

# A Two-Player Price Impact Game

Moritz Voss

UC Santa Barbara

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# Single Player Problem

Bank, Soner, V. ('17)

For a given predictable  $\xi \in L^2(\mathbb{P} \otimes dt)$  and given  $x \in \mathbb{R}$ ,  $\sigma > 0$ ,  $\lambda > 0$ ,  $\gamma > 0$ , find an absolutely continuous, adapted process  $X = x + \int_0^\cdot \alpha_t dt$  with  $\alpha \in L^2(\mathbb{P} \otimes dt)$  which minimizes

$$\mathbb{E} \left[ \int_0^T (X_t - \xi_t)^2 \sigma dt + \lambda \int_0^T \alpha_t^2 dt + \gamma \int_0^T \alpha_t \cdot (X_t - x) dt \right]$$

subject to  $X_T = \Xi_T$  for some given  $\Xi_T \in L^2(\mathcal{F}_{T-}, \mathbb{P})$ .

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## References:

Kohlmann/Tang ('02), Rogers/Singh ('10), Frei/Westray ('13), Horst/Naujokat ('14), Almgren/Li ('14), Cartea/Jaimungal ('15), Bank/Soner/V. ('17), ...

# Single Player Problem: Solution

## Theorem (Bank, Soner, V. ('17))

Under suitable assumptions the optimal control  $\hat{\alpha}$  with strategy  $\hat{X} = x + \int_0^\cdot \hat{\alpha}_t dt$  is given by

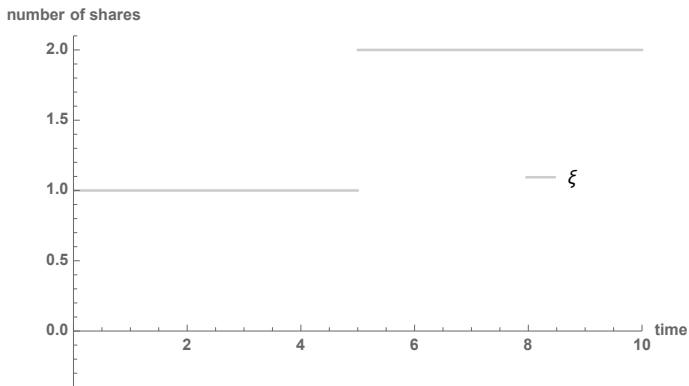
$$\hat{\alpha}_t = \tilde{c}_t \cdot \left( \hat{\xi}_t - \hat{X}_t \right)$$

with deterministic function  $\tilde{c}_t > 0$  satisfying  $\lim_{t \uparrow T} \tilde{c}_t = +\infty$  and

$$\hat{\xi}_t = \tilde{w}_t^1 \cdot \mathbb{E}[\Xi_T | \mathcal{F}_t] + \tilde{w}_t^2 \cdot \mathbb{E} \left[ \int_t^T \xi_s \cdot \tilde{K}(t, s) ds \mid \mathcal{F}_t \right]$$

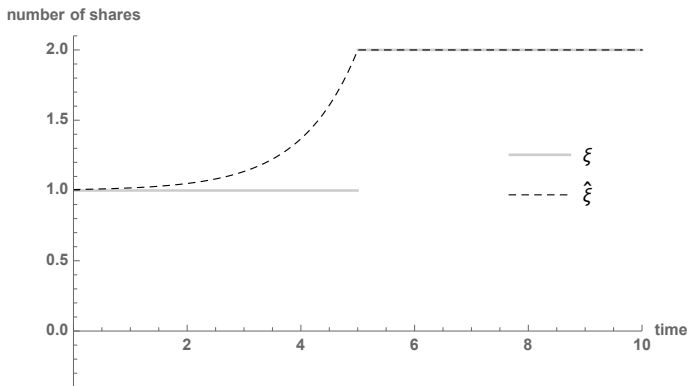
with deterministic nonnegative weights  $\tilde{w}_t^1 + \tilde{w}_t^2 = 1$ ,  $\lim_{t \uparrow T} \tilde{w}_t^1 = 1$ ,  $\lim_{t \uparrow T} \tilde{w}_t^2 = 0$  and deterministic kernel  $\tilde{K}$ .

# Illustration: Stock-Buying Schedule



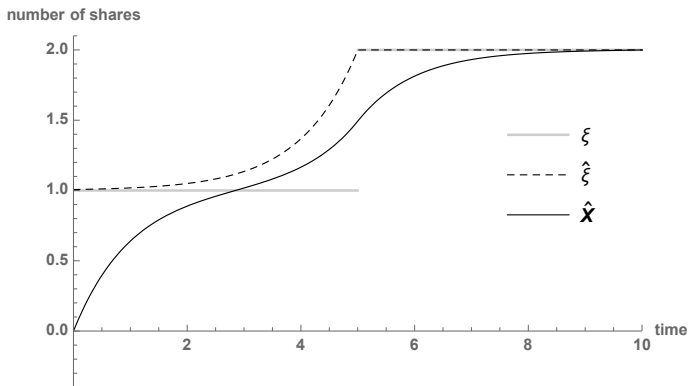
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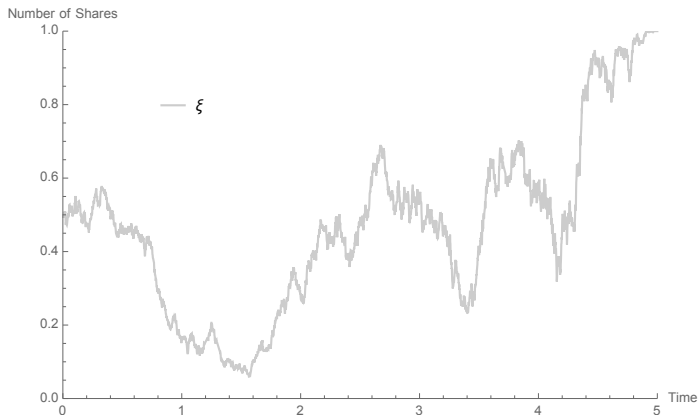
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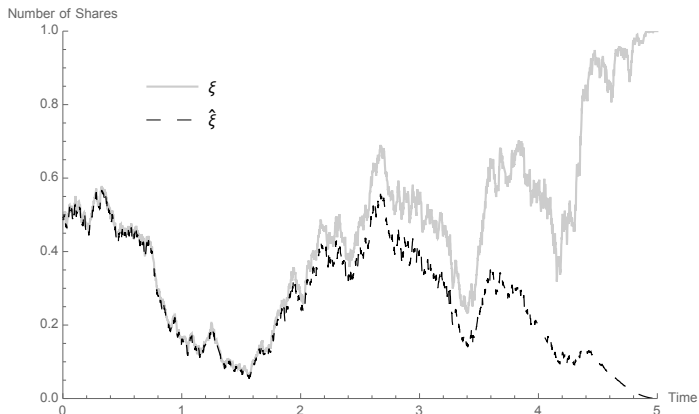


# Illustration: Running after the delta



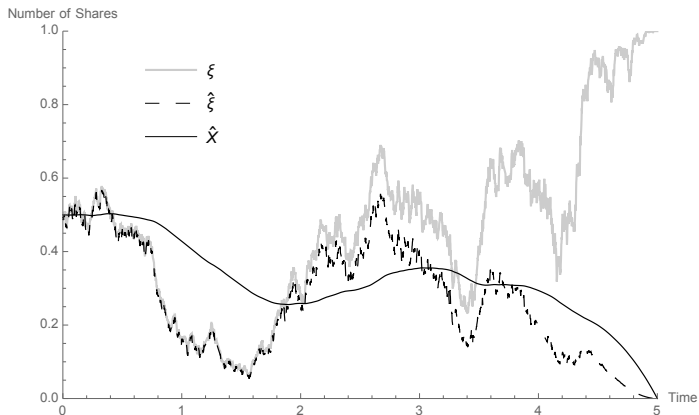
Single player:  $x = 1/2$ ,  $\xi_t = \Phi\left(\frac{P_t - P_0}{\sqrt{\sigma(T-t)}}\right)$ ,  $\Xi_T = 0$ .

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## Two-Player Problem

**Player 1:**  $X^1 = x^1 + \int_0^{\cdot} \alpha_t^1 dt$  and targets  $\xi^1, \Xi_T^1$

$$J^1(\alpha^1) \triangleq \mathbb{E} \left[ \int_0^T (X_t^1 - \xi_t^1)^2 \sigma dt + \lambda \int_0^T \alpha_t^1 \cdot (\alpha_t^1) dt \right] \rightarrow \min_{\alpha^1} \text{ s.t. } X_0^1 = x^1, X_T^1 = \Xi_T^1$$

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Two strategic agents are competing for liquidity!



# Nash equilibrium

A pair of admissible strategies  $(\hat{\alpha}^1, \hat{\alpha}^2)$  is called a **Nash equilibrium** for the game if for all admissible strategies  $\alpha^1, \alpha^2$  we have

$$J^1(\hat{\alpha}^1, \hat{\alpha}^2) \leq J^1(\alpha^1, \hat{\alpha}^2) \quad \text{and} \quad J^2(\hat{\alpha}^1, \hat{\alpha}^2) \leq J^2(\hat{\alpha}^1, \alpha^2),$$

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**Note:** We will only consider an **open loop** Nash equilibrium!

## Related Literature: Optimal Portfolio Liquidation

Player 1 (“distressed trader”):  $x^1 > 0, \Xi_T^1 = 0, \xi^1 \equiv 0$

Player 2 (“predator”):  $x^2 = \Xi_T^2 = 0, \xi^2 \equiv 0$

## Related Literature: Optimal Portfolio Liquidation

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Explicit results:

1. **Carlin, Lobo, Viswanathan ('07)**: risk-neutral ( $\sigma = 0$ ), deterministic open loop
2. **Schied & Zhang ('14)**: mean-variance optimization ( $\sigma > 0$ ), deterministic open loop
3. **Schöneborn & Schied ('07)**: extend Carlin et al. by allowing for longer time horizon  $\tilde{T} > T$  for predator (two stage model), deterministic open loop

Numerical analysis:

4. **Carmona & Yang ('08)**: adopt optimization problem of Carlin et al., stochastic closed loop strategies, noise traders, also allow longer time horizon  $\tilde{T} > T$  for predator (two stage model)

# Nash equilibrium in optimal liquidation

Qualitative property of Nash equilibrium:

predatory trading vs. liquidity provision

# Nash equilibrium in optimal liquidation

Qualitative property of Nash equilibrium:

predatory trading vs. liquidity provision

- ▶ Carlin et al.: preying **always** occurs
- ▶ Schöneborn & Schied, Carmona & Yang: Two-stage framework extending Carlin et al.
  - ▶ preying in plastic market ( $\gamma \gg \lambda$ )
  - ▶ liquidity provision in elastic market ( $\lambda \gg \gamma$ )
- ▶ Schied & Zhang: predatory trading or liquidity provision occurs

## Two-Player Problem: Solution

### Theorem

Under suitable assumptions there exists a unique open loop Nash equilibrium  $(\hat{\alpha}^1, \hat{\alpha}^2)$  with **Player 1's** control  $\hat{\alpha}^1$  given by

$$\hat{\alpha}_t^1 = c_t \cdot \left( \hat{\xi}_t^1 - w_t^5 \cdot \hat{X}_t^2 - \hat{X}_t^1 \right)$$

with deterministic function  $c_t > 0$  satisfying  $\lim_{t \uparrow T} c_t = +\infty$  and

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$$\begin{aligned} \hat{\xi}_t^1 &= w_t^1 \cdot \mathbb{E}[\Xi_T^1 + \Xi_T^2 \mid \mathcal{F}_t] + w_t^2 \cdot \mathbb{E}[\Xi_T^1 - \Xi_T^2 \mid \mathcal{F}_t] \\ &\quad + w_t^3 \cdot \mathbb{E} \left[ \int_t^T (\xi_s^1 + \xi_s^2) \cdot K^1(t, s) ds \mid \mathcal{F}_t \right] \\ &\quad + w_t^4 \cdot \mathbb{E} \left[ \int_t^T (\xi_s^1 - \xi_s^2) \cdot K^2(t, s) ds \mid \mathcal{F}_t \right] \end{aligned}$$

with deterministic nonnegative weights  $w_t^1 + w_t^2 + w_t^3 + w_t^4 = 1$ ,  $w_t^5 \in [-1, 1]$ ,  $\lim_{t \uparrow T} w_t^{3,4,5} = 0$ ,  $\lim_{t \uparrow T} w_t^{1,2} = 1/2$ , and deterministic kernels  $K^1, K^2$ .



## Two-Player Problem: Solution

### Theorem (cont.)

And, similarly, with **Player 2's** control  $\hat{\alpha}^2$  given by

$$\hat{\alpha}_t^2 = c_t \cdot \left( \hat{\xi}_t^2 - w_t^5 \cdot \hat{X}_t^1 - \hat{X}_t^2 \right)$$

and

$$\begin{aligned} \hat{\xi}_t^2 &= w_t^1 \cdot \mathbb{E}[\Xi_T^1 + \Xi_T^2 \mid \mathcal{F}_t] + w_t^2 \cdot \mathbb{E}[\Xi_T^2 - \Xi_T^1 \mid \mathcal{F}_t] \\ &\quad + w_t^3 \cdot \mathbb{E} \left[ \int_t^T (\xi_s^1 + \xi_s^2) \cdot K^1(t, s) ds \mid \mathcal{F}_t \right] \\ &\quad + w_t^4 \cdot \mathbb{E} \left[ \int_t^T (\xi_s^2 - \xi_s^1) \cdot K^2(t, s) ds \mid \mathcal{F}_t \right]. \end{aligned}$$

# Optimal Portfolio Liquidation Revisited

Schied & Zhang ('17)

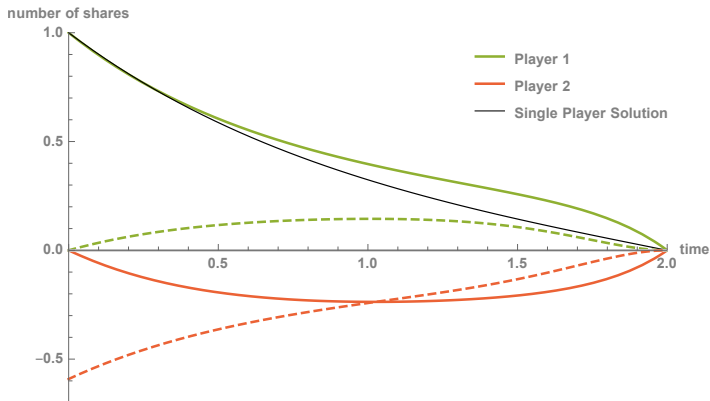
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Player 2 (“predator”):  $x^2 = 0, \Xi_T^2 = 0, \xi_t^2 \equiv 0$ .

Parameters:  $\sigma = 1, \lambda = 1, \gamma = 2$ .

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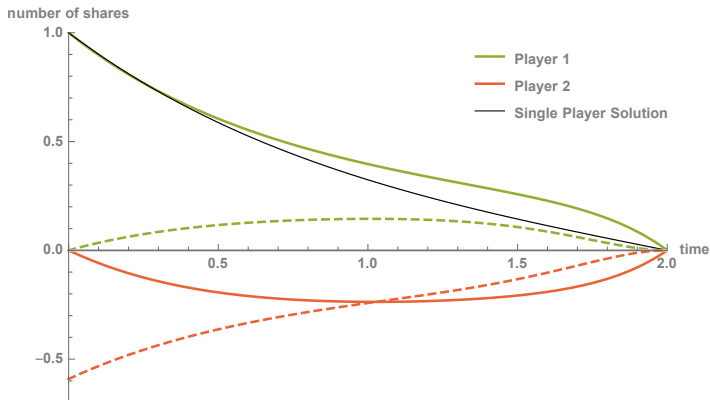
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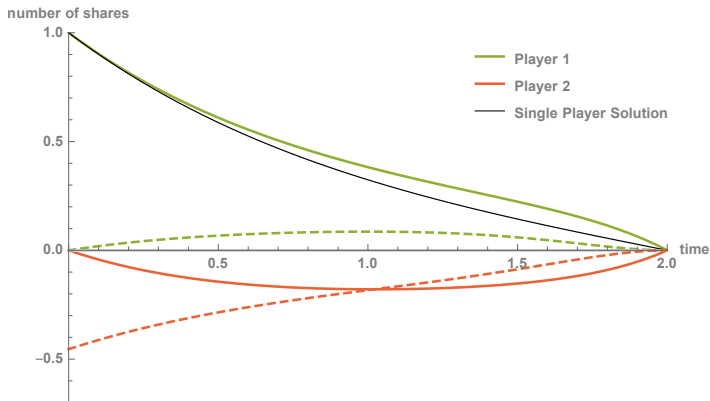
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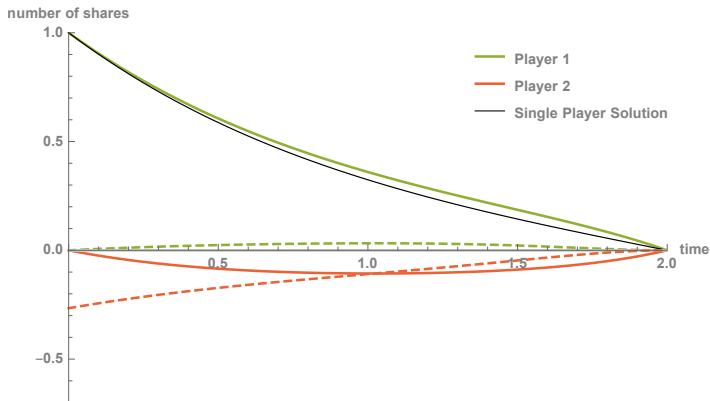
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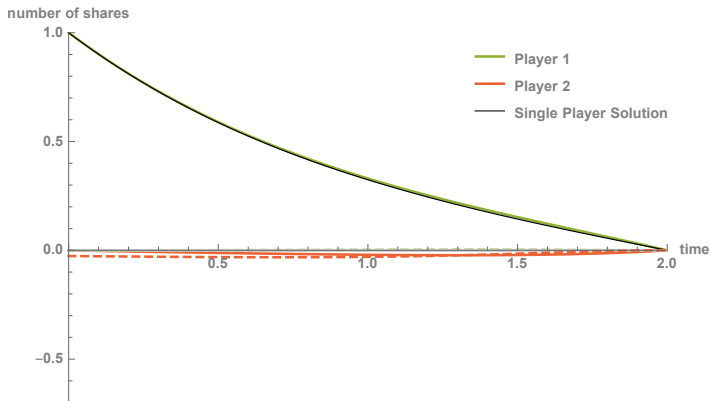
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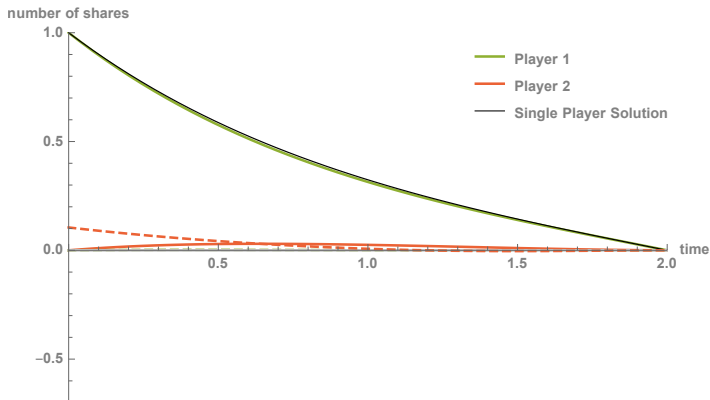
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Schied & Zhang ('17)



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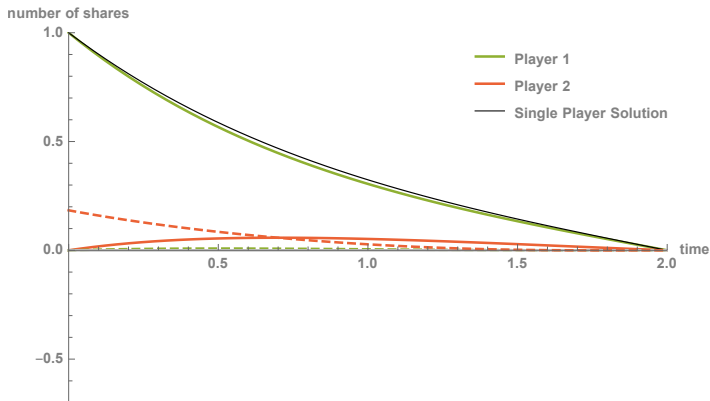
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Parameters:  $\sigma = 1$ ,  $\lambda = 1$ ,  $\gamma = 0.25$ .



# Optimal Portfolio Liquidation Revisited

Schied & Zhang ('17)



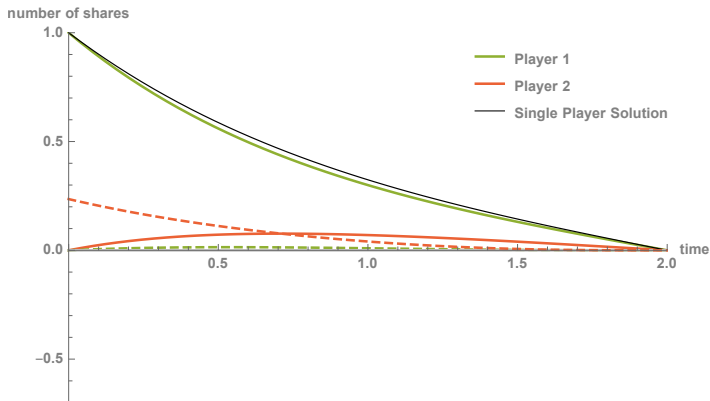
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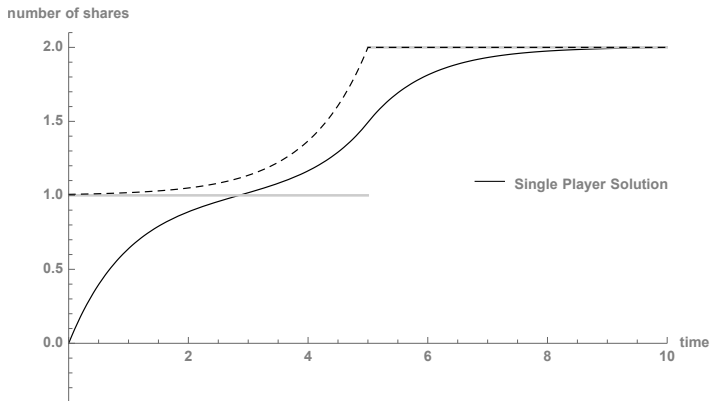


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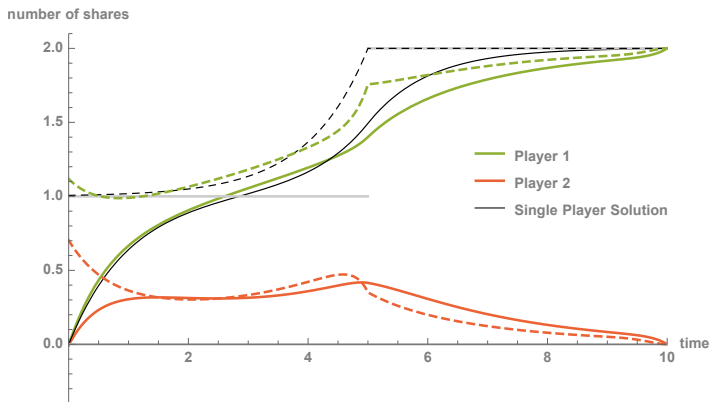


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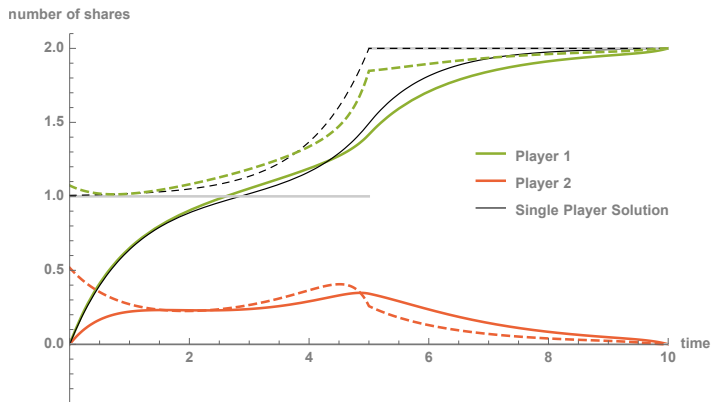


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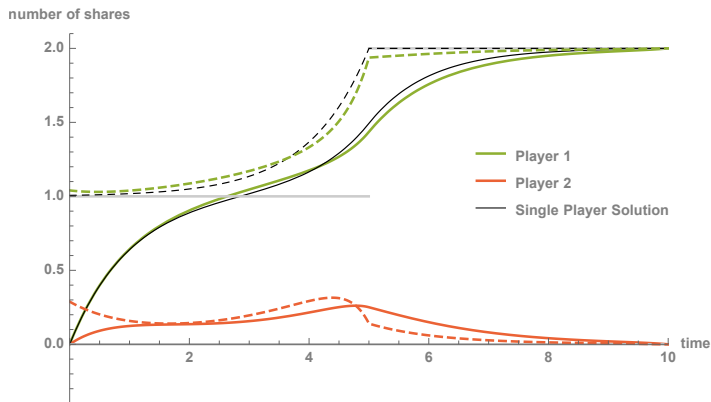


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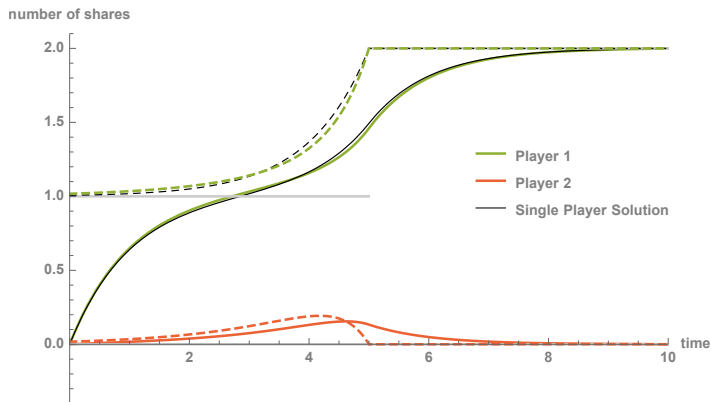


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Parameters:  $\sigma = 1$ ,  $\lambda = 1$ ,  $\gamma = 1$ .

# Stock-Buying Schedule

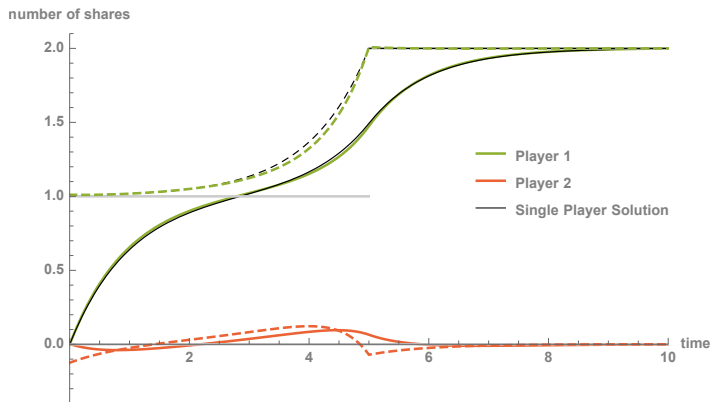


Player 1:  $x^1 = 0$ ,  $\xi_t^1 = 1 \cdot 1_{\{0 \leq t < 5\}} + 2 \cdot 1_{\{5 \leq t \leq 10\}}$ ,  $\Xi_T^1 = 2$ .

Player 2 ("predator"):  $x^2 = 0$ ,  $\xi_t^2 \equiv 0$ ,  $\Xi_T^2 = 0$ .

Parameters:  $\sigma = 1$ ,  $\lambda = 1$ ,  $\gamma = 0.5$ .

# Stock-Buying Schedule



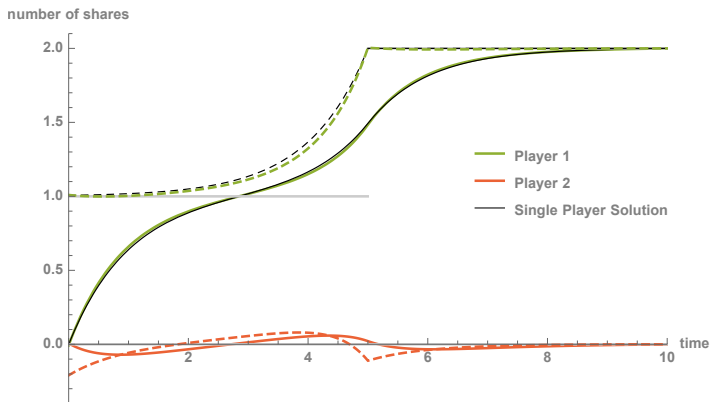
Player 1:  $x^1 = 0$ ,  $\xi_t^1 = 1 \cdot 1_{\{0 \leq t < 5\}} + 2 \cdot 1_{\{5 \leq t \leq 10\}}$ ,  $\Xi_T^1 = 2$ .

Player 2 ("predator"):  $x^2 = 0$ ,  $\xi_t^2 \equiv 0$ ,  $\Xi_T^2 = 0$ .

Parameters:  $\sigma = 1$ ,  $\lambda = 1$ ,  $\gamma = 0.25$ .



# Stock-Buying Schedule

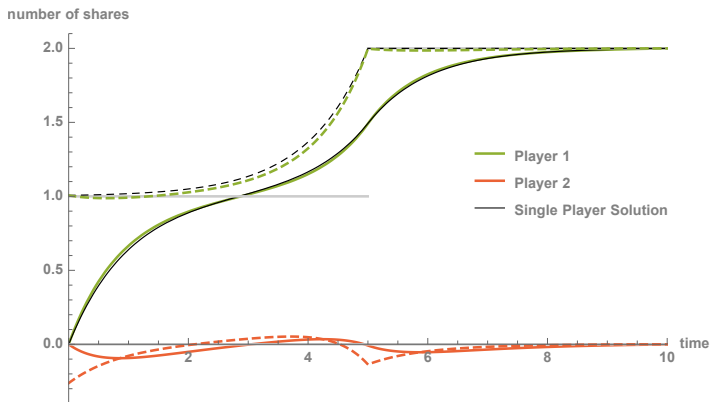


Player 1:  $x^1 = 0$ ,  $\xi_t^1 = 1 \cdot 1_{\{0 \leq t < 5\}} + 2 \cdot 1_{\{5 \leq t \leq 10\}}$ ,  $\Xi_T^1 = 2$ .

Player 2 ("predator"):  $x^2 = 0$ ,  $\xi_t^2 \equiv 0$ ,  $\Xi_T^2 = 0$ .

Parameters:  $\sigma = 1$ ,  $\lambda = 1$ ,  $\gamma = 0.1$ .

# Stock-Buying Schedule

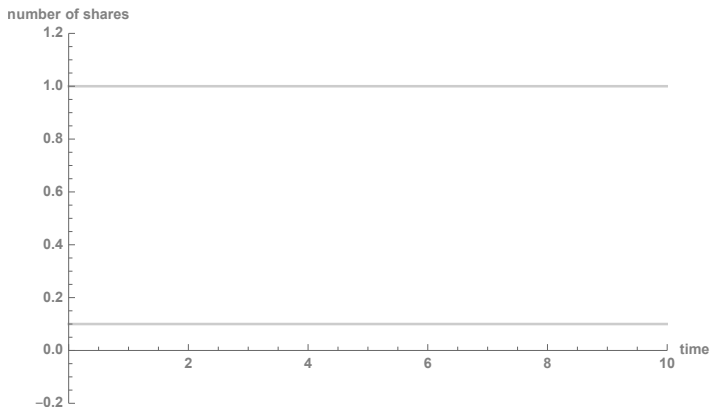


Player 1:  $x^1 = 0$ ,  $\xi_t^1 = 1 \cdot 1_{\{0 \leq t < 5\}} + 2 \cdot 1_{\{5 \leq t \leq 10\}}$ ,  $\Xi_T^1 = 2$ .

Player 2 ("predator"):  $x^2 = 0$ ,  $\xi_t^2 \equiv 0$ ,  $\Xi_T^2 = 0$ .

Parameters:  $\sigma = 1$ ,  $\lambda = 1$ ,  $\gamma = 0$ .

# Constant Inventory Targets

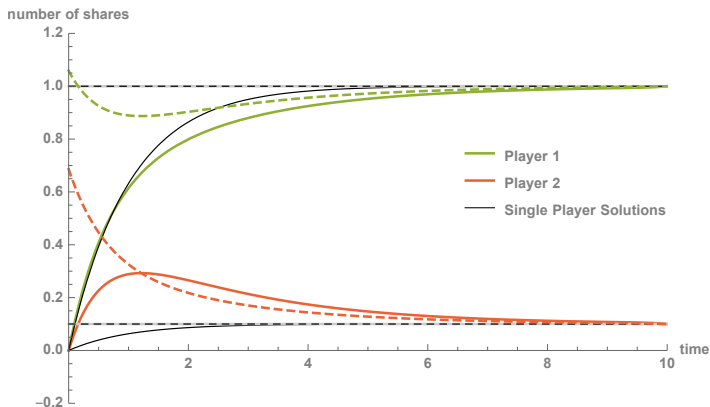


Player 1:  $x^1 = 0$ ,  $\xi_t^1 \equiv 1$ ,  $\Xi_T^1 = 1$ .

Player 2:  $x^2 = 0$ ,  $\xi_t^2 \equiv 0.1$ ,  $\Xi_T^2 = 0.1$ .

Parameters:  $\sigma = 1$ ,  $\lambda = 1$ ,  $\gamma = 2$ .

# Constant Inventory Targets

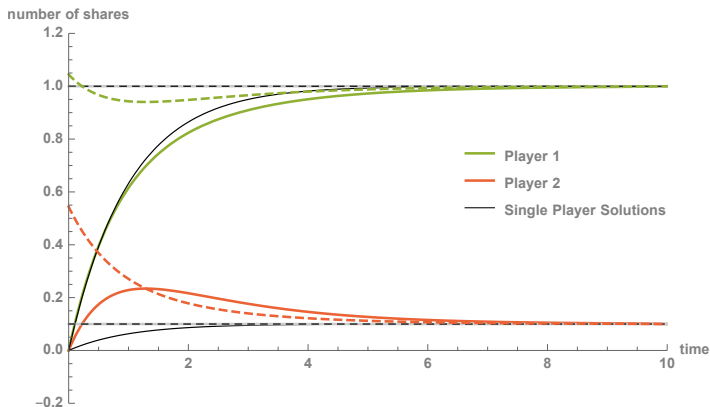


Player 1:  $x^1 = 0$ ,  $\xi_t^1 \equiv 1$ ,  $\Xi_T^1 = 1$ .

Player 2:  $x^2 = 0$ ,  $\xi_t^2 \equiv 0.1$ ,  $\Xi_T^2 = 0.1$ .

Parameters:  $\sigma = 1$ ,  $\lambda = 1$ ,  $\gamma = 2$ .

# Constant Inventory Targets

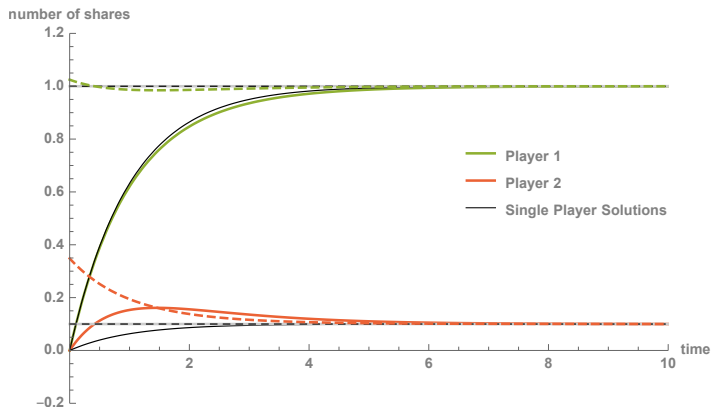


Player 1:  $x^1 = 0$ ,  $\xi_t^1 \equiv 1$ ,  $\Xi_T^1 = 1$ .

Player 2:  $x^2 = 0$ ,  $\xi_t^2 \equiv 0.1$ ,  $\Xi_T^2 = 0.1$ .

Parameters:  $\sigma = 1$ ,  $\lambda = 1$ ,  $\gamma = 1.5$ .

# Constant Inventory Targets

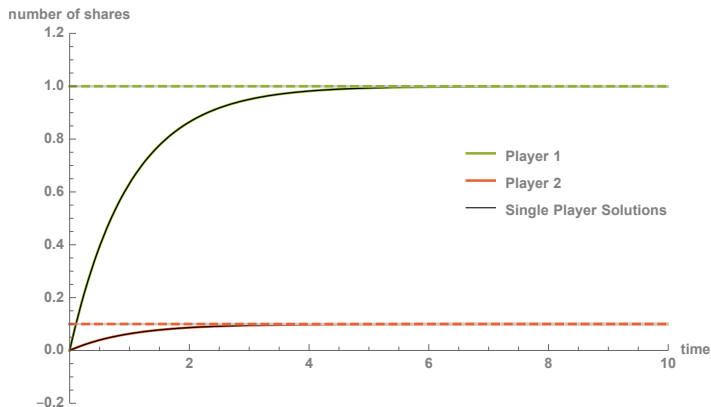


Player 1:  $x^1 = 0$ ,  $\xi_t^1 \equiv 1$ ,  $\Xi_T^1 = 1$ .

Player 2:  $x^2 = 0$ ,  $\xi_t^2 \equiv 0.1$ ,  $\Xi_T^2 = 0.1$ .

Parameters:  $\sigma = 1$ ,  $\lambda = 1$ ,  $\gamma = 1$ .

# Constant Inventory Targets

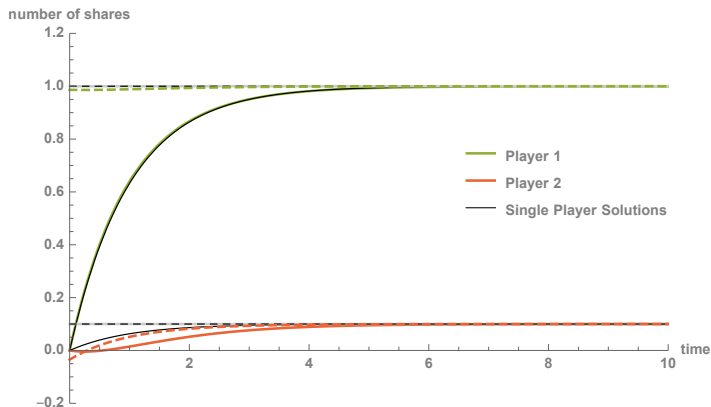


Player 1:  $x^1 = 0$ ,  $\xi_t^1 \equiv 1$ ,  $\Xi_T^1 = 1$ .

Player 2:  $x^2 = 0$ ,  $\xi_t^2 \equiv 0.1$ ,  $\Xi_T^2 = 0.1$ .

Parameters:  $\sigma = 1$ ,  $\lambda = 1$ ,  $\gamma = 0.5$ .

# Constant Inventory Targets



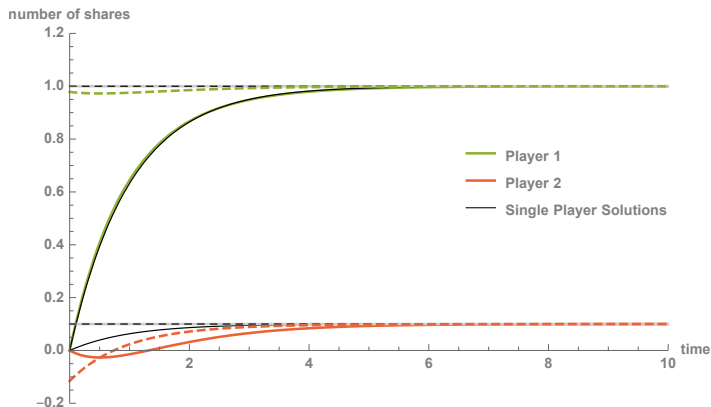
Player 1:  $x^1 = 0$ ,  $\xi_t^1 \equiv 1$ ,  $\Xi_T^1 = 1$ .

Player 2:  $x^2 = 0$ ,  $\xi_t^2 \equiv 0.1$ ,  $\Xi_T^2 = 0.1$ .

Parameters:  $\sigma = 1$ ,  $\lambda = 1$ ,  $\gamma = 0.25$ .



# Constant Inventory Targets

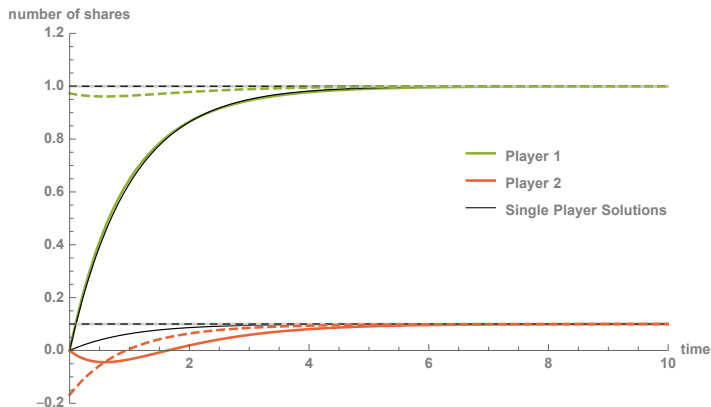


Player 1:  $x^1 = 0$ ,  $\xi_t^1 \equiv 1$ ,  $\Xi_T^1 = 1$ .

Player 2:  $x^2 = 0$ ,  $\xi_t^2 \equiv 0.1$ ,  $\Xi_T^2 = 0.1$ .

Parameters:  $\sigma = 1$ ,  $\lambda = 1$ ,  $\gamma = 0.1$ .

# Constant Inventory Targets

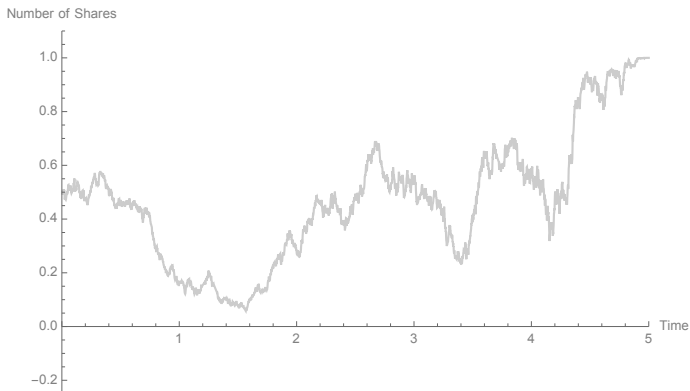


Player 1:  $x^1 = 0$ ,  $\xi_t^1 \equiv 1$ ,  $\Xi_T^1 = 1$ .

Player 2:  $x^2 = 0$ ,  $\xi_t^2 \equiv 0.1$ ,  $\Xi_T^2 = 0.1$ .

Parameters:  $\sigma = 1$ ,  $\lambda = 1$ ,  $\gamma = 0$ .

## Running after the delta

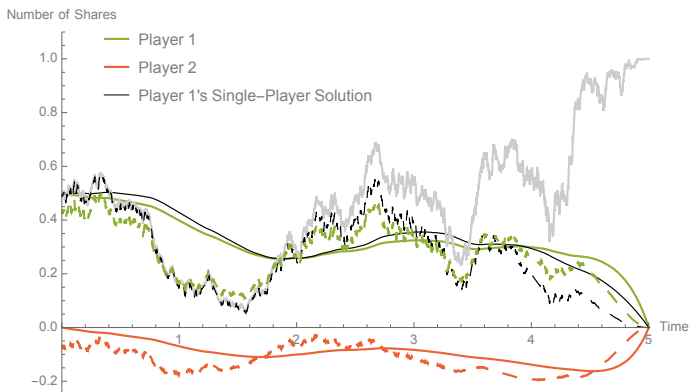


Player 1:  $x^1 = 1/2$ ,  $\xi_t^1 = \text{delta-hedge}$ ,  $\Xi_T^1 = 0$ .

Player 2 ("predator"):  $x^2 = 0$ ,  $\xi_t^2 \equiv 0$ ,  $\Xi_T^2 = 0$ .

Parameters:  $\sigma = 1$ ,  $\lambda = 1$ ,  $\gamma = 2$ .

# Running after the delta

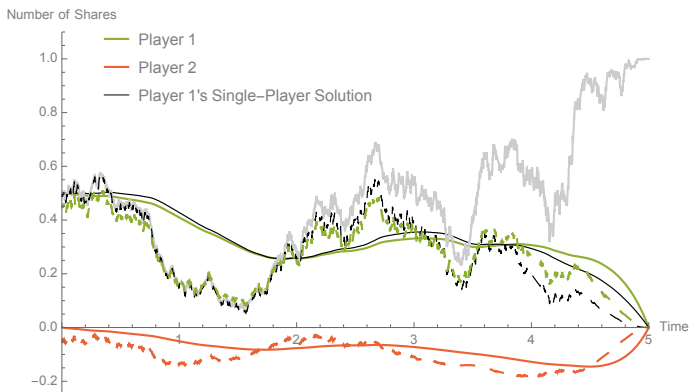


Player 1:  $x^1 = 1/2$ ,  $\xi_t^1 = \text{delta-hedge}$ ,  $\Xi_T^1 = 0$ .

Player 2 ("predator"):  $x^2 = 0$ ,  $\xi_t^2 \equiv 0$ ,  $\Xi_T^2 = 0$ .

Parameters:  $\sigma = 1$ ,  $\lambda = 1$ ,  $\gamma = 2$ .

# Running after the delta

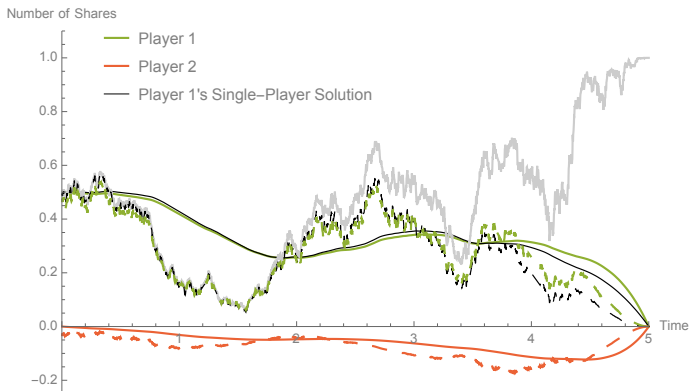


Player 1:  $x^1 = 1/2$ ,  $\xi_t^1$  = delta-hedge,  $\Xi_T^1 = 0$ .

Player 2 (“predator”):  $x^2 = 0$ ,  $\xi_t^2 \equiv 0$ ,  $\Xi_T^2 = 0$ .

Parameters:  $\sigma = 1$ ,  $\lambda = 1$ ,  $\gamma = 1.5$ .

# Running after the delta

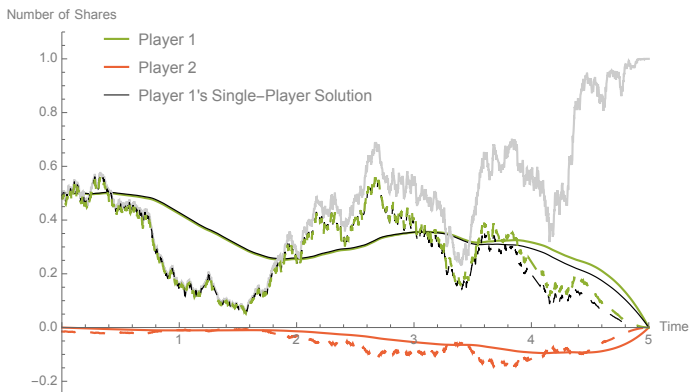


Player 1:  $x^1 = 1/2$ ,  $\xi_t^1 = \text{delta-hedge}$ ,  $\Xi_T^1 = 0$ .

Player 2 ("predator"):  $x^2 = 0$ ,  $\xi_t^2 \equiv 0$ ,  $\Xi_T^2 = 0$ .

Parameters:  $\sigma = 1$ ,  $\lambda = 1$ ,  $\gamma = 1.0$ .

# Running after the delta

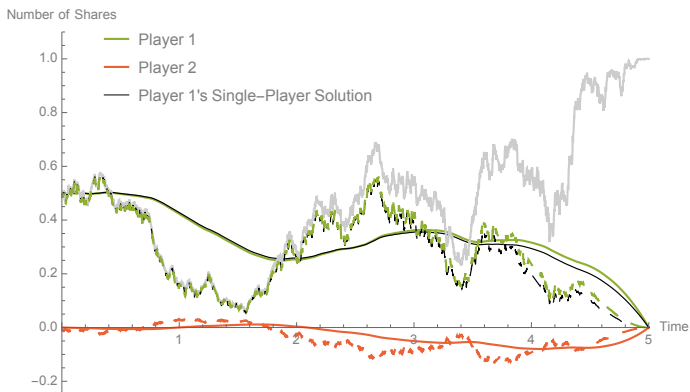


Player 1:  $x^1 = 1/2$ ,  $\xi_t^1 = \text{delta-hedge}$ ,  $\Xi_T^1 = 0$ .

Player 2 ("predator"):  $x^2 = 0$ ,  $\xi_t^2 \equiv 0$ ,  $\Xi_T^2 = 0$ .

Parameters:  $\sigma = 1$ ,  $\lambda = 1$ ,  $\gamma = 0.5$ .

# Running after the delta



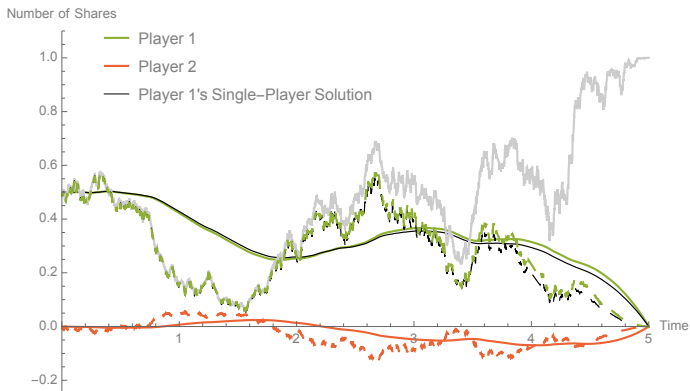
Player 1:  $x^1 = 1/2$ ,  $\xi_t^1 = \text{delta-hedge}$ ,  $\Xi_T^1 = 0$ .

Player 2 ("predator"):  $x^2 = 0$ ,  $\xi_t^2 \equiv 0$ ,  $\Xi_T^2 = 0$ .

Parameters:  $\sigma = 1$ ,  $\lambda = 1$ ,  $\gamma = 0.25$ .



# Running after the delta

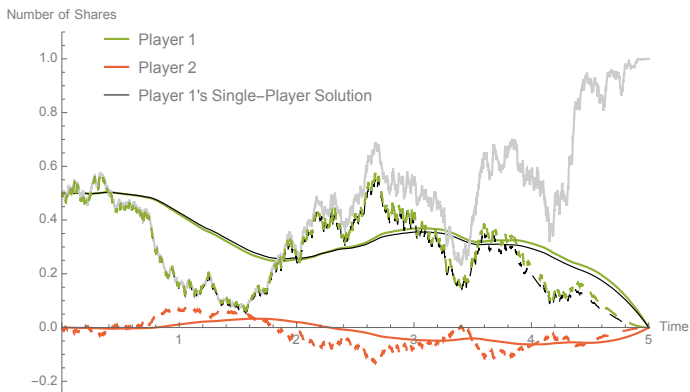


Player 1:  $x^1 = 1/2$ ,  $\xi_t^1 = \text{delta-hedge}$ ,  $\Xi_T^1 = 0$ .

Player 2 ("predator"):  $x^2 = 0$ ,  $\xi_t^2 \equiv 0$ ,  $\Xi_T^2 = 0$ .

Parameters:  $\sigma = 1$ ,  $\lambda = 1$ ,  $\gamma = 0.1$ .

# Running after the delta



Player 1:  $x^1 = 1/2$ ,  $\xi_t^1 = \text{delta-hedge}$ ,  $\Xi_T^1 = 0$ .

Player 2 ("predator"):  $x^2 = 0$ ,  $\xi_t^2 \equiv 0$ ,  $\Xi_T^2 = 0$ .

Parameters:  $\sigma = 1$ ,  $\lambda = 1$ ,  $\gamma = 0$ .

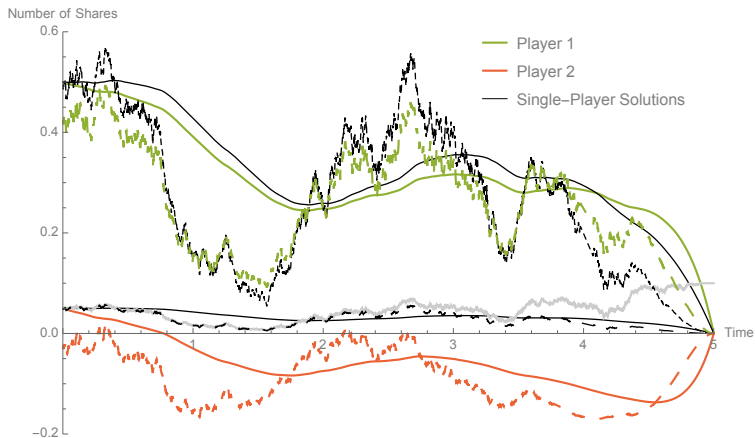
## Running after the delta

Player 1:  $x^1 = 1/2$ ,  $\xi_t^1 = \text{delta-hedge}$ ,  $\Xi_T^1 = 0$ .

Player 2:  $x^2 = 1/20$ ,  $\xi_t^2 = 0.1 \cdot \xi_t^1$ ,  $\Xi_T^2 = 0$ .

Parameters:  $\sigma = 1$ ,  $\lambda = 1$ ,  $\gamma = 2$ .

# Running after the delta

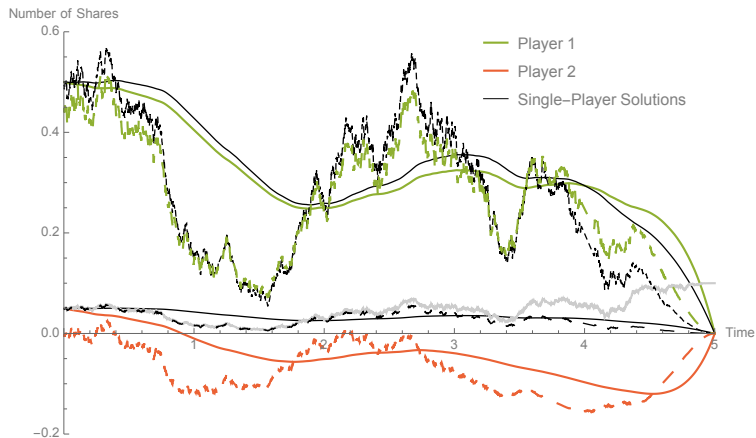


Player 1:  $x^1 = 1/2$ ,  $\xi_t^1 = \text{delta-hedge}$ ,  $\Xi_T^1 = 0$ .

Player 2:  $x^2 = 1/20$ ,  $\xi_t^2 = 0.1 \cdot \xi_t^1$ ,  $\Xi_T^2 = 0$ .

Parameters:  $\sigma = 1$ ,  $\lambda = 1$ ,  $\gamma = 2$ .

# Running after the delta

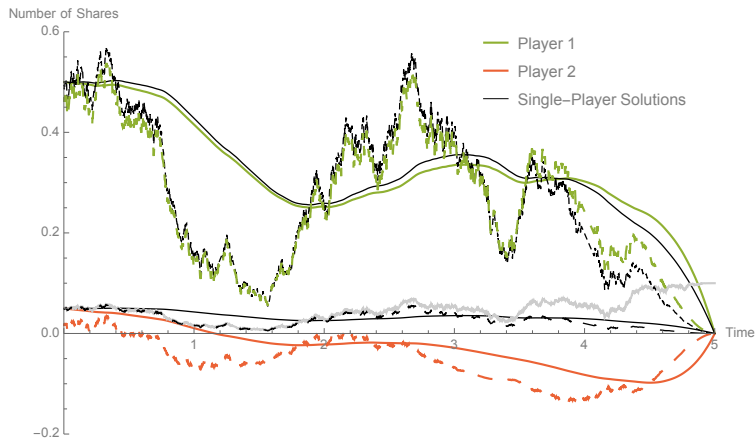


Player 1:  $x^1 = 1/2$ ,  $\xi_t^1 = \text{delta-hedge}$ ,  $\Xi_T^1 = 0$ .

Player 2:  $x^2 = 1/20$ ,  $\xi_t^2 = 0.1 \cdot \xi_t^1$ ,  $\Xi_T^2 = 0$ .

Parameters:  $\sigma = 1$ ,  $\lambda = 1$ ,  $\gamma = 1.5$ .

# Running after the delta

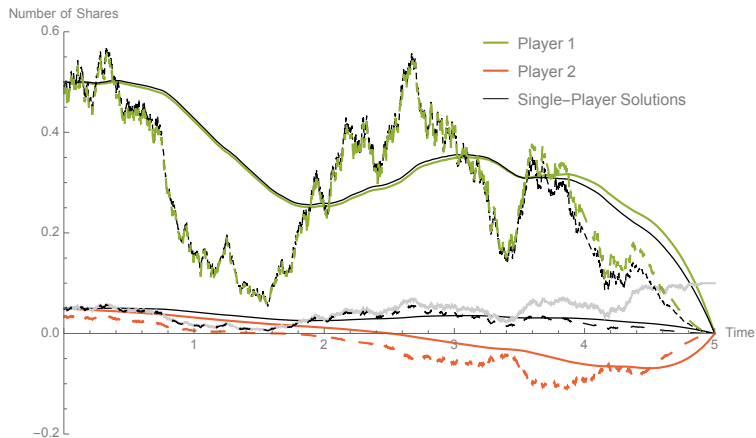


Player 1:  $x^1 = 1/2$ ,  $\xi_t^1 = \text{delta-hedge}$ ,  $\Xi_T^1 = 0$ .

Player 2:  $x^2 = 1/20$ ,  $\xi_t^2 = 0.1 \cdot \xi_t^1$ ,  $\Xi_T^2 = 0$ .

Parameters:  $\sigma = 1$ ,  $\lambda = 1$ ,  $\gamma = 1.0$ .

## Running after the delta

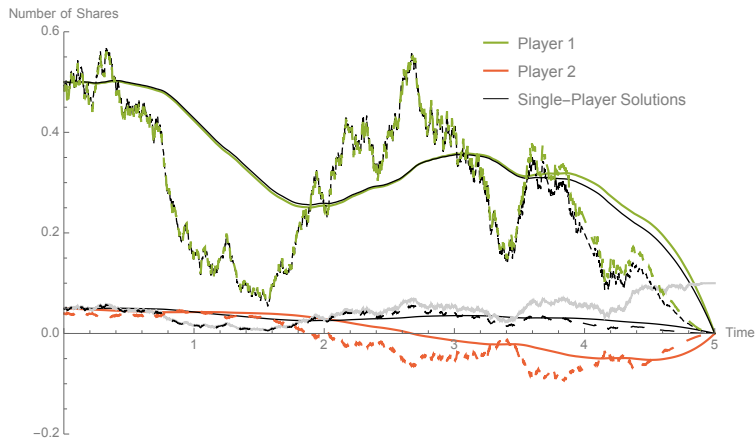


Player 1:  $x^1 = 1/2$ ,  $\xi_t^1 = \text{delta-hedge}$ ,  $\Xi_T^1 = 0$ .

Player 2:  $x^2 = 1/20$ ,  $\xi_t^2 = 0.1 \cdot \xi_t^1$ ,  $\Xi_T^2 = 0$ .

Parameters:  $\sigma = 1$ ,  $\lambda = 1$ ,  $\gamma = 0.5$ .

# Running after the delta



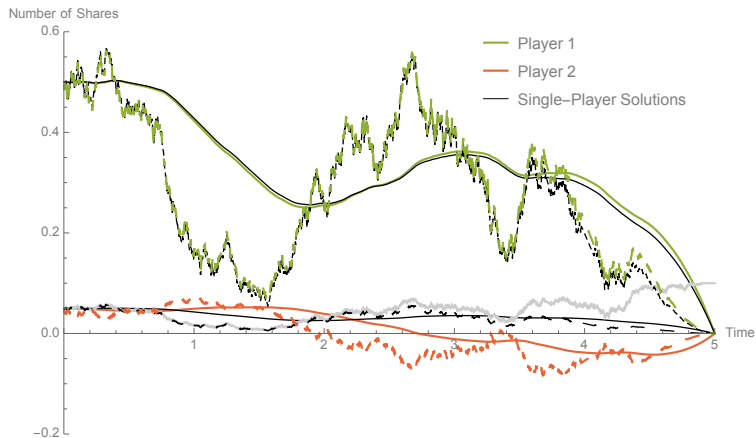
Player 1:  $x^1 = 1/2$ ,  $\xi_t^1 = \text{delta-hedge}$ ,  $\Xi_T^1 = 0$ .

Player 2:  $x^2 = 1/20$ ,  $\xi_t^2 = 0.1 \cdot \xi_t^1$ ,  $\Xi_T^2 = 0$ .

Parameters:  $\sigma = 1$ ,  $\lambda = 1$ ,  $\gamma = 0.25$ .



# Running after the delta

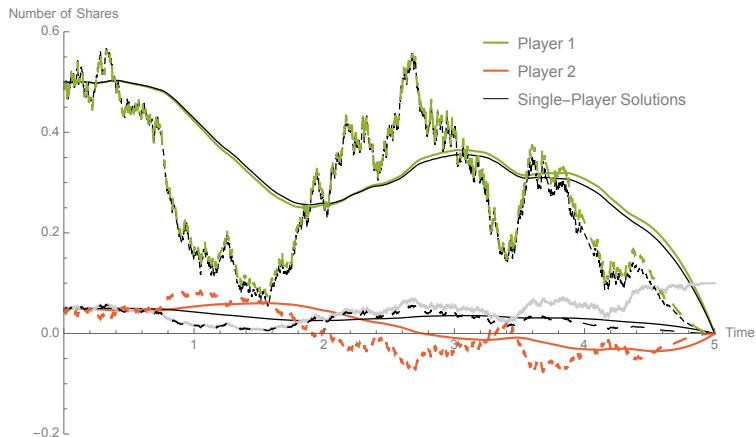


Player 1:  $x^1 = 1/2$ ,  $\xi_t^1 = \text{delta-hedge}$ ,  $\Xi_T^1 = 0$ .

Player 2:  $x^2 = 1/20$ ,  $\xi_t^2 = 0.1 \cdot \xi_t^1$ ,  $\Xi_T^2 = 0$ .

Parameters:  $\sigma = 1$ ,  $\lambda = 1$ ,  $\gamma = 0.1$ .

# Running after the delta



Player 1:  $x^1 = 1/2$ ,  $\xi_t^1 = \text{delta-hedge}$ ,  $\Xi_T^1 = 0$ .

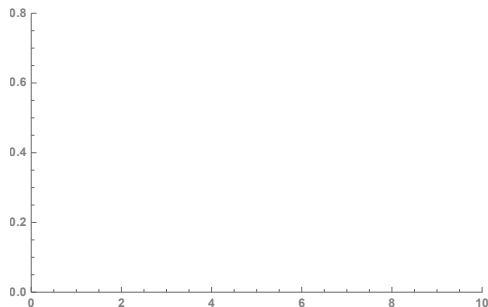
Player 2:  $x^2 = 1/20$ ,  $\xi_t^2 = 0.1 \cdot \xi_t^1$ ,  $\Xi_T^2 = 0$ .

Parameters:  $\sigma = 1$ ,  $\lambda = 1$ ,  $\gamma = 0$ .

# Illustration: Weights

Recall:

$$\begin{aligned}\hat{\xi}_t^1 &= w_t^1 \cdot \mathbb{E}[\Xi_T^1 + \Xi_T^2 \mid \mathcal{F}_t] + w_t^2 \cdot \mathbb{E}[\Xi_T^1 - \Xi_T^2 \mid \mathcal{F}_t] \\ &+ w_t^3 \cdot \mathbb{E} \left[ \int_t^T (\xi_s^1 + \xi_s^2) \cdot K^1(t, s) ds \mid \mathcal{F}_t \right] + w_t^4 \cdot \mathbb{E} \left[ \int_t^T (\xi_s^1 - \xi_s^2) \cdot K^2(t, s) ds \mid \mathcal{F}_t \right]\end{aligned}$$

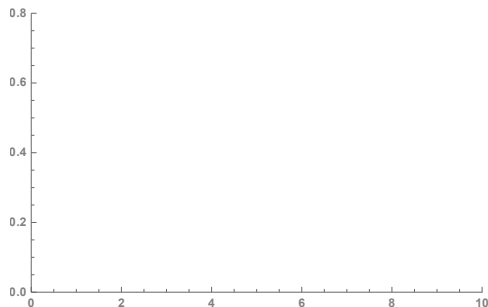


Parameters:  $\sigma = 1$ ,  $\lambda = 1$ ,  $\gamma = 2$ .

# Illustration: Weights

Recall:

$$\begin{aligned}\hat{\xi}_t^1 &= w_t^1 \cdot \mathbb{E}[\Xi_T^1 + \Xi_T^2 \mid \mathcal{F}_t] + w_t^2 \cdot \mathbb{E}[\Xi_T^1 - \Xi_T^2 \mid \mathcal{F}_t] \\ &+ w_t^3 \cdot \mathbb{E} \left[ \int_t^T (\xi_s^1 + \xi_s^2) \cdot K^1(t, s) ds \mid \mathcal{F}_t \right] + w_t^4 \cdot \mathbb{E} \left[ \int_t^T (\xi_s^1 - \xi_s^2) \cdot K^2(t, s) ds \mid \mathcal{F}_t \right]\end{aligned}$$

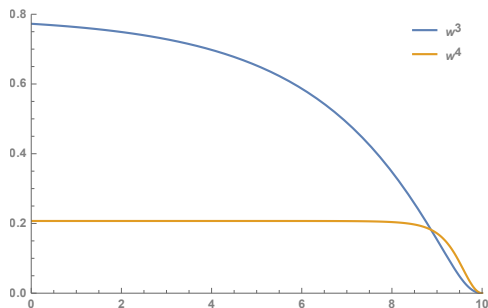


Parameters:  $\sigma = 1$ ,  $\lambda = 1$ ,  $\gamma = 2$ .

# Illustration: Weights

Recall:

$$\hat{\xi}_t^1 = w_t^1 \cdot \mathbb{E}[\Xi_T^1 + \Xi_T^2 \mid \mathcal{F}_t] + w_t^2 \cdot \mathbb{E}[\Xi_T^1 - \Xi_T^2 \mid \mathcal{F}_t] \\ + w_t^3 \cdot \mathbb{E} \left[ \int_t^T (\xi_s^1 + \xi_s^2) \cdot K^1(t, s) ds \mid \mathcal{F}_t \right] + w_t^4 \cdot \mathbb{E} \left[ \int_t^T (\xi_s^1 - \xi_s^2) \cdot K^2(t, s) ds \mid \mathcal{F}_t \right]$$

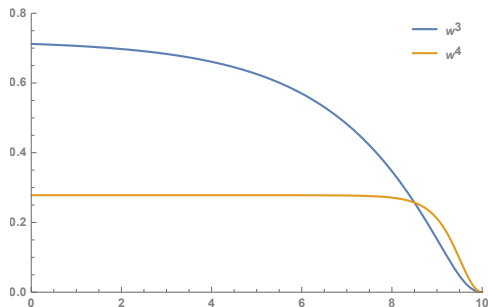


Parameters:  $\sigma = 1$ ,  $\lambda = 1$ ,  $\gamma = 2$ .

# Illustration: Weights

Recall:

$$\hat{\xi}_t^1 = w_t^1 \cdot \mathbb{E}[\Xi_T^1 + \Xi_T^2 \mid \mathcal{F}_t] + w_t^2 \cdot \mathbb{E}[\Xi_T^1 - \Xi_T^2 \mid \mathcal{F}_t] \\ + w_t^3 \cdot \mathbb{E} \left[ \int_t^T (\xi_s^1 + \xi_s^2) \cdot K^1(t, s) ds \mid \mathcal{F}_t \right] + w_t^4 \cdot \mathbb{E} \left[ \int_t^T (\xi_s^1 - \xi_s^2) \cdot K^2(t, s) ds \mid \mathcal{F}_t \right]$$

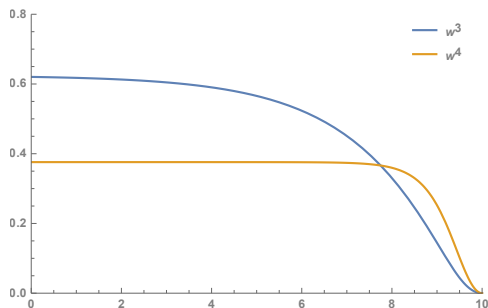


Parameters:  $\sigma = 1$ ,  $\lambda = 1$ ,  $\gamma = 1.5$ .

# Illustration: Weights

Recall:

$$\begin{aligned}\hat{\xi}_t^1 &= w_t^1 \cdot \mathbb{E}[\Xi_T^1 + \Xi_T^2 \mid \mathcal{F}_t] + w_t^2 \cdot \mathbb{E}[\Xi_T^1 - \Xi_T^2 \mid \mathcal{F}_t] \\ &+ w_t^3 \cdot \mathbb{E} \left[ \int_t^T (\xi_s^1 + \xi_s^2) \cdot K^1(t, s) ds \mid \mathcal{F}_t \right] + w_t^4 \cdot \mathbb{E} \left[ \int_t^T (\xi_s^1 - \xi_s^2) \cdot K^2(t, s) ds \mid \mathcal{F}_t \right]\end{aligned}$$

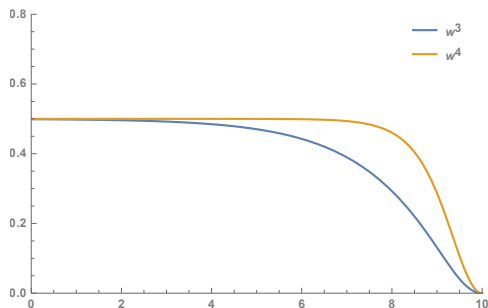


Parameters:  $\sigma = 1$ ,  $\lambda = 1$ ,  $\gamma = 1$ .

# Illustration: Weights

Recall:

$$\hat{\xi}_t^1 = w_t^1 \cdot \mathbb{E}[\Xi_T^1 + \Xi_T^2 \mid \mathcal{F}_t] + w_t^2 \cdot \mathbb{E}[\Xi_T^1 - \Xi_T^2 \mid \mathcal{F}_t] \\ + w_t^3 \cdot \mathbb{E} \left[ \int_t^T (\xi_s^1 + \xi_s^2) \cdot K^1(t, s) ds \mid \mathcal{F}_t \right] + w_t^4 \cdot \mathbb{E} \left[ \int_t^T (\xi_s^1 - \xi_s^2) \cdot K^2(t, s) ds \mid \mathcal{F}_t \right]$$



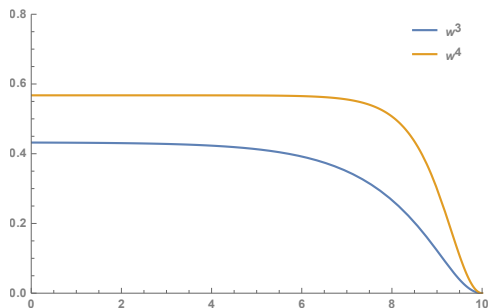
Parameters:  $\sigma = 1$ ,  $\lambda = 1$ ,  $\gamma = 0.5$ .



# Illustration: Weights

Recall:

$$\hat{\xi}_t^1 = w_t^1 \cdot \mathbb{E}[\Xi_T^1 + \Xi_T^2 \mid \mathcal{F}_t] + w_t^2 \cdot \mathbb{E}[\Xi_T^1 - \Xi_T^2 \mid \mathcal{F}_t] \\ + w_t^3 \cdot \mathbb{E} \left[ \int_t^T (\xi_s^1 + \xi_s^2) \cdot K^1(t, s) ds \mid \mathcal{F}_t \right] + w_t^4 \cdot \mathbb{E} \left[ \int_t^T (\xi_s^1 - \xi_s^2) \cdot K^2(t, s) ds \mid \mathcal{F}_t \right]$$

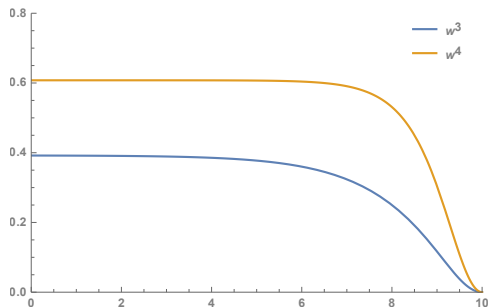


Parameters:  $\sigma = 1$ ,  $\lambda = 1$ ,  $\gamma = 0.25$ .

# Illustration: Weights

Recall:

$$\hat{\xi}_t^1 = w_t^1 \cdot \mathbb{E}[\Xi_T^1 + \Xi_T^2 \mid \mathcal{F}_t] + w_t^2 \cdot \mathbb{E}[\Xi_T^1 - \Xi_T^2 \mid \mathcal{F}_t] \\ + w_t^3 \cdot \mathbb{E} \left[ \int_t^T (\xi_s^1 + \xi_s^2) \cdot K^1(t, s) ds \mid \mathcal{F}_t \right] + w_t^4 \cdot \mathbb{E} \left[ \int_t^T (\xi_s^1 - \xi_s^2) \cdot K^2(t, s) ds \mid \mathcal{F}_t \right]$$

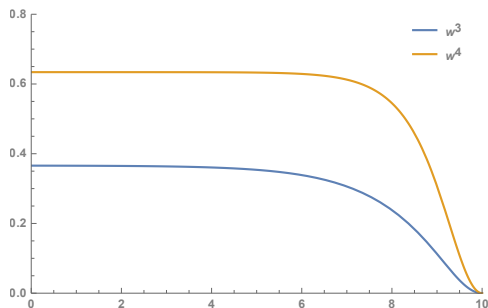


Parameters:  $\sigma = 1$ ,  $\lambda = 1$ ,  $\gamma = 0.1$ .

# Illustration: Weights

Recall:

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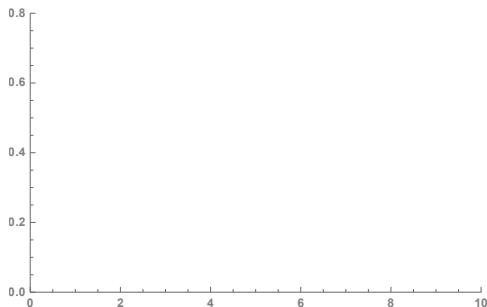


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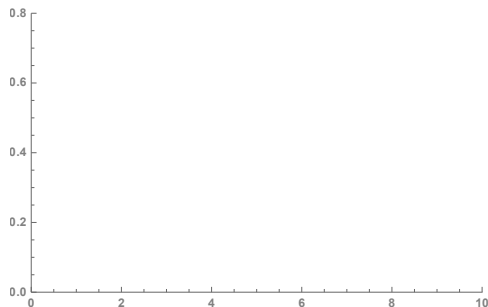


Parameters:  $\sigma = 1$ ,  $\lambda = 1$ ,  $\gamma = 2$ .

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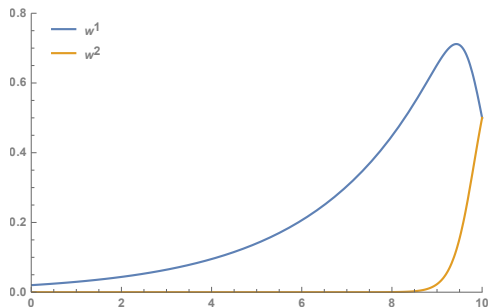


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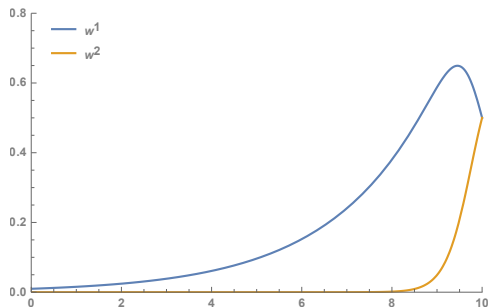


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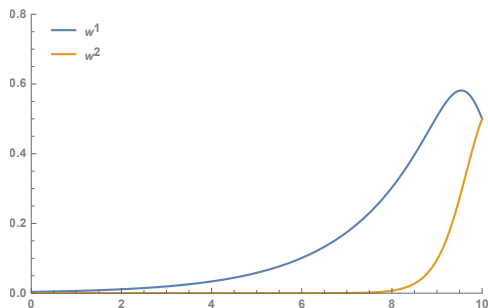


Parameters:  $\sigma = 1$ ,  $\lambda = 1$ ,  $\gamma = 1.5$ .

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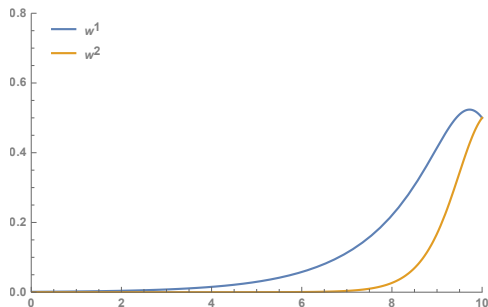
Parameters:  $\sigma = 1$ ,  $\lambda = 1$ ,  $\gamma = 1$ .



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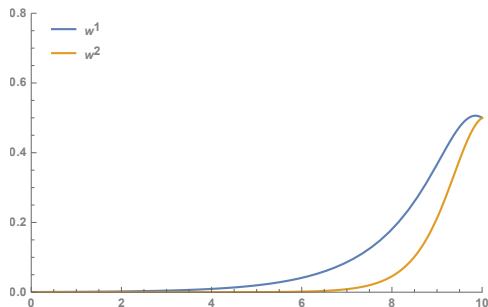


Parameters:  $\sigma = 1$ ,  $\lambda = 1$ ,  $\gamma = 0.5$ .

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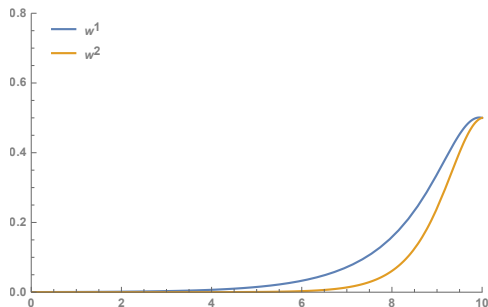


Parameters:  $\sigma = 1$ ,  $\lambda = 1$ ,  $\gamma = 0.25$ .

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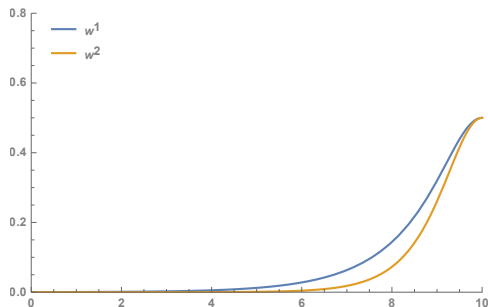


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Parameters:  $\sigma = 1$ ,  $\lambda = 1$ ,  $\gamma = 0$ .

# Sketch of Proof

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For fixed  $\tilde{\alpha}^2 \in \mathcal{A}^2$ ,  $\tilde{\alpha}^1 \in \mathcal{A}^1$  the mappings

$$\alpha^1 \mapsto J^1(\alpha^1, \tilde{\alpha}^2) \quad \alpha^2 \mapsto J^2(\tilde{\alpha}^1, \alpha^2)$$

are strictly convex (over convex admissible sets  $\mathcal{A}^1$ ,  $\mathcal{A}^2$ ).

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### Lemma

*There exists at most one Nash equilibrium.*

## Sketch of Proof

Given two controls  $\tilde{\alpha}^1 \in \mathcal{A}^1$ ,  $\tilde{\alpha}^2 \in \mathcal{A}^2$  we can introduce the Gâteaux derivatives of the mappings:

$$\alpha^1 \mapsto J^1(\alpha^1, \tilde{\alpha}^2) \quad \alpha^2 \mapsto J^2(\tilde{\alpha}^1, \alpha^2)$$

at  $\alpha^1 \in \mathcal{A}^1$  and  $\alpha^2 \in \mathcal{A}^2$ , respectively, in any (suitable) directions  $\beta^1, \beta^2 \in \mathcal{A}^0$ :

$$\langle \nabla_1 J^1(\alpha^1, \tilde{\alpha}^2), \beta^1 \rangle \triangleq \lim_{\varepsilon \rightarrow 0} \frac{J^1(\alpha^1 + \varepsilon \beta^1, \tilde{\alpha}^2) - J^1(\alpha^1, \tilde{\alpha}^2)}{\varepsilon},$$
$$\langle \nabla_2 J^2(\tilde{\alpha}^1, \alpha^2), \beta^2 \rangle \triangleq \lim_{\varepsilon \rightarrow 0} \frac{J^2(\tilde{\alpha}^1, \alpha^2 + \varepsilon \beta^2) - J^2(\tilde{\alpha}^1, \alpha^2)}{\varepsilon}.$$



# Sketch of Proof

## Lemma

For  $\alpha^1 \in \mathcal{A}^1, \alpha^2 \in \mathcal{A}^2$  we have

$$\begin{aligned} & \langle \nabla_1 J^1(\alpha^1, \alpha^2), \beta^1 \rangle \\ &= \mathbb{E} \left[ \int_0^T \beta_s^1 \left( \lambda \alpha_s^1 + \frac{\lambda}{2} \alpha_s^2 + \gamma (X_s^2 - x^2) + \int_s^T (X_t^1 - \xi_t^1) \sigma dt \right) ds \right] \end{aligned}$$

and

$$\begin{aligned} & \langle \nabla_2 J^2(\alpha^1, \alpha^2), \beta^2 \rangle \\ &= \mathbb{E} \left[ \int_0^T \beta_s^2 \left( \lambda \alpha_s^2 + \frac{\lambda}{2} \alpha_s^1 + \gamma (X_s^1 - x^1) + \int_s^T (X_t^2 - \xi_t^2) \sigma dt \right) ds \right] \end{aligned}$$

for any  $\beta^1, \beta^2 \in \mathcal{A}^0$ .

# Sketch of Proof

## Lemma

Suppose that  $(\hat{X}^1, \hat{X}^2)$  with controls  $(\hat{\alpha}^1, \hat{\alpha}^2) \in \mathcal{A}^1 \times \mathcal{A}^2$  solves following coupled forward backward SDE system

$$\left\{ \begin{array}{l} dX_t^1 = \alpha_t^1 dt, \quad X_0^1 = x^1, \\ dX_t^2 = \alpha_t^2 dt, \quad X_0^2 = x^2, \\ d\alpha_t^1 = \frac{\sigma}{\lambda}(X_t^1 - \xi_t^1)dt - \frac{\gamma}{\lambda}\alpha_t^2 dt - \frac{1}{2}d\alpha_t^2 + dM_t^1, \quad X_T^1 = \Xi_T^1, \\ d\alpha_t^2 = \frac{\sigma}{\lambda}(X_t^2 - \xi_t^2)dt - \frac{\gamma}{\lambda}\alpha_t^1 dt - \frac{1}{2}d\alpha_t^1 + dM_t^2, \quad X_T^2 = \Xi_T^2, \end{array} \right.$$

for two suitable square integrable martingales  $(M_t^1)_{0 \leq t < T}$  and  $(M_t^2)_{0 \leq t < T}$ . Then  $(\hat{\alpha}^1, \hat{\alpha}^2)$  is a Nash equilibrium.

# Conclusion

- ▶ study competition of two strategic agents for liquidity in a financial market
- ▶ agents interact through common aggregated temporary and permanent price impact à la Almgren & Chriss
- ▶ resulting stochastic linear quadratic differential game with terminal state constraints allows for an explicitly available open loop Nash equilibrium
- ▶ closed-form solution reveals how the equilibrium strategies of the two players take into account the other agent's trading targets
- ▶ rich set of phenomena occurring in equilibrium: coexistence of cooperation and predation (depending on the ratio between temporary and permanent price impact)

## Future Research

- ▶ opens door to study  $N$ -player stochastic differential game with mean field interaction through common price impact (Game with Major and Minor Players)
- ▶ study the Mean-Field limit  $N \rightarrow +\infty$

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- ▶ non-homogeneous agents: different inventory risk-aversion, different temporary price impact parameters  $\lambda^1, \lambda^2$
- ▶ different information structure (agents have private filtrations)
- ▶ incorporate transient price impact à la Obizhaeva & Wang

# Reference



A Two-Player Price Impact Game

*Preprint available on arXiv:1911.05122.*



Hedging with Temporary Price Impact

with Peter Bank, H. Mete Soner

*Mathematics and Financial Economics, 2017.*

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**Thank you very much!**