017], [Rudloff and Ulus, 2020]
- Mathematical nature: Extension of multi-criteria optimization (or set optimization) into infinite stochastic dimensions
- Expectations: A full characterization of optimal policies and how they reshape under categorization of goods, dynamic ranges of asset prices

Motivation and contribution

Incomplete preferences: Imprecise beliefs vs imprecise tastes

- Representation of incomplete preferences by multifunctions: [Aumann, 1962], [Ok, 2002], [Dubra et al., 2004], [Evren and Ok, 2011], [Evren, 2014]
- Multi-utility proposed for incomplete preferences from imprecise tastes – distinguishable from beliefs (Knightian uncertainty) (e.g., [Bewley, 2002], [Rigotti and Shannon, 2005], [Galaabaatar and Karni, 2012])
- Knightian uncertainty – shortage of quantifiable knowledge about market aspects (model parameters); Imprecise tastes – deep-rooted in preferences, quantifiable by specified multi-utility
- Comparison: Imprecise beliefs – a pool of probability measures; imprecise tastes – a multi-utility function ([Nau, 2006], [Ok et al., 2012])

Motivation and contribution

Consumption-investment choices with preferential incompleteness

- Existing research on consumption-investment problems under imprecise beliefs (in Markovian settings) – robust utility maximization for “worst-case scenario” due to ambiguity aversion: [Fouque et al., 2016], [Biagini, and Pınar, 2017], [Liang and Ma, 2020]
- New problem under imprecise tastes – multi-utility maximization due to preferences for randomization; ample empirical evidence: [Danan and Ziegelmeyer, 2006], [Deparis et al., 2012], [Agranov and Ortoleva, 2017], [Sautua, 2017], [Cettolin and Riedl, 2019]
- Rules: Multi-criteria comparison, set comparison; importance in tracking down all equivalently optimal policies

Motivation and contribution

Time-varying imprecise tastes: Challenges and novelty

- Usual time preferences: Time-varying patience (stochastic discounting) ([Roelofsma and Read, 2000])
- Material changes in incomplete preferences: Other time-varying, possibly stochastic preference parameters – attention degrees, risk aversion degrees, etc.; leading to intransitive time preferences from a mutual effect of patience and shifted tastes ([Mandler, 2005], [Ok and Masatlioglu, 2007], [Dubra, 2009])

Motivation and contribution

Time-varying imprecise tastes: Challenges and novelty

- **A motivating example:** An investor in an int'l economy faces two goods (domestic (A) and foreign (B)). Rules of comparison: (A,B) is preferred over another pair only if A-amount has not decreased but B-amount can, depending on the level of substitution. MRS of B relative to A is subject to an inexact-valued scaling factor $i \in [0, \chi]$ ("consumption home bias," [Coeurdacier and Rey, 2012]). Multi-utility:

$$u(c_{\mathrm{A}}, c_{\mathrm{B}}) = \{u_i(c_{\mathrm{A}}, c_{\mathrm{B}}) := \breve{u}_1(c_{\mathrm{A}}) + i\breve{u}_2(c_{\mathrm{B}}) : i \in [0, \chi]\},$$

  where $\breve{u}_1$ and $\breve{u}_2$ are univariate utility functions independent of i. Investor's problem: To maximize

$$\mathbb{E}[u(c_{\mathrm{A}}, c_{\mathrm{B}})] = \mathbb{E}[\{u_0(c_{\mathrm{A}}, c_{\mathrm{B}}), u_\chi(c_{\mathrm{A}}, c_{\mathrm{B}})\}]$$

  by seeking admissible investment policies. Over time, the interval $[0, \chi]$ may change due to various factors (optimism of foreign technologies, exchange rate increase, etc.); actual formation of interval comes from many possible realizations
- Methodology: Set-valued random variable, or set-valued stochastic processes in the time flow (referring to [Zhang et al., 2009], [Li et al., 2010], [Kisielewicz, 2012], [Kisielewicz, 2020]); duality results also available by employing [Hamel et al., 2015]

Motivation and contribution

Main contributions

- To construct a hybrid set-valued stochastic process that encompasses arbitrary patterns of parametric changes in incomplete preferences
- To provide a formulation for multi-utility maximization involving consumption and investment in continuous time, which gives rise to a new multi-stochastic criteria optimization problem
- To refine scalarization techniques to account for both infiniteness and randomness in dimensions for a complete characterization of optimal consumption policies and propose a novel stochastic geometry-based method to identify corresponding optimal investment policies
- To characterize the composition of optimal consumption-investment policies and according to empirically evidenced psychological effects

Problem formulation

Market setup

- Uncertainty structure: $(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{F} \equiv \{\mathscr{F}_t\}_{t\in[0,T]})$, with $\mathbb{F}$ - augmented natural filtration of W (m-D Brownian motion); all processes $\mathbb{F}$-non-anticipating; $\mathcal{F} = \mathscr{F}_T$
- Financial market: One risk-free asset with return $r \equiv (r_t)_{t\in[0,T]} > 0$ bounded; m risky assets with return

$$Z_t = Z_0 + \int_0^t \mu_s \mathrm{d}s + \int_0^t \sigma_s \mathrm{d}W_s, \quad t \in [0, T],$$

  with $Z_0, \mu \in \mathbb{R}^m$, $\sigma \in \mathbb{R}^{m\otimes m}$, μ integrable, σ square-integrable and invertible.
- Commodity market: $n \geq 2$ distinct goods – flexible characterization, e.g., tangible vs intangible, convenience vs shopping vs specialty ([Bucklin, 1963])

Problem formulation

Construction of time-varying incomplete preferences

Definition

Let $\mathcal{I}$ be a nonempty closed convex subset of the Euclidean space $\mathbb{R}^d$ with $d \in \mathbb{N}_{++}$ and define a set of utility elements $\{u_i : i \in \mathcal{I}\}$. Suppose that $c \succeq c'$ if and only if the utility elements $u_i(c) \geq u_i(c')$ for every $i \in \mathcal{I}$; then $\mathcal{I}$ is referred to as a (multi-utility representation) index set for the preference relation $\succeq$.

- $\mathcal{I}$ for easy labeling of utility elements by d different parameters; $\mathrm{card}\mathcal{I} \leq \mathfrak{c}$
- $\mathcal{R} := \prod_{k=1}^{d} R_k$ - global product space of d parameters, $i_k \in R_k$; $\mathcal{I} \in \mathrm{Cl}(\mathcal{R})$
- In the time flow, $\mathcal{I} \equiv (\mathcal{I}_t)_{t \in [0,T]}$ is a (set-valued) stochastic process.

Problem formulation

Construction of time-varying incomplete preferences

- External preferential changes: Driven by market characteristic (non-monotone), e.g., time-varying risk aversion ([Guiso et al., 2018]); driven by psychological effects under sophistication (known monotonicity), e.g., stochastic patience ([Read and Roelofsma, 2003]) or time-varying attention with socialization (status consumption, perceived valuation, tech. development) ([Janssen and Jager, 2001], [Mrad et al., 2020], [Çanakoğlu and Özekici, 2012], [Wu et al., 2018])
- Corresponding components: A non-monotone component plus two monotone (decreasing vs increasing) components – generally constructed from set-valued Itô processes
- Aggregate effects: Minkowski summation – (1) nice mathematical properties (including convexity preservation); (2) isolability of different channels of indecisiveness changes (element-wise vector addition in sets)
- Non-redundancy: Restricting $\mathcal{I}$ to valid parameter spaces

Problem formulation

Construction of time-varying incomplete preferences

- Multi-utility index dynamics: For $(t,\omega) \in [0,T] \times \Omega$,

$$\mathcal{I}_t(\omega) := \mathcal{R} \cap \mathrm{cl}_{\mathbb{R}^d}\Bigg(I_{1,t}(\omega) + \bigcap_{s\in[0,t\wedge\tau(\omega)]} I_{2,s}(\omega) + \overline{\mathrm{co}}_{\mathbb{R}^d} \bigcup_{s\in[0,t]} I_{3,s}(\omega)\Bigg);$$

$\tau(\omega) := \sup\left\{t \in [0,T] : \mathrm{card} \bigcap_{s\in[0,t]} I_{2,s}(\omega) > 0\right\}$; for each $q \in \{1,2,3\}$,

$$I_{q,t} = \mathrm{cl}_{\mathbb{L}^1}\left(I_{q,0} + \int_0^t f_{q,s}\mathrm{d}s + \int_0^t \overline{\mathrm{co}}_{\mathbb{L}^2} G_{q,s}\mathrm{d}W_s\right), \quad t \in [0,T]$$

as the sum of an Aumann stochastic integral and a set-valued Itô integral

- Further details: $I_{q,0}$ - $\mathscr{F}_0$-measurable nonempty closed convex subset of $\mathbb{R}^d$; $f_q : [0,T] \times \Omega \mapsto \mathrm{Cl}(\mathbb{R}^d)$ - closed convex set-valued stochastic process; $G_q := \{(g_{q,k} : [0,T] \times \Omega \mapsto \mathbb{R}^{d\times m}) : k \in \mathbb{N}_{++}\}$ - collection of continuous $(d \times m)$-dimensional processes, satisfying some suitable integrability conditions

Problem formulation

Construction of time-varying incomplete preferences

- Interpretations: $I_{q,0}$ - initial space of imprecise tastes; $\int_0^\iota f_{q,s}\mathrm{d}s$ - long-term momenta; $\int_0^\iota \overline{\mathrm{co}}_{\mathbb{L}^2} G_{q,s}\mathrm{d}W_s$ - short-term noises
- Generality: Time-varying indecisiveness span by multi-valued f_q's and G_q's; reduced to time-invariance (as in [Hamel and Wang, 2017] and [Rudloff and Ulus, 2020]) if $f_q = \{\mathbf{0}\}$ and $G_q = \{\mathbf{0}\}$
- Limitations: No endogenous preferential changes (mere-exposure effect, [Bornstein, 1989]) – addressed in a second paper

Proposition

The set-valued process $\mathcal{I}$ is $\mathbb{F}$-non-anticipating, integrably bounded, and continuous $\mathbb{P}$-a.s.

Problem formulation

Construction of time-varying incomplete preferences

- Multi-utility representation:

$$u(t,c) \equiv u(t,c|\mathcal{I}_t) = \begin{cases} \{u_i(t,c) : i \in \mathcal{I}_t\}, & \text{if } c \in \mathbb{R}^n_{++}, \\ -\infty, & \text{o.w.}, \end{cases} \quad t \in [0,T];$$

  u_i's are $\mathcal{B}([0,T] \times \mathbb{R}^n_{++})$-measurable real-valued utility elements that are càdlàg in time; the index map $\mathbb{R}^d \ni i \mapsto u_i \in \mathbb{R}$ is also $\mathcal{B}(\mathbb{R}^d)$-measurable, continuous, and bounded at infinity.

- Dimensionality reduction: If $\exists J \subsetneq \mathcal{I}$ finite with fixed cardinality $\mathrm{card} J$, $\mathbb{F}$-non-anticipating such that $u(t,c|\mathcal{I}_t) = \overline{\mathrm{co}}_{\mathcal{C}_b} u(t,c|J_t)$, $\mathbb{P}$-a.s. in the space $\mathcal{C}_b(\mathbb{R}^d;\mathbb{R})$ of continuous functions bounded at infinity, then replace $\mathcal{I}$ by J.

- Bequest function: A univariate standard utility function U

Problem formulation

Construction of time-varying incomplete preferences

- Ordering cones: $\mathcal{K}_t \subseteq \bigcup_{s\in[0,t]} \prod_{i\in\mathcal{I}_s} \mathrm{im}(u_i(s,\cdot)+U) \ni \mathbf{0}$, $t \in [0,T]$, taking values in $\mathrm{Cl}(\mathcal{C}_\mathrm{b}(\mathbb{R}^d;\mathbb{R}))$, convex, assumably pointed; interpreted as a region of comparability where utility differences over consumption-bequest quantities can be ranked; upon dim. reduction, $\mathcal{K}_t \subseteq \prod_{i=1}^{\mathrm{card} J_t} \mathrm{im}(u_i(t,\cdot)+U)$.
- $\mathcal{K}$ can be designed to be $\mathbb{F}$-non-anticipating – knowing the ability to rank bundles contemporaneously

Problem formulation

Construction of time-varying incomplete preferences

Assumption

(i) (Monotonicity): For any $c, c' \in \mathbb{R}^n_+$ *with* $c - c' \in \mathbb{R}^n_+$ *and any* $x \geq x' \geq 0$,

$$u(t,c) - u(t,c') \in \mathcal{K}_t \quad \text{and} \quad U(x) - U(x') \geq 0.$$

(ii) (Concavity): For any $\alpha \in [0,1]$, $c, c' \in \mathbb{R}^n_+$, *and* $x, x' \geq 0$,

$$u(t, \alpha c + (1-\alpha)c') \in \alpha u(t,c) + (1-\alpha)u(t,c') + \mathcal{C}_{\mathrm{b}}(\mathcal{I}_t; \mathbb{R}_+)$$

and

$$U(\alpha x + (1-\alpha)x') \geq \alpha U(x) + (1-\alpha)U(x').$$

(iii) (Non-redundancy): $u_i \equiv 0$ *for every* $i \in \mathcal{R}^{\complement}$.

- Indecisiveness can be thought of as being fixed in the universe $\mathfrak{U}_{\mathbb{R}^d}$ of all possible types of tastes while process $\mathcal{I}$ controls which types are in force (effectively) over time.

Problem formulation

Construction of time-varying incomplete preferences

Proposition

For any fixed $t \in [0, T]$ and a given multi-utility function $u \in \mathfrak{U}_{\mathcal{I}_t}$, define the $u(t, \cdot)$-induced preference relation on $\mathbb{R}^n_+$ as the set

$$\succeq_t := \{(c, c') \in \mathbb{R}^n_+ \times \mathbb{R}^n_+ : u(t, c) - u(t, c') \in \mathcal{K}_t\}.$$

Then $\succeq_t$ is reflexive and transitive but not necessarily complete.

- Key properties: $\succeq$ statically incomplete, dynamically (possibly) intransitive (“max-min” multi-utility, [Nishimura and Ok, 2016])
- Bequest preference simply governed by $\geq$ (complete).

Definition

For any fixed $t \in [0, T]$, given a multi-utility function $u(t, \cdot) \in \mathfrak{U}_{\mathcal{I}_t}$, the incomplete part of the induced preference relation $\succeq_t$ is defined as

$$\ominus_t := \{(c, c') \in \mathbb{R}^n_+ \times \mathbb{R}^n_+ : u(t, c) - u(t, c') \in \pm\mathcal{K}_t\}^{\complement}.$$

Problem formulation

Multi-utility maximization problem

- Investor's wealth: For $t \in [0, T]$,

$$X_t \equiv X_t^{(c,\Pi)} = X_0 + \int_0^t (r_s X_s - C_s)\mathrm{d}s + \int_0^t \langle \Pi_s, (\mu_s - r_s \mathbf{1})\mathrm{d}s + \sigma_s \mathrm{d}W_s \rangle_m;$$

  c - n-D consumption process, $C := \langle c, \mathbf{1} \rangle_n$ its total, Π - m-D portfolio process in dollar amounts, $X_0 > 0$ - given initial wealth

Assumption

(i) $c_t \in \mathbb{R}_+^n$, $\forall t \in [0, T]$, *and* $\int_0^T C_s \mathrm{d}s < \infty$, $\mathbb{P}$-*a.s.*

(ii) $\int_0^T \|\Pi_s^{\intercal} \sigma_s\|_2^2 \mathrm{d}s < \infty$ *and* $\int_0^T |\langle \Pi_s, \mu_s - r_s \mathbf{1} \rangle_m| \mathrm{d}s < \infty$, $\mathbb{P}$-*a.s.*

- Notation: $c \in \mathfrak{C}_n$ and $\Pi \in \mathfrak{P}_m$

Problem formulation

Multi-utility maximization problem

- Investor's problem:

$$\sup_{(c,\Pi)\in\mathfrak{A}(X_0)} V(c,\Pi), \quad V(c,\Pi) := \mathbb{E}\left[\int_0^T u(t,c_t)\mathrm{d}t + U(X_T)\right]$$

within admissibility set (given $X_0 > 0$)
$\mathfrak{A}(X_0) := \{(c,\Pi) \in \mathfrak{C}_n \times \mathfrak{P}_m : X_t \geq 0,\ t \in [0,T],\ \mathbb{P}\text{-a.s.}\}$
Meaning: V valued in $\mathcal{C}_{\mathrm{b}}(\bar{\mathcal{I}};\mathbb{R})$, with $\bar{\mathcal{I}}$ $\mathscr{F}_0$-measurable such that $d_{\mathrm{H}}(\mathrm{cl}_{\mathbb{R}^d} \bigcup_{t\in[0,T]} \mathcal{I}_t, \{\mathbf{0}\}) \leq d_{\mathrm{H}}(\bar{\mathcal{I}}, \{\mathbf{0}\})$, $\mathbb{P}$-a.s. (as a domain extension); integral in the sense of Bochner (not Aumann); maximality w.r.t. some chosen $\mathscr{F}_0$-measurable pointed closed convex cone $\bar{\mathcal{K}} \supseteq \overline{\mathrm{co}}_{\mathcal{C}_{\mathrm{b}}} \bigcup_{t\in[0,T]} \mathcal{K}_t$ ($\mathbb{P}$-a.s.) over $\bar{\mathcal{I}}$; $\mathcal{K}_t = \bar{\mathcal{K}}$ $\forall t$ if dimensionality is reduced to $\mathrm{card}J$

Definition

Say that $(c,\Pi) \in \mathfrak{A}(X_0)$ is a $\bar{\mathcal{K}}$-maximal solution of the above problem if $(V(c,\Pi) + \bar{\mathcal{K}}) \cap V(\mathfrak{A}(X_0)) = \{V(c,\Pi)\}$. On the other hand, it is said to be weakly $\bar{\mathcal{K}}$-maximal if $\mathrm{int}\bar{\mathcal{K}} \neq \varnothing$ and $(V(c,\Pi) + \mathrm{int}\bar{\mathcal{K}}) \cap V(\mathfrak{A}(X_0)) = \varnothing$.

Problem formulation

Multi-utility maximization problem

- State price density (market completeness):

$$\xi_t := \exp\left(-\int_0^t \left(r_s + \frac{1}{2}\|\theta_s\|_2^2 \right) \mathrm{d}s - \int_0^t \langle \theta_s, \mathrm{d}W_s \rangle_m \right), \quad t \in [0, T]$$

- Static problem:

$$\sup_{(c, X_T) \in \mathfrak{B}(X_0)} V(c, X_T),$$

within (budget set)

$$\mathfrak{B}(X_0) := \left\{ (c, X_T) \in \mathfrak{C}_n \times \mathbb{L}^1_{\mathcal{F}}(\Omega; \mathbb{R}_+) : \mathbb{E}\left[\int_0^T \xi_s C_s \mathrm{d}s + \xi_T X_T \right] \leq X_0 \right\}$$

Theorem

(i) If $(c, \Pi) \in \mathfrak{A}(X_0)$, then $(c, X_T) \in \mathfrak{B}(X_0)$.

(ii) If $(c, X_T) \in \mathfrak{B}(X_0)$, then there exists $\Pi \in \mathfrak{P}_m$ such that $(c, \Pi) \in \mathfrak{A}(X_0)$.

Solution by way of scalarization

A modified Gass-Satty method

- Nature: A method to project all objective functions into $\mathbb{R}$, a.k.a. weighted-sum method ([Gass and Satty, 1955]), modified to accommodate infinite stochastic dimensions
- Single-criterion problem:

$$\sup_{(c,X_T)\in\mathfrak{B}(X_0)} V(c,X_T|w), \quad w\in\mathcal{K}^\dagger, \sup_{t\in[0,T]} \|w(t)\|_1 > 0;$$

  w being a weight functional, with $\| \, \|_1$ the TV norm (reducible to the Taxicab norm) on finite Radon measures, and

$$V(c,X_T|w) := \mathbb{E}\left[\int_0^T \left\langle w(t), u(t,c_t) + \frac{U(X_T)}{T}\right\rangle_{\mathcal{I}_t} \mathrm{d}t\right],$$

  real-valued; $\mathcal{K}_t^\dagger := \{z\in(\mathcal{C}_\mathrm{b}(\mathcal{I}_t;\mathbb{R}))^\dagger : \langle z,k\rangle_{\mathcal{I}_t} \geq 0, \ \forall k\in\mathcal{K}_t\}$ - (topological) dual cone of $\mathcal{K}_t$, $\forall t\in[0,T]$
- Remark: $\mathcal{K}^\dagger$ is an $\mathbb{F}$-non-anticipating closed convex-valued process.
- Interpretation: w - floating totaling rule applied inter-temporally to the multi-utility u augmented by the time-scaled bequest utility U/T; indefiniteness of w for imprecision of tastes

Solution by way of scalarization

A modified Gass-Satty method

Theorem

(i) If (c^*, X_T^*) *is a* $\bar{\mathcal{K}}$*-maximal solution of the multi-criteria problem, then there exists* $w(t) \in \mathcal{K}_t^\dagger$ *for every* $t \in [0, T]$ *with* $\sup_{t\in[0,T]} \|w(t)\|_1 > 0$ *such that* $(c^*, X_T^* | w)$ *is a maximal solution of the single-criterion problem.*

(ii) If $(c^*, X_T^* | w)$ *is a maximal solution of the single-criterion problem then* $(c^*, X_T^* | w)$ *is at least a weakly* $\bar{\mathcal{K}}$*-maximal solution of the multi-criteria problem.*

Proposition

Let $(c^*, X_T^* | w)$ *be a maximal solution of the single-criterion problem conditional on* $w \in \mathcal{K}^\dagger$*; then the set of* $\bar{\mathcal{K}}$*-maximal solutions of the multi-criteria problem is precisely equal to*

$$\mathcal{S}^* = \left\{ (c^*, X_T^* | w) : w \in \mathcal{K}^\dagger, \sup_{t \in [0,T]} \|w(t)\|_1 = 1 \right\}.$$

Solution by way of scalarization

A modified Gass-Satty method

- Remarks: $\text{int}\mathcal{K} \neq \emptyset$ ensures necessity; a convex criterion space $V(\mathfrak{B}(X_0))$ ensures sufficiency; $\mathcal{S}^*$ gives rise to a $\mathcal{B}([0,T]) \otimes \mathcal{F}$-measurable w-parameterized augmented set-valued process (c^*, X_T^*) valued in $\text{Cl}\big(\mathfrak{C}_n \times \mathbb{L}^1_{\mathcal{F}}(\Omega; \mathbb{R}_+)\big)$; $\mathcal{K}^\dagger \equiv \bar{\mathcal{K}}^\dagger$ becomes (fixed) finite-dimensional with dimensionality reduction.
- Implication: The modified Gass-Satty method is capable of recovering all the optimal consumption-bequest policies.

Solution by way of scalarization

Solution procedures

Assumption

For every $t \in [0, T]$ *and* $i \in \mathcal{I}_t$, $u_i(t, \cdot) \in \mathcal{C}^\infty(\mathbb{R}^n_{++}; \mathbb{R})$, *which satisfies the Inada conditions that the first-order derivatives with respect to the jth consumption quantity,* c_j, $\lim_{c_j \searrow 0} u_i^{(j)}(t, c) = \infty$ *and* $\lim_{c_j \to \infty} u_i^{(j)}(t, c) = 0$, *for any* $j \in \mathbb{N} \cap [1, n]$; *similarly,* $U \in \mathcal{C}^\infty(\mathbb{R}_{++}; \mathbb{R})$, *satisfying that* $\lim_{x \searrow 0} U(x) = \infty$ *and* $\lim_{x \to \infty} U(x) = 0$.

- Step 1: Construct the index set process $\mathcal{I} \subseteq \mathbb{R}^d$ according to established recipes (by specifying $I_{q,0}$, f_q, and G_q) and compute its unconditional superset $\bar{\mathcal{I}}$.
- Step 2: Specify utility elements u_i's according to interests, set up multi-utility u, and check if dimensionality reduction holds (if so change $\mathcal{I}$ to J and consider $\bar{\mathcal{I}}$ as $\mathbb{N} \cap [1, \mathrm{card} J]$); specify ordering cones $\mathcal{K}$ and $\bar{\mathcal{K}}$ accordingly.

Solution by way of scalarization

Solution procedures

- Step 3: Specify the following optimality conditions (sufficient and necessary)

$$\eta\xi = \langle w, u^{(j)}(\iota, c)\rangle_{\mathcal{I}}, \quad j \in \mathbb{N} \cap [1, n],$$

$$\eta\xi_T = \frac{U'(X_T)}{T} \int_0^T \langle w(t), \mathbf{1}\rangle_{\mathcal{I}_t} \mathrm{d}t,$$

$$X_0 = \mathbb{E}\left[\int_0^T \xi_t C_t \mathrm{d}t + \xi_T X_T\right];$$

  $u^{(j)}$ - multifunction of jth derivatives; attainability ensured by Inada conditions

- Comments: $(n+2)$-dimensional nonlinear systems for (c, X_T) and η given w; feedback forms: $c = \psi_{\mathcal{I}}(\eta\xi|w)$, $X_T = (U')^{-1}(\eta\xi_T T / \int_0^T \langle w(t), \mathbf{1}\rangle_{\mathcal{I}_t} \mathrm{d}t)$; occasionally a single solution
- Step 4: Take union over $\bar{\mathcal{K}}^\dagger \ni w$ (s.t. normalization) to get to the full set of solutions.

Solution by way of scalarization

Example 1: Invariant indecisiveness

- Setting: $m = 1$, $n = 2$, $d = 1$; constant market coefficients (r, μ, σ); $I_{2,0} = I_{3,0} = \{0\}$, $f_q \equiv \{0\}$ and $G_q \equiv \{0\}$, $\forall q \in \{1, 2, 3\}$ – constant $\bar{\mathcal{I}} = \mathcal{I} = I_{1,0}$; no bequest utility $U \equiv 0$
- Multi-utility: Totally incomparable goods, independent assessment;

$$u_i(\iota, c) \equiv u_i(c) = \begin{cases} \dfrac{c_i^{1-p} - 1}{1 - p}, & \text{if } i \in \{1, 2\}, \\ 0, & \text{o.w.,} \end{cases} \tag{I}$$

  with $p \in \mathbb{R}_{++} \setminus \{1\}$ constant risk aversion degree
- One good more essential than the other, imprecise attention;

$$u_i(\iota, c) \equiv u_i(c) = \begin{cases} \dfrac{i(c_1^{1-p} - 1) + (c_2^{1-p} - 1)}{1 - p}, & \text{if } i \in [0, \chi], \\ 0, & \text{o.w.,} \end{cases} \tag{II}$$

  with $\chi > 0$ upper bound of attention degree i

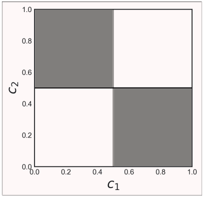
Solution by way of scalarization

Example 1: Invariant indecisiveness

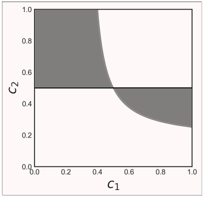
- Adequate substitutes, imprecise interaction;

$$u_i(\iota, c) \equiv u_i(c) = \begin{cases} \dfrac{c_1^{1-p} + c_2^{1-p}}{1-p} - \dfrac{i(c_1 c_2)^{1-p}}{(1-p)^2}, & \text{if } i \in [\varkappa_1, \varkappa_2] \subsetneq \mathbb{R}_{++}, \\ 0, & \text{o.w.}; \end{cases} \tag{III}$$

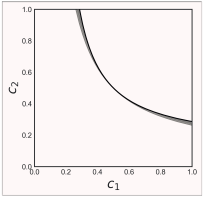
$\varkappa_1, \varkappa_2$ the lower/upper bounds for interaction degrees.



Case (I)



Case (II)



Case (III)

Solution by way of scalarization

Example 1: Invariant indecisiveness

- Solutions: Case (I). With $\rho_p(r,\theta) := (1/p-1)r + (1-p)/(2p^2)\theta^2$,

$$\mathcal{S}^* = \left\{ c \in \mathfrak{C}_2 : C^* = \frac{\rho_p(r,\theta)X_0}{\xi^{1/p}(e^{\rho_p(r,\theta)T}-1)} \right\}$$

  - focus on total consumption, randomization over combinations

- Case (II).

$$\mathcal{S}^* = \left\{ c^* \in \mathfrak{C}_2 : C^* = \frac{\rho_p(r,\theta)X_0}{\xi^{1/p}(e^{\rho_p(r,\theta)T}-1)};\ \frac{c_1^*}{c_2^*} \in [0,\chi^{1/p}] \right\}$$

  - focus on total consumption, limited consumption of the first good

- Case (III). $\psi(\eta\xi|w) \rightsquigarrow 1 - x^{1-p}(w_1\varkappa_1 + w_2\varkappa_2)/(1-p) = \eta\xi x^p,\ x \geq 0;$

$$\mathcal{S}^* = \bigcup_{w\in\mathbb{R}_+^2,\ \|w\|_1=1} \left\{ c^* \in \mathfrak{C}_2 : c_1^* = c_2^* = \psi(\eta\xi|w); \right.$$

$$\left. \eta \rightsquigarrow 2\int_0^T \mathbb{E}[\xi_t\psi(\eta\xi_t|w)]\mathrm{d}t = X_0 \right\}$$

  - inexact total consumption, equal weights of two goods

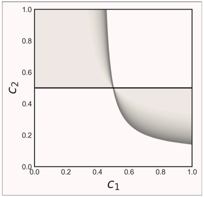
Solution by way of scalarization

Example 2: Socialization and increasing indecisiveness

- Setting: Same as Case (II) of Ex. 1 except $I_{1,0} = I_{2,0} = \{0\}$, $I_{3,0} = [0,1]$, $f_q \equiv \{0\}$, $\forall q \in \{1,2,3\}$, $G_1 \equiv G_2 \equiv \{0\}$, $G_3 \equiv \{\lambda\}$, $\lambda > 0$ constant, so $\mathcal{I}_t = \left[0, \lambda W_t^{\uparrow} + 1\right]$, $t \in [0,T]$, with $W^{\uparrow} := \sup_{s\in[0,t]} W_s$ the running maximum of W; $\bar{\mathcal{I}} = \mathbb{R}_+$
- Multi-utility: Increasing attention degrees due to socialization;

$$u_i(t,c) = \begin{cases} \dfrac{\chi_i(c_1^{1-p}-1)+(c_2^{1-p}-1)}{e^{\beta t}(1-p)}, & \text{if } i \in \left[0, \lambda W_t^{\uparrow}+1\right], \\ 0, & \text{o.w.}; \end{cases}$$

$\chi : \mathbb{R}_+ \mapsto \mathbb{R}_+$ attention degree function (bounded, nondecreasing); λ - attention increase acceleration



Solution by way of scalarization

Example 2: Socialization and increasing indecisiveness

- Solution:

$$
\mathcal{S}^* = \bigcup_{w \in \mathbb{R}^2_+,\ \|w\|_1 = 1} \Bigg\{ c^* \in \mathfrak{C}_2 : C^* = \frac{\left(w_1 \chi_0 + w_2 \chi_{\lambda W^{\uparrow}+1}\right)^{1/p} + 1}{\left(\eta \xi e^{\beta \iota}\right)^{1/p}};
$$

$$
\frac{c_1^*}{c_2^*} \in \left[\chi_0^{1/p}, \chi_{\lambda W^{\uparrow}+1}^{1/p}\right];
$$

$$
\eta^{1/p} = \frac{1}{X_0} \int_0^T \iint_{\mathbb{R}_+ \times (\infty, x_1]} \sqrt{\frac{2}{\pi t^3}} (2x_1 - x_2) \exp\left(\left(\frac{1}{p} - 1 \right) \left(r + \frac{\theta^2}{2} \right) t \right.
$$

$$
\left. + \theta x_2 - \frac{\beta t}{p} - \frac{(2x_1 - x_2)^2}{2t} \right) \left(\left(w_1 \chi_0 + w_2 \chi_{\lambda x_1 + 1}\right)^{1/p} + 1 \right) \mathrm{d}(x_1, x_2) \mathrm{d}t \Bigg\}
$$

- inexact total consumption, stochastic limit of first good's consumption

Solution by way of scalarization

Example 3: Socialization, market volatility, and changing indecisiveness

- Setting: Similar to Ex. 2 except stochastic market volatility (exponential OU)

$$\sigma_t = \exp\left((\log \sigma_0) e^{-\kappa t} + \varsigma \int_0^t e^{-\kappa(t-s)} \mathrm{d}W_s \right), \quad t \in [0, T],$$

  with parameters $\sigma_0 > 0$, $\kappa > 0$, and $\varsigma > 0$; $\mathcal{R} = \mathbb{R}_+ \times [1, \infty)$, $I_{1,0} = \{(0, \sigma_0)\}$, $I_{2,0} = \{\mathbf{0}\}$, $I_{3,0} = [0, 1] \times [1, 2]$, $f_1 = \{(0, \kappa(\varsigma^2/(2\kappa) - \log \sigma)\sigma)\}$, $f_2 = f_3 = \{\mathbf{0}\}$, $G_1 = \{(0, \varsigma\sigma)\}$, $G_2 \equiv \{\mathbf{0}\}$, $G_3 = \{(\lambda, 0)\}$, so $\mathcal{I}_t = [0, \lambda W_t^{\uparrow} + 1] \times [\sigma_t + 1, \sigma_t + 2]$, $t \in [0, T]$, $\bar{\mathcal{I}} = \mathbb{R}_+ \times [1, \infty)$; bequest utility $U(x) = e^{-\beta T}(x^{1-p_\circ} - 1)/(1 - p_\circ)$, $x > 0$, $p_\circ \in \mathbb{R}_{++} \setminus \{1\}$ given

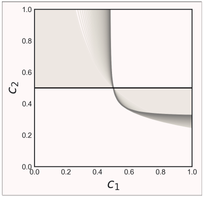
Solution by way of scalarization

Example 3: Socialization, market volatility, and changing indecisiveness

- Multi-utility: Increasing attention plus volatility-driven risk aversion;

$$u_i(t,c) = \begin{cases} \dfrac{\chi_{i_1}(c_1^{1-p_{i_2}} - 1) + (c_2^{1-p_{i_2}} - 1)}{e^{\beta t}(1-p_{i_2})}, & \text{if } i \in \mathcal{I}_t, \\ 0, & \text{o.w.}; \end{cases}$$

$p : [1,\infty) \mapsto \mathbb{R}_{++} \setminus \{1\}$ - some bounded nondecreasing function



Solution by way of scalarization

Example 3: Socialization, market volatility, and changing indecisiveness

- Solution: $\mathbb{R}_+ \ni x \mapsto \vartheta_{p,w_2;\sigma}(x) := \int_{\sigma+1}^{\sigma+2} w_{2,i_2} x^{-p_{i_2}} \mathrm{d}i_2 \in [0,\infty]$,

$$\mathcal{S}^* = \bigcup_{\substack{\bar{w}\in(\mathcal{C}_\mathrm{b}(\mathbb{R}_+\times[1,\infty);\mathbb{R}_+))^\dagger,\\ \|\bar{w}\|_1=1}} \Bigg\{ (c^*, X_T^*) : c_1^* = \vartheta^{-1}_{p,\bar{w}_2;\sigma}\left(\frac{\eta\xi e^{\beta\iota}}{\int_0^{\lambda W^\uparrow+1} \bar{w}_{1,i_1}\chi_{i_1}\mathrm{d}i_1}\right);$$

$$c_2^* = \vartheta^{-1}_{p,\bar{w}_2;\sigma}\left(\frac{\eta\xi e^{\beta\iota}}{\int_0^{\lambda W^\uparrow+1} \bar{w}_{1,i_1}\mathrm{d}i_1}\right);\ X_T^* = \left(\frac{\int_0^T \int_{[0,\lambda W_t^\uparrow+1]\times[\sigma_t+1,\sigma_t+2]} \bar{w}_\iota \mathrm{d}\iota \mathrm{d}t}{\eta\xi_T T e^{\beta T}}\right)^{1/p_\circ};$$

$$\eta \rightsquigarrow \int_0^T \mathbb{E}\left[\xi_t\left(\vartheta^{-1}_{p,\bar{w}_2;\sigma_t}\left(\frac{\eta\xi_t e^{\beta t}}{\int_0^{\lambda W_t^\uparrow+1} \bar{w}_{1,i_1}\chi_{i_1}\mathrm{d}i_1}\right) + \vartheta^{-1}_{p,\bar{w}_2;\sigma_t}\left(\frac{\eta\xi_t e^{\beta t}}{\int_1^{\lambda W_t^\uparrow+2} \bar{w}_{1,i_1}\mathrm{d}i_1}\right)\right)\right]\mathrm{d}t$$

$$+\mathbb{E}\left[\xi_T^{1-1/p_\circ}\left(\frac{\int_0^T \int_{[0,\lambda W_t^\uparrow+1]\times[\sigma_t+1,\sigma_t+2]} \bar{w}_\iota \mathrm{d}\iota \mathrm{d}t}{\eta T e^{\beta T}}\right)^{1/p_\circ}\right] = X_0 \Bigg\}$$

- same as in Ex. 2, plus “effectively” shrunk indecisiveness due to increased risk aversion

Optimal investment

Portfolio structure

- Idea: To compute Π^* given each $\mathbb{F}$-non-anticipating selector of C^* as a set-valued process; C^* single-valued occasionally
- Optimal wealth:

$$X_t^* = \xi_t^{-1}\mathbb{E}\left[\int_t^T \xi_s C_s^* \mathrm{d}s + \xi_T X_T^* \middle| \mathscr{F}_t\right], \quad t \in [0,T]$$

  for every (C^*, X_T^*) optimal
- Feedback expressions:

$$C^* = \Psi_{\mathcal{I}}(\eta\xi|w) := \langle \psi_{\mathcal{I}}(\eta\xi|w), \mathbf{1}\rangle_n, \quad X_T^* = (U')^{-1}\left(\frac{\eta\xi_T T}{\int_0^T \langle w(t), \mathbf{1}\rangle_{\mathcal{I}_t} \mathrm{d}t}\right)$$

Optimal investment

Portfolio structure

Theorem

The optimal investment policy is given by the set-valued process

$$\Pi_t^* = \mathrm{cl}_{\mathbb{L}^1}\Big\{\xi_t^{-1}\mathbb{E}\Big[\int_t^T \xi_s\gamma_{\mathcal{I}_s}(\eta\xi_s|w(s))\mathrm{d}s + \xi_T\Gamma(\eta\xi_T|w)\Big|\mathscr{F}_t\Big](\sigma_t^\intercal)^{-1}\theta_t$$

$$-\xi_t^{-1}(\sigma_t^\intercal)^{-1}\mathbb{E}\Big[\int_t^T \xi_s((\Psi_{\mathcal{I}_s}(\eta\xi_s|w(s)) - \gamma_{\mathcal{I}_s}(\eta\xi_s|w(s)))H_{t,s} + \upsilon(t,s|w(s)))\mathrm{d}s$$

$$\xi_T\bigg(\bigg((U')^{-1}\bigg(\frac{\eta\xi_T T}{\int_0^T\langle w(s),\boldsymbol{1}\rangle_{\mathcal{I}_s}\mathrm{d}s}\bigg) - \Gamma(\eta\xi_T|w)\bigg)H_{t,T} + \Upsilon(t,T|w)\bigg)\bigg|\mathscr{F}_t\Big]:$$

$$w(s)\in\mathcal{K}_s^\dagger,\ \forall s\in[t,T];\ \sup_{s\in[0,T]}\|w(s)\|_1 > 0\Big\},\quad t\in[0,T],$$

where

$$\gamma_{\mathcal{I}}(\eta\xi|w) := -\eta\xi\boldsymbol{1}^\intercal\big(\langle w,(u^{(j)})^{(j')}(\iota,\psi_{\mathcal{I}}(\eta\xi|w))\rangle\big)^{-1}_{j,j'\in\mathbb{N}\cap[1,n]}\boldsymbol{1},$$

...

Optimal investment

Portfolio structure

Theorem (cont'd)

$$\Gamma(\eta\xi_T|w) := -\frac{\eta\xi_T T}{\int_0^T \langle w(s), \boldsymbol{1}\rangle_{\mathcal{I}_s} \mathrm{d}s}((U')^{-1})'\left(\frac{\eta\xi_T T}{\int_0^T \langle w(s), \boldsymbol{1}\rangle_{\mathcal{I}_s} \mathrm{d}s}\right),$$

$$\upsilon(t,\iota|w) := \Bigg(\left(\langle w, (u^{(j)})^{(j')}(\iota, \psi_{\mathcal{I}}(\eta\xi|w))\rangle\right)^{-1}_{j,j'\in\mathbb{N}\cap[1,n]} \times \left(\int_{\partial\mathcal{I}} \boldsymbol{v}(i,W) \lrcorner (w_i u_i^{(j)}(\iota, \psi_{\mathcal{I}}(\eta\xi|w))\mathrm{d}i)\mathbb{1}_{(0,\infty)}(\|w\|_1)\right)_{j\in\mathbb{N}\cap[1,n]}\Bigg)^{\intercal}\boldsymbol{1},$$

and

$$\Upsilon(t,T|w) := \eta\xi_T T((U')^{-1})'\left(\frac{\eta\xi_T T}{\int_0^T \langle w(s), \boldsymbol{1}\rangle_{\mathcal{I}_s} \mathrm{d}s}\right)\frac{\int_t^T \int_{\partial\mathcal{I}_s} \boldsymbol{v}(i,W_s)\lrcorner(w_i(s)\mathrm{d}i)\mathrm{d}s}{\left(\int_0^T \langle w(s), \boldsymbol{1}\rangle_{\mathcal{I}_s} \mathrm{d}s\right)^2};$$

$\boldsymbol{v}(i,W_s)$ *- velocity vector field of the* $\mathbb{R}^d$*-boundary* $\partial\mathcal{I}_s$ *for* $s\in(t,T]$ *on the classical Wiener space* $\mathbb{C}_0([0,T];\mathbb{R}^m)$*;* $\lrcorner$ *- interior product*

Optimal investment

Portfolio structure

- Interpretations: $\gamma_{\mathcal{I}}$, Γ - parameterized risk tolerance functions corresponding to consumption multi-utility resp. bequest utility; H - determined $\mathbb{F}$-non-anticipating processes for (r, θ)-hedging; $\upsilon_{\mathcal{I}}$, Υ - parameterized psychological effect functions for indecisiveness
- Dimensionality reduction: $\mathcal{I}$ replaced by J; duality pairing over $\operatorname{card} J$; $\partial J = J$
- Portfolio decomposition:

$$\Pi^* = \mathrm{cl}_{\mathbb{L}^1}\left\{(\Pi^{(\theta)} + \Pi^{(H)} + \Pi^{(\maltese)}|w) : w \in \mathcal{K}^\dagger, \sup_{t\in[0,T]} \|w(t)\|_1 = 1\right\}$$

  for a mean-variance portfolio, a market risk-hedging portfolio, and an indecisiveness risk-hedging portfolio, in sequence; readily reducible to classical decomposition without time-varying incompleteness
- Threefold effects: (1) Alteration of optimal consumption policy structures – dynamic intransitivity of preferences – new Wiener functionals; (2) a new portfolio component $\Pi^{(\maltese)}$ for hedging motives for imprecise taste risks; (3) multiplicity in optimal investment policies – static incompleteness of preferences

Optimal investment

Portfolio structure

- Simplification: If each u_i is consumption-additive, i.e.,

$$u_i(t,c) = \langle \alpha_i(t), \breve{u}(c) \rangle_n \equiv \sum_{j=1}^{n} \alpha_{ij}(t) \breve{u}_j(c_j), \quad i \in \mathcal{I}$$

  for suitable functions $\{\breve{u}_j : j \in \mathbb{N} \cap [1,n]\} \subseteq \mathcal{C}^\infty(\mathbb{R}^n_+; \mathbb{R})$ and time-dependent coefficients α_{ij}'s and $d = 1$, an easier-to-implement formula is available (next corollary).

Optimal investment

Portfolio structure

Corollary

$$
\begin{aligned}
\Pi_t^* = \mathrm{cl}_{\mathbb{L}^1}\Bigg\{ & \xi_t^{-1}\mathbb{E}\Bigg[\int_t^T \xi_s \sum_{j=1}^n \gamma_j(\eta\xi_s|w(s))\mathrm{d}s + \xi_T\Gamma(\eta\xi_T|w)\Bigg|\mathscr{F}_t\Bigg](\sigma_t^{\intercal})^{-1}\theta_t \\
& -\xi_t^{-1}(\sigma_t^{\intercal})^{-1}\mathbb{E}\Bigg[\int_t^T \xi_s\Bigg(\sum_{j=1}^n\Bigg((\breve{u}_j')^{-1}\Bigg(\frac{\eta\xi_s}{\langle w(s),\alpha_j(s)\rangle_{\mathcal{I}_s}}\Bigg) - \gamma_j(\eta\xi_s|w(s))\Bigg)H_{t,s} \\
& \quad + \eta\xi_s\sum_{j=1}^n((\breve{u}_j')^{-1})'\Bigg(\frac{\eta\xi_s}{\langle w(s),\alpha_j(s)\rangle_{\mathcal{I}_s}}\Bigg) \\
& \quad \times\frac{w_{\check{Y}_{+,s}}(s)\alpha_{\check{Y}_{+,s}j}(s)\mathcal{D}_t\check{Y}_{+,s}\mathbb{1}_{\mathcal{R}}(\check{Y}_{+,s}) - w_{\check{Y}_{-,s}}(s)\alpha_{\check{Y}_{-,s}j}(s)\mathcal{D}_t\check{Y}_{-,s}\mathbb{1}_{\mathcal{R}}(\check{Y}_{-,s})}{\langle w(s),\alpha_j(s)\rangle_{\mathcal{I}_s}^2}\Bigg)\mathrm{d}s \\
& \quad + \xi_T\Bigg(\Bigg((U')^{-1}\Bigg(\frac{\eta\xi_T T}{\int_0^T\langle w(s),\mathbf{1}\rangle_{\mathcal{I}_s}\mathrm{d}s}\Bigg) - \Gamma(\eta\xi_T|w)\Bigg)H_{t,T} + \eta\xi_T T \cdots
\end{aligned}
$$

Optimal investment

Portfolio structure

Corollary (cont'd)

$$\times ((U')^{-1})'\left(\frac{\eta\xi_T T}{\int_0^T \langle w(s), \mathbf{1}\rangle_{\mathcal{I}_s} \mathrm{d}s}\right)\frac{1}{\left(\int_0^T \langle w(s), \mathbf{1}\rangle_{\mathcal{I}_s} \mathrm{d}s\right)^2}$$

$$\times \int_t^T \left(w_{\check{Y}_{+,s}}(s)\mathcal{D}_t\check{Y}_{+,s}\mathbb{1}_{\mathcal{R}}(\check{Y}_{+,s}) - w_{\check{Y}_{-,s}}(s)\mathcal{D}_t\check{Y}_{-,s}\mathbb{1}_{\mathcal{R}}(\check{Y}_{-,s})\right)\mathrm{d}s\Bigg)\Bigg|\mathscr{F}_t\Bigg] :$$

$$w(s) \in \mathcal{K}_s^\dagger,\ \forall s \in [t, T];\ \sup_{s\in[0,T]} \|w(s)\|_1 > 0\Bigg\},\quad t \in [0, T],$$

with

$$\gamma_j(\eta\xi|w) = -\frac{\eta\xi}{\langle w, \alpha_j\rangle_{\mathcal{I}}}((\check{u}_j')^{-1})'\left(\frac{\eta\xi}{\langle w, \alpha_j\rangle_{\mathcal{I}}}\right),\quad j \in \mathbb{N} \cap [1, n],$$

$\{\check{Y}_\pm\} = \partial\check{\mathcal{I}}$, $\check{\mathcal{I}}$ *the unrefined index set such that* $\mathcal{I} = \mathcal{R} \cap \check{\mathcal{I}}$

Optimal investment

Solution procedures

- Step 1: Check consumption additivity and parameter space dimensionality unity – if so apply Corollary – directly find two endpoints $\check{Y}_\pm$ constituting the boundary $\partial\check{\mathcal{I}}$ of the unrefined index set.
- Step 2: If Step 1 fails, apply Theorem; in so doing, check if dimensionality reduction holds – if so use J as new boundary, or else identify a representation Castaing (Mallavin differentiable) of $\partial\mathcal{I}_t$ for a given $t \in [0, T]$ (not difficult for regular d-polytopes, sufficient for applications)
- Step 3: Vary w in the dual cone for the full set of optimal investment policies and (optionally) take limits (in $\mathbb{L}^1$) as needed.

Example 1 revisited

- Case (I) and Case (II):

$$\Pi_t^* = \left\{ \frac{X_0 \theta (e^{\rho_p(r,\theta)(T-t)} - 1)}{p\sigma \xi_t^{1/p} (e^{\rho_p(r,\theta)T} - 1)} \right\}, \quad t \in [0,T]$$

- single-valued mean-variance portfolio

- Case (III):

$$\Pi_t^* = \mathrm{cl}_{\mathbb{L}^1} \left\{ \frac{\theta\eta}{\sigma\xi_t} \int_t^T \right.$$
$$\mathbb{E}\left[\frac{2(1-p)\xi_s^2 \psi(\eta\xi_s|w)^{2p+1}}{p(1-p)\psi(\eta\xi_s|w)^p + (1-2p)(w_1\varkappa_1 + w_2\varkappa_2)\psi(\eta\xi_s|w)} \middle| \mathscr{F}_t \right] \mathrm{d}s :$$
$$\left. \eta \text{ given; } w \in \mathbb{R}_+^2,\ \|w\|_1 = 1 \right\}, \quad t \in [0,T]$$

- (strict) set-valued mean-variance portfolio (simulation needed)

Examples 2 and 3 revisited

- Example 2:

$$\Pi_t^* = \mathrm{cl}_{\mathbb{L}^1}\Bigg\{ \frac{\theta}{p\sigma\eta^{1/p}\xi_t} \int_t^T \mathbb{E}\Bigg[\frac{\xi_s^{1-1/p}}{e^{\beta s/p}} \Big((w_1\chi_0 + w_2\chi_{\lambda W_s^{\uparrow}+1})^{1/p} + 1 \Big) \Bigg| \mathscr{F}_s \Bigg] \mathrm{d}s$$
$$+ \frac{\lambda w_2}{p\sigma\eta^{1/p}\xi_t} \int_t^T \mathbb{E}\Bigg[\frac{\xi_s^{1-1/p}}{e^{\beta s/p}} \Bigg(\frac{\chi'_{\lambda W_s^{\uparrow}+1}}{(w_1\chi_0 + w_2\chi_{\lambda W_s^{\uparrow}+1})^{1-1/p}} + 1 \Bigg)$$
$$\times\, \mathrm{erfc}\frac{W_t^{\uparrow} - W_t}{\sqrt{2(s-t)}} \Bigg| \mathscr{F}_t \Bigg] \mathrm{d}s : \ \eta \text{ given; } w \in \mathbb{R}_+^2,\ \|w\|_1 = 1 \Bigg\},\ t \in [0,T],$$

  \- (strict) set-valued portfolio with a mean-variance component and an indecisiveness risk-hedging component (simulation optional)

- Example 3: Formula too long to show (albeit easily computable) - (strict) set-valued portfolio with a mean-variance component, a market risk-hedging component, and an indecisiveness risk-hedging component (simulation needed)

Simulation techniques

Euler-type discretization

- Finite time partition: $P_K = \{t_{l|K} : l \in \mathbb{N} \cap [0, K]\}$, $K \in \mathbb{N}_{++}$, of generic time interval $[0, t]$, $t \in (0, T]$
- Approximate index set: For every $l \leq K$ and $q \in \{1, 2, 3\}$,

$$\hat{I}^{(K)}_{q,t_{l|K}} := I_{q,0} + \sum_{\iota=0}^{l-1} (t_{\iota+1|K} - t_{\iota|K}) f_{q,t_{\iota|K}} + \overline{\mathrm{co}}_{\mathbb{L}^2} \left\{ \sum_{\iota=0}^{l-1} g_{q,k,t_{\iota|K}} (W_{t_{\iota+1|K}} - W_{t_{\iota|K}}) : k \in \mathbb{N}_{++} \right\}$$

and subsequently

$$\hat{\mathcal{I}}^{(K)}_t := \mathcal{R} \cap \left(\hat{I}^{(K)}_{1,t_{K|K}} + \bigcap_{l=0}^{(K-1)\wedge\hat{\ell}} \hat{I}^{(K)}_{2,t_{l+1|K}} + \mathrm{co}_{\mathbb{R}^d} \bigcup_{l=0}^{K-1} \hat{I}^{(K)}_{3,t_{l+1|K}} \right),$$

with ($t_{K|K} \equiv t$)

$$\hat{\ell} := \inf \left\{ l \in \mathbb{N} \cap [1, K] : \mathrm{card} \bigcap_{l=0}^{K-1} \hat{I}^{(K)}_{2,t_{l+1}|K} = 0 \right\} - 1;$$

Simulation techniques

Euler-type discretization

Proposition

For any $t \in [0, T]$ with corresponding partition $P_K \subsetneq [0, t]$ it holds that

$$\lim_{K \to \infty} \mathbb{E}\left[d_{\mathrm{H}}\left(\mathcal{I}_t, \hat{\mathcal{I}}_t^{(K)}\right)\right] = 0.$$

- Remarks: Numerical integration over $\mathcal{I}$ is implementable via Gauss-Kronrod quadratures ([Ma et al., 1996]); ξ is single-valued – standard Euler discretization schemes apply.

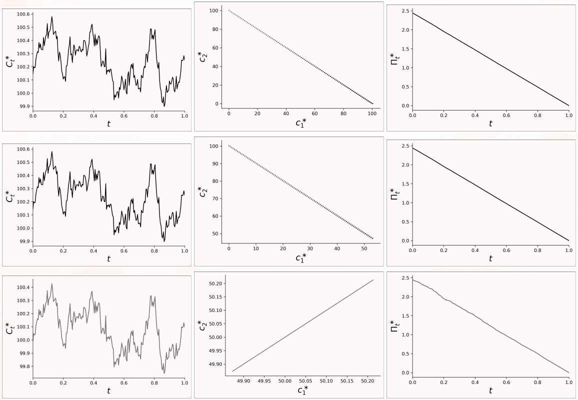
Numerical experiments

- Common parameter values: $r = 0.001$, $\mu = 0.02$, $T = 1$ (year), $X_0 = \$100.00$; $K = 200$ (uniform)

Specific parameter values

Category	parameter	dual cone (approximation)
Ex. 1 Case (I)	$\sigma = 0.36$, $p = 6$	$\mathbb{R}^2_+$ (100-partition)
Ex. 1 Case (II)	$\sigma = 0.36$, $p = 6$, $\chi = 3$	$\mathbb{R}^2_+$ (100-partition)
Ex. 1 Case (III)	$\sigma = 0.36$, $p = 6$, $[\varkappa_1, \varkappa_2] = [1, 2]$	$\mathbb{R}^2_+$ (100-partition)
Ex. 2	$\sigma = 0.36$, $p = 6$, $\beta = 0.05$, $\lambda = 0.2$, $\chi = \mathrm{id} \wedge 5$	$\mathbb{R}^2_+$ (100-partition)
Ex. 3	$\sigma_0 = 0.36$, $\kappa = 0.1$, $\varsigma = 0.8$, $\beta = 0.05$, $\lambda = 0.2$, $p_\circ = 5$, $\chi = p/3 = \mathrm{id} \wedge 5$	$(\mathcal{C}_\mathrm{b}(\mathbb{R}_+ \times [1, \infty); \mathbb{R}_+))^\dagger$ (100-w_1-partition, $w_2 = \delta_{\{2.2\}}$)

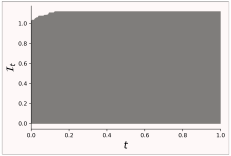
Numerical experiments
Ex. 1: Case (I), Case (II), Case (III)

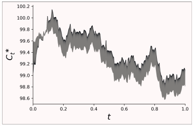


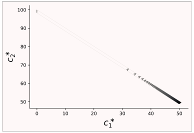
Simulation techniques

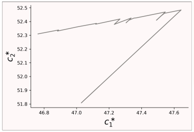
Numerical experiments

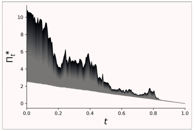
<u>Ex. 2:</u>

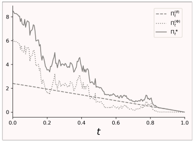








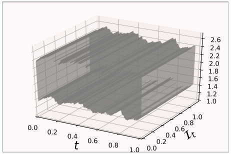


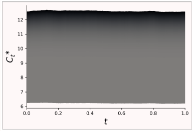


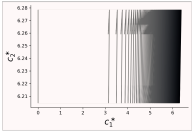
Simulation techniques

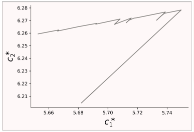
Numerical experiments

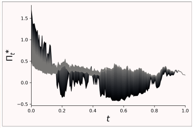
Ex. 3:

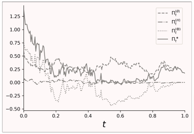












Conclusions

Concluding remarks

- Study: Martingale solutions to Merton's optimal consumption-investment problem with time-varying incomplete preferences
- Considerations: Important external psychological factors with abundant empirical evidence; a model of time-varying incomplete preferences by a stochastic multi-utility index set following a d-dimensional set-valued Itô process with two monotone components
- Methodology: A unified approach towards characterizing the pool of optimal consumption-bequest policies by defining weight functionals in a stochastic dual cone; the associated optimal investment policy achieved via exploiting Malliavin calculus in a set-valued setting, alongside techniques from stochastic geometry
- Implementation of optimal policies: (1) Pick a weight functional w from its identified dual cone for computing consumption(-bequest) policies following established procedures; (2) Using the same functional compute the corresponding optimal investment policies (limiting cases considerable, or utopian policies)

Conclusions

Economic implications

- Time-varying incomplete preferences: Imprecise tastes and preferences for randomization; externally fluctuating preferences – stochastic patience, increasing/decreasing attention, time-varying risk aversion, etc.
- Multi-utility maximization: To seek all equivalently optimal (admissible) consumption policies, with a floating rule for inflicting conflicts; flexible (switching) optimal policies over time
- Optimal investment: A new portfolio decomposition highlighting a hedging demand for indecisiveness risks (esp. for material preference changes); flexible policies matched to consumption policies
- Equilibrium outlook: Capacity to unravel the dynamic variability of asset prices (puzzles); fuzzy equilibria

Conclusions

Further research

- Coalescence of imprecise tastes into beliefs ([Nau, 2006]): A partially robust multi-utility maximization problem, e.g., as in Motivating Ex., with objective function

$$\inf_{\mathbb{P}\in\P} \mathbb{E}^{\mathbb{P}}\left[u(c_{\mathrm{A}}, c_{\mathrm{B}}) - \mathcal{C}_{\mathrm{b}}([0,\chi];\mathbb{R}_+)\right] = \bigcap_{\mathbb{P}\in\P} \left(\mathbb{E}^{\mathbb{P}}\left[\{u_0(c_{\mathrm{A}}, c_{\mathrm{B}}), u_\chi(c_{\mathrm{A}}, c_{\mathrm{B}})\}\right] - \mathbb{R}^2_+\right);$$

  infimum taken over a convex space ¶ of equivalent probability measures each admitting a bijective correspondence with a duplet $(\mu^{\mathbb{P}}, \sigma^{\mathbb{P}})$ characterizing the market; to be understood in a set-valued sense ([Hamel et al., 2015])
- Habitual indecisiveness: Mere-exposure effect; habit formation; internally driven incomplete preferences; endogenous multi-utility index set; nonlinear scalarization methods; set-valued BSDE (*an ongoing study*)
- More versatile market characteristics: Jumps ([Michelbrink and Le, 2012]), memory effects – fractionally integrated Lévy-Itô processes in a set-valued setting (*model construction already studied*)
- General incomplete-market equilibrium framework (*in preparation*)

Q&A

Thank you!

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