Stochastic Games with Delay: a Toy Model

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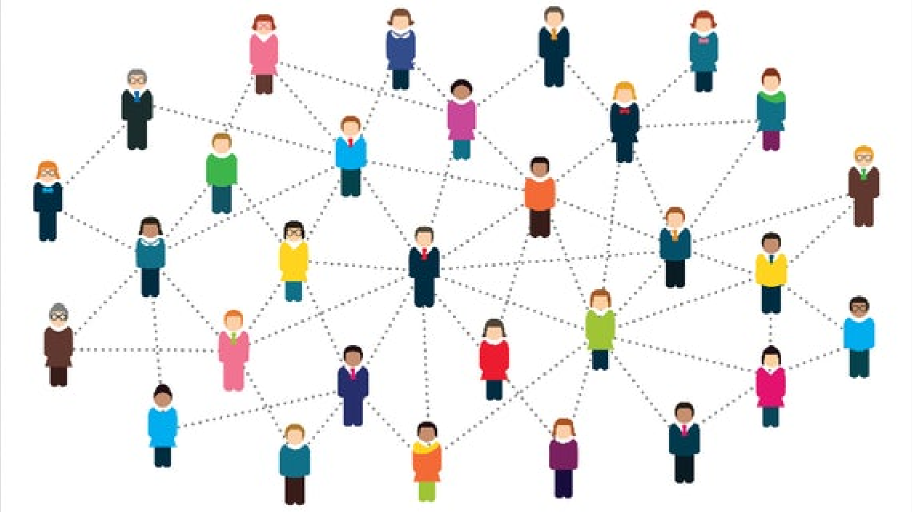
Networking



Fish Schooling

Literature Review

- Mean Field Games:
  - R. Carmona, J.-P. Fouque, L.-H. Sun [2015].
  - R. Carmona, F. Delarue [2018].
  - P. Cardaliaguet, F. Delarue, J.-M. Lasery, P.-L. Lions [2015].
  - C. Wu and J. Zhang [2018].
  - etc.
- Control Problems with Delay:
  - Y. Alekal, P. Brunovsky, DH. Chyung, and EB. Lee [1971].
  - RB Vinter and RH Kwong [1981].
  - F. Gozzi and C. Marinelli [2004].
  - S. Peng and Z. Yang [2009].
  - Y. Saporito and J. Zhang [2018]
  - etc.
- Stochastic Games with Delay:
  - R. Carmona, J.-P.Fouque, M. Mousavi, L.-H. Sun [2016].
  - J.-P.Fouque, Z. Zhang [2018].
  - etc.

Introduction: Linear Quadratic Stochastic Games

Linear Quadratic Game

Bank i for $i = 1, \cdots, N$ is borrowing from and lending to a central bank:

$$dX_t^i = \alpha_t^i dt + \sigma dW_t^i, \quad X_0^i = \xi^i.$$

where

- X_t^i represents the log-monetary reserves of the ith bank,
- W_t^i are independent standard Brownian motions,
- $\sigma > 0$, the diffusion coefficients are constant and identical,
- Bank i controls its rate of borrowing $(\alpha_t^i > 0)$/lending $(\alpha_t^i < 0)$ to a central bank through the **control** α_t^i.

The Cost Functional

Bank i wants to **minimize**

$$J^i(\alpha) = \mathbb{E}\left\{ \int_0^T f_i(X_t, \alpha_t^i)dt + g_i(X_T) \right\},$$

with **running cost**

$$f_i(x, \alpha^i) = \frac{1}{2}(\alpha^i)^2 + \frac{\epsilon}{2}(\overline{x} - x^i)^2,$$

and **terminal cost**

$$g_i(x) = \frac{c}{2}\left(\overline{x} - x^i\right)^2.$$

Value function

$$V^i(t, x) = \inf_{\alpha} J^i(\alpha).$$

Solving for an Exact Nash Equilibrium

Definition

A set of admissible strategy profiles $\hat{\alpha} = (\hat{\alpha}^1, \cdots, \hat{\alpha}^N) \in \mathbb{A}^{(N)}$ is said to be a Nash equilibrium for the game if:

$$\forall i \in \{1, \cdots, N\}, \forall \alpha^i \in \mathbb{A}^i, \quad J^i(\hat{\alpha}) \leq J^i(\alpha^i, \hat{\alpha}^{-i}),$$

where $(\alpha^i, \hat{\alpha}^{-i})$ stands for the strategy profile $(\hat{\alpha}^1, \cdots, \hat{\alpha}^{i-1}, \alpha^i, \hat{\alpha}^{i+1})$, in which the player i chooses the strategy α^i while the others keep the original ones $\hat{\alpha}^j$.

- Probabilistic Approach (N-coupled Forward-Backward SDEs)
- PDE Approach (N-coupled Hamilton-Jacobi-Bellman (HJB) PDEs)
- This is an example of **Mean Field Game (MFG)** studied extensively by P.L. Lions and collaborators, R. Carmona and F. Delarue, ...

Stochastic Game/Mean Field Game with Delay

Stochastic Game with Delay

Banks are borrowing from and lending to a central bank and money is returned at maturity τ:

$$dX_t^i = \left[\alpha_t^i - \alpha_{t-\tau}^i\right] dt + \sigma dW_t^i, \quad i = 1, \cdots, N$$

where α^i is the control of bank i which wants to **minimize**

$$\begin{aligned} J^i(\alpha) &= \boldsymbol{E}\left\{\int_0^T f_i(X_t, \alpha_t^i) dt + g_i(X_T)\right\}, \\ f_i(x, \alpha^i) &= \frac{1}{2}(\alpha^i)^2 + \frac{\epsilon}{2}(\overline{x} - x^i)^2, \\ g_i(x) &= \frac{c}{2}\left(\overline{x} - x^i\right)^2, \\ X_0^i &= \xi^i, \qquad \alpha_t^i = 0, \quad t \in [-\tau, 0). \end{aligned}$$

Case $\tau = 0$: no lending/borrowing $\longrightarrow$ no liquidity.
Case $\tau = T$: no return/delay $\longrightarrow$ full liquidity.

Mean Field Game with Delay

- Mean field game theory is the study of strategic decision making in very **large populations** of small interacting agents, i.e., a game with infinite many **indistinguishable** players.
- All players are rational, i.e., each player tries to minimize their cost against the mass of other players.
- The running cost and terminal cost only depend on ith player's state x^i and the empirical distribution of $(x^j)_{j\neq i}$.
- As $N \to \infty$, denote $m_t = \int_R x d\mu_t(x)$,

$$
\begin{aligned}
f(X_t, \mu_t, \alpha_t) &= \frac{1}{2}(\alpha_t)^2 + \frac{\epsilon}{2}\left(m_t - X_t\right)^2 , \\
g(X_T, \mu_T) &= \frac{c}{2}\left(m_T - X_T\right)^2 .
\end{aligned}
$$

Probabilistic Approach

Forward-Advanced-Backward SDEs

Theorem. The strategy $\hat{\alpha}$ given by

$$\hat{\alpha}_t = -Y_t + \boldsymbol{E}^{\mathcal{F}_t}(Y_{t+\tau})$$

is a **open-loop Nash equilibrium** where (X, Y, Z) is the unique solution to the following system of **FABSDEs**:

$$\begin{aligned} X_t &= \xi + \int_0^t (\hat{\alpha}_s - \hat{\alpha}_{s-\tau})\, ds + \sigma W_t, \quad t \in [0, T], \\ Y_t &= c\,(X_T - m_T) + \int_t^T \epsilon\,(X_s - m_s)\, ds - \int_t^T Z_s dW_s, \quad t \in [0, T], \\ Y_t &= 0, \quad t \in (T, T+\tau], \end{aligned}$$

where the processes Z_t are adapted and square integrable, and $\boldsymbol{E}^{\mathcal{F}_t}$ denotes the conditional expectation with respect to the filtration generated by the Brownian motions.

Outline of the Proof

Proof.
Let $\alpha' \in \mathbb{A}$ be a generic admissible control, and $X' = X^{\alpha'}$ the corresponding controlled state.

$$J(\hat{\alpha}) - J(\alpha') = \boldsymbol{E}\Big\{ \int_0^T (f(X_t, \mu_t, \hat{\alpha}_t) - f(X'_t, \mu'_t, \alpha'_t))\, dt \\ + g(X_T, \mu_T) - g(X'_T, \mu'_T) \Big\}.$$

Since g is L-convex in (x, μ),

$$\begin{aligned} & \boldsymbol{E}(g(X_T, \mu_T) - g(X'_T, \mu'_T)) \\ \leq\ & \boldsymbol{E}[(\partial_x g(X_T, \mu_T) + \tilde{\boldsymbol{E}}[\partial_\mu g(\tilde{X}_T, \mu_T)(X_T)]) \cdot (X_T - X'_T)] \\ =\ & \boldsymbol{E}[Y_T(X_T - X'_T)]. \end{aligned}$$

Outline of the Proof

Applying Itô's formula, we have

$$
\begin{aligned}
&\boldsymbol{E}(Y_T(X_T - X_T')) \\
&= \boldsymbol{E}\left[\int_0^T (X_t - X_t')dY_t + \int_0^T Y_t d(X_t - X_t')\right] \\
&= \boldsymbol{E}\int_0^T \left\{ -\epsilon(X_t - m_t)(X_t - X_t') + Y_t\left(\hat{\alpha}_t - \alpha_t' - (\hat{\alpha}_{t-\tau} - \alpha_{t-\tau}')\right) \right\} dt \\
&= \boldsymbol{E}\int_0^T \left\{ -\epsilon(X_t - m_t)(X_t - X_t') + \left(Y_t - \boldsymbol{E}^{\mathcal{F}_t}(Y_{t+\tau})\right)(\hat{\alpha}_t - \alpha_t') \right\} dt.
\end{aligned}
$$

due to the change of time,

$$
\boldsymbol{E}\int_0^T Y_t\left(\hat{\alpha}_{t-\tau} - \alpha_{t-\tau}'\right) dt = \boldsymbol{E}\int_{-\tau}^{T-\tau} Y_{s+\tau}\left(\hat{\alpha}_s - \alpha_s'\right) ds
$$

$$
= \boldsymbol{E}\int_0^T Y_{s+\tau}\left(\hat{\alpha}_s - \alpha_s'\right) ds = \boldsymbol{E}\int_0^T \boldsymbol{E}^{\mathcal{F}_s}(Y_{s+\tau})\left(\hat{\alpha}_s - \alpha_s'\right) ds.
$$

since $\hat{\alpha}_t = \alpha_t' = 0$ for $t \in [-\tau, 0)$ and $Y_t = 0$ for $t \in (T, T+\tau]$.

Outline of the Proof

Convexity of f in (x, μ, α), we deduce

$$
\begin{aligned}
& J(\hat\alpha) - J(\alpha') \\
\leq \; & \boldsymbol{E}\int_0^T \Big[\Big(\partial_x f(X_t, \mu_t, \hat\alpha_t) + \tilde{\boldsymbol{E}}[\partial_\mu f(\tilde X_t, \mu_t, \hat\alpha_t)(X_t) \Big) (X_t - X_t')]) \\
& + \partial_\alpha f(X_t, \hat\alpha_t)(\hat\alpha_t - \alpha_t') \Big] dt + \boldsymbol{E}[Y_T(X_T - X_T')] \\
= \; & \boldsymbol{E}\int_0^T \Big\{ \epsilon(X_t - m_t)(X_t - X_t') + (\partial_\alpha f(X_t, \mu_t, \hat\alpha))(\hat\alpha_t - \alpha_t') \Big\} dt \\
& + \boldsymbol{E}\int_0^T \Big\{ -\epsilon(X_t - X_t')(X_t - m_t) + (Y_t - \boldsymbol{E}^{\mathcal{F}_t}[Y_{t+\tau}])(\hat\alpha_t - \alpha_t') \Big\} dt. \\
= \; & \boldsymbol{E}\int_0^T (\hat\alpha_t^i - \alpha_t') \times \Big[\hat\alpha_t + (Y_t - \boldsymbol{E}^{\mathcal{F}_t}(Y_{t+\tau})) \Big] dt \\
= \; & 0
\end{aligned}
$$

Forward-Advanced-Backward SDEs

Theorem. The strategy $\hat{\alpha}$ given by

$$\hat{\alpha}_t = -Y_t + \boldsymbol{E}^{\mathcal{F}_t}(Y_{t+\tau})$$

is a **open-loop Nash equilibrium** where (X, Y, Z) is the unique solution to the following system of **FABSDEs**:

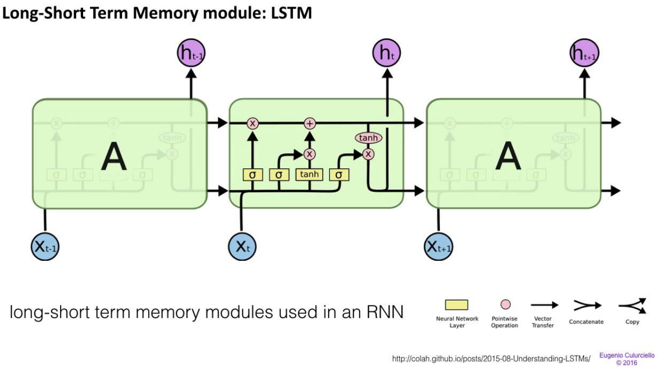
$$\begin{aligned} X_t &= \xi + \int_0^t (\hat{\alpha}_s - \hat{\alpha}_{s-\tau})\, ds + \sigma W_t, \quad t \in [0, T], \\ Y_t &= c\,(X_T - m_T) + \int_t^T \epsilon\,(X_s - m_s)\, ds - \int_t^T Z_s dW_s, \quad t \in [0, T], \\ Y_t &= 0, \quad t \in (T, T+\tau], \end{aligned}$$

where the processes Z_t are adapted and square integrable, and $\boldsymbol{E}^{\mathcal{F}_t}$ denotes the conditional expectation with respect to the filtration generated by the Brownian motions.

Existence, no Uniqueness

No simple explicit formula for the optimal strategy $\hat{\alpha}$.

Recurrent Neural Network



- $Y_t \approx \phi(t, (W_s)_{0 \le s \le t} | \Theta_t)$
- $\mathbb{E}[Y_{t+\tau} | \mathcal{F}_t] \approx \psi(t, (W_s)_{0 \le s \le t} | \Lambda_t)$
- $Z_t \approx \chi(t, (W_s)_{0 \le s \le t} | \Gamma_t)$

$$
\begin{aligned}
f_t &= \sigma_g(W_f x_t + U_f h_{t-1} + b_f) \\
i_t &= \sigma_g(W_i x_t + U_i h_{t-1} + b_i) \\
o_t &= \sigma_g(W_o x_t + U_o h_{t-1} + b_o) \\
c_t &= f_t \circ c_{t-1} + i_t \circ \sigma_c(W_c x_t + U_c h_{t-1} + b_c) \\
h_t &= o_t \circ \sigma_h(c_t)
\end{aligned}
$$

Algorithm

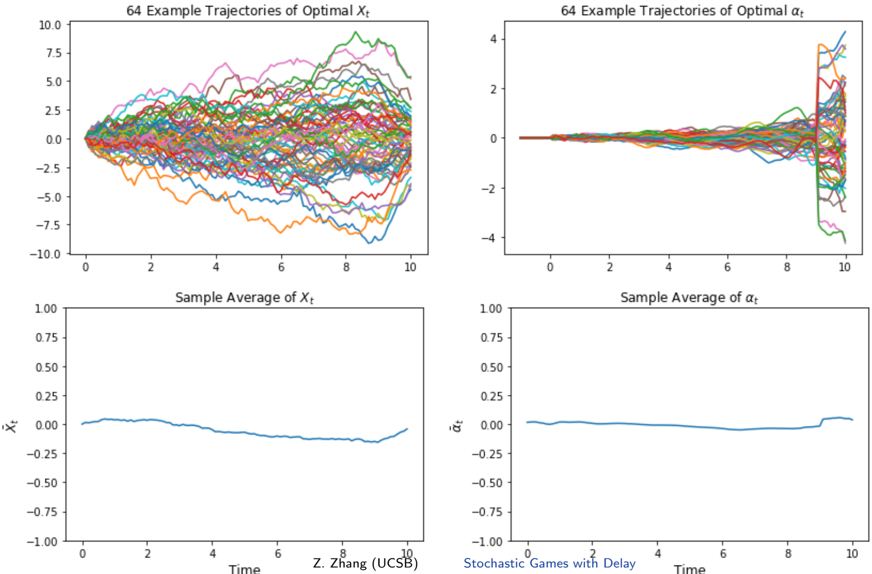
- Time discretization. $h = T/N$, $D = \tau/h$.
  $-\tau = t_{-D} \leq \cdots \leq t_{-1} \leq t_0 = 0 = t_0 \leq t_1 \leq \cdots \leq t_N \leq T$.
- Initial states. $X_0 = 0$, $Y_0 \approx \phi(0, W_0|\Theta_0)$, $\boldsymbol{E}[Y_\tau|\mathcal{F}_0] \approx \psi(0, W_0|\Lambda_0)$, $Z_0 \approx \chi(0, W_0|\Gamma_0)$. $\alpha_0 = -Y_0 + \boldsymbol{E}[Y_\tau|\mathcal{F}_0]$.
- Euler–Maruyama method.

$$
\begin{aligned}
X_{t_{k+1}} &= X_{t_k} + (\alpha_{t_k} - \alpha_{t_{k-D}}) * h + \sigma \Delta W_{t_{k+1}}, \text{where } \Delta W_{t_{k+1}} \sim N(0, h). \\
\tilde{Y}_{t_{k+1}} &= Y_{t_k} - \epsilon X_{t_k} * h + Z_{t_k} \Delta W_{t_{k+1}}. \\
Y_{t_{k+1}} &\approx \phi(t_{k+1}, (W_s)_{0 \leq s \leq t_{k+1}} | \Theta_{t_{k+1}})
\end{aligned}
$$

- Loss $= \sum_{m=1}^{M} \sum_{k=1}^{N} (Y_{t_k}^m - \tilde{Y}_{t_k}^m)^2 + \sum_{m=1}^{M} (Y_{t_N}^m - cX_{T_N}^m)^2 + \sum_{m=1}^{M} \sum_{k=0}^{N-D} (Y_{t_{k+D}}^m - \boldsymbol{E}[Y_{t_{k+D}}^m | \mathcal{F}_{t_k}])^2$.
- Apply stochastic gradient descent (SGD) to minimize loss and update parameters.

Results

$$T = 10, \tau = 1, h = 0.1, \epsilon = 1, c = 1, M = 2560.$$



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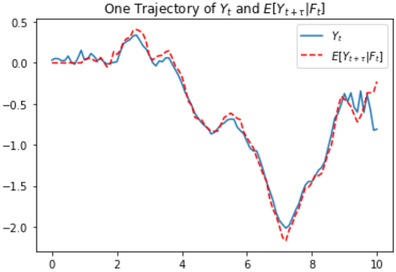

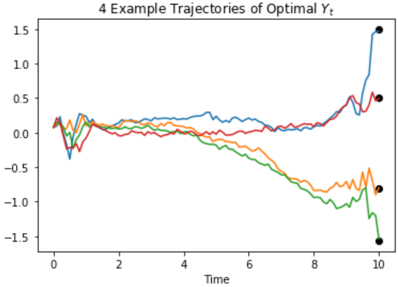

PDE Approach

Infinite-dimensional HJB Approach

- Denote $\boldsymbol{H} := L^2([-\tau, 0]; \boldsymbol{R})$.
- Given $z := (z_0, z_1) \in \boldsymbol{R} \times \boldsymbol{H}$, where $z_0 \in \boldsymbol{R}$, and $z_1 \in \boldsymbol{H}$. The inner product on $\boldsymbol{R} \times \boldsymbol{H}$ will be denoted by $\langle \cdot, \cdot \rangle$, and it is defined by

$$\langle z, \tilde{z} \rangle = z_0 \tilde{z}_0 + \int_{-\tau}^{0} z_1(s) \tilde{z}_1(s) ds.$$

- Therefore, the new state is denoted by $Z_t = (Z_{0,t}, Z_{1,t}(s)),\ s \in [-\tau, 0]$, which corresponds to $(X_t, \alpha_{t-\tau-s})$, i.e., the states and the past of the strategies in our case.

Infinite-dimensional HJB Approach

In order to use the **dynamic programming principle** for stochastic game in search of a **closed-loop Nash equilibrium**, at time $t \in [0, T]$, given the initial state $Z_0 = z$, one representative bank chooses the control α to minimise its objective function $J(t, z, \alpha)$.

$$J(t, z, \alpha) = \boldsymbol{E}\left\{ \int_t^T f(Z_{0,s}, \mu_{0,s}, \alpha_s) dt + g(Z_{0,T}, \mu_{0,T}) \mid Z_t = z \right\},$$

The value function $V(t, z)$ is

$$V(t, z) = \inf_{\alpha} J(t, z, \alpha).$$

subject to

$$dZ_t = (AZ_t + B\alpha_t)dt + GdW_t.$$

Coupled HJB Equations

The **value functions** $V(t,z)$ is the unique solution (in a suitable sense) of the following **HJB equations**:

$$\partial_t V + \frac{1}{2} Tr(Q\partial_{zz} V) + \langle Az, \partial_z V\rangle + H_0(\partial_z V) = 0,$$
$$V(T) = g(Z_{0,T}, \mu_{0,T}),$$

$$\begin{aligned}
Q &= G * G, \quad G : z_0 \to (\sigma z_0, 0), \\
A &: (z_0, z_1(\gamma)) \to (z_1(0), -\frac{dz_1(\gamma)}{d\gamma}) \quad a.e., \quad \gamma \in [-\tau, 0], \\
H_0(p) &= \inf_{\alpha}[\langle B\alpha, p\rangle + f(z_0, \alpha)], \quad p \in \boldsymbol{R} \times \boldsymbol{H}, \\
B &: u \to (u, -\delta_{-\tau}(\gamma)u), \quad \gamma \in [-\tau, 0].
\end{aligned}$$

Forward Kolmogorov Equation

- Next, since we "lift" the original non-Markovian optimization problem into a infinite dimensional Markovian control problem. We are able to write the corresponding generator , which is denoted by $(Ł_t)_{t\in[0,T]}$,

$$Ł\varphi(t,z) = \langle (AZ + B\hat{\alpha}), \partial_z \varphi \rangle + \frac{1}{2} Tr(G^* G \partial_{zz}\varphi).$$

- **Forward Kolmogorov Equation**

$$\begin{aligned}
\partial_t \nu = & \int_{-\tau}^{0} \partial_{z_1} \left(\frac{d}{ds} z_1 \nu \right) ds - \int_{-\tau}^{0} \partial_{z_1}(z_1\nu)(\delta_0(s) - \delta_{-\tau}(s)) ds \\
& + \partial_{z_0}\{(\partial_{z_0} V - [\partial_{z_1} V](-\tau))\nu\} \\
& - \int_{-\tau}^{0} \partial_{z_1}\{(\partial_{z_0} V - [\partial_{z_1} V](-\tau))\nu\}\delta_{-\tau}(s) ds + \frac{1}{2}\sigma^2 \partial_{z_0 z_0}\nu, \\
\nu_0 = & \boldsymbol{P}(\xi, \phi(s)_{s\in[-\tau,0]}).
\end{aligned}$$

Derivative in $\boldsymbol{P}(\boldsymbol{H})$

Definition

We say that $F : \boldsymbol{P}(\boldsymbol{H}) \to \boldsymbol{H}$ is $\mathcal{C}^1$ if there exists an operator $\frac{\delta F}{\delta \nu} : \boldsymbol{P}(\boldsymbol{H}) \times \boldsymbol{H} \to \boldsymbol{H}$ such that for any μ_1 and $\mu_1' \in \boldsymbol{P}(\boldsymbol{H})$

$$\lim_{\epsilon \to 0^+} \frac{F(\mu_1 + \epsilon(\mu_1' - \mu_1)) - F(\mu_1)}{\epsilon} = \int_{\boldsymbol{H}} \frac{\delta F}{\delta \mu_1}(\mu_1, y_1) d(\mu_1' - \mu_1)(y_1).$$

Definition

If $\frac{\delta F}{\delta \mu_1}(\mu_1, y_1)$ is of class $\mathcal{C}^1$ with respect to y_1, the marginal derivative $D_{\mu_1} F : \boldsymbol{P}(\boldsymbol{H}) \times \boldsymbol{H} \to \boldsymbol{H}$ is defined in the sense of Fréchet derivative:

$$D_{\mu_1} F(\mu_1, y_1) := D_{y_1} \frac{\delta F}{\delta \mu_1}(\mu_1, y_1).$$

Derivative in $\boldsymbol{P}(\boldsymbol{H})$

Remark

Usually we will encounter a map $U : \boldsymbol{P}(\boldsymbol{H}) \to \boldsymbol{R}$. In this case, U can be expressed in a form of composition $\tilde{U} \circ F$, where $\tilde{U} : \boldsymbol{H} \to \boldsymbol{R}$, and $F : \boldsymbol{P}(\boldsymbol{H}) \to \boldsymbol{H}$, i.e., $U = (\tilde{U} \circ F)(\mu_1)$.

If $\frac{\delta F}{\delta \mu_1}$ is $\mathcal{C}^1$ with respect to y_1, and $\tilde{U}$ is Fréchet differentiable, then $\frac{\delta U}{\delta \mu_1} : \boldsymbol{P}(\boldsymbol{H}) \times \boldsymbol{H} \to \boldsymbol{H}$, and $D_{\mu_1} U : \boldsymbol{P}(\boldsymbol{H}) \times \boldsymbol{H} \to \boldsymbol{H}$ are defined by

$$\frac{\delta U}{\delta \mu_1}(\mu_1, y_1) := (D_F \tilde{U}) \left(\frac{\delta F}{\delta \mu_1} \right), \text{ and } D_{\mu_1} U(\mu_1, y_1) := \left(D_F \tilde{U} \right) (D_{\mu_1} F) .$$

The Master Equation

For any $(t_0, \nu_0) \in [0, T] \times \boldsymbol{P}(\boldsymbol{R} \times \boldsymbol{H})$, we define

$$U(t_0, \cdot, \nu_0) := V(t_0, \cdot),$$

where (V, ν) is a classical solution to the system of forward-backward equations. Then U must satisfy the following master equation

$$\begin{aligned}
&\partial_t U(t, z_0, z_1, \nu) + \frac{1}{2}\sigma^2 \partial_{z_0 z_0} U(t, z_0, z_1, \nu) + \frac{1}{2}\sigma^2 \int_{\boldsymbol{R}} \partial_{y_0} D_{\mu_0} U(t, z_0, z_1, \nu, y_0) d\mu_0(y_0) \\
&+ \int_{-\tau}^{0} z_1 \frac{d}{ds} \partial_{z_1} U(t, z_0, z_1, \nu) ds + \int_{-\tau}^{0} \int_{\boldsymbol{H}} y_1 \frac{d}{ds} \left[D_{\mu_1} U(t, z_0, z_1, \nu, y_1) \right](s) d\mu_1(y_1) ds \\
&- \int_{\boldsymbol{R}\times\boldsymbol{H}} \left(\partial_{y_0} U(t, y_0, y_1, \nu) - [\partial_{y_1} U(t, y_0, y_1, \nu)](-\tau) \right) D_{\mu_0} U(t, z_0, z_1, \nu, y_0) d\nu(y) \\
&+ \int_{\boldsymbol{R}\times\boldsymbol{H}} \left(\partial_{y_0} U(t, y_0, y_1, \nu) - [\partial_{y_1} U(t, y_0, y_1, \nu)](-\tau) \right) [D_{\mu_1} U(t, z_0, z_1, \nu, y_1)](-\tau) d\nu(y) \\
&- \frac{1}{2} \left(\partial_{z_0} U(t, z_0, z_1, \nu) - [\partial_{z_1} U(t, z_0, z_1, \nu)](-\tau) \right)^2 + \frac{\epsilon}{2} \left(\int_{\boldsymbol{R}} y_0 d\mu_0(y_0) - z_0 \right)^2 = 0,
\end{aligned}$$

where μ_0 and μ_1 are the marginal law for Z_0 and Z_1 respectively.

Explicit Solution of the Master Equation with Delay

It turns out that this master equation can be solved explicitly by making the following ansatz.

We define $m_0 := \int_{\boldsymbol{R}} y_0 d\mu_0(y_0)$ and $m_1 := \int_{\boldsymbol{H}} y_1 d\mu_1(y_1)$ for convenience, then

$$U(t, z_0, z_1, \nu) = E_0(t)(m_0 - z_0)^2 - 2(m_0 - z_0) \int_{-\tau}^{0} E_1(t, -\tau - s)(m_1 - z_1) ds + \int_{-\tau}^{0} \int_{-\tau}^{0} E_2(t, -\tau - s, -\tau - r)(m_1 - z_1)(m_1 - z_1) ds dr + E_3(t).$$

Explicit Solution of the Master Equation with Delay

We compute the partial derivatives needed in the master equation explicitly, we have

$$\partial_t U = \frac{dE_0(t)}{dt}(m_0 - z_0)^2 - 2(m_0 - z_0)\int_{-\tau}^{0} \frac{\partial E_1(t,-\tau-s)}{\partial t}(m_1 - z_1)ds$$
$$+ \int_{-\tau}^{0}\int_{-\tau}^{0} \frac{\partial E_2(t,-\tau-s,-\tau-r)}{\partial t}(m_1 - z_1)(m_1 - z_1)dsdr + \frac{dE_3(t)}{dt},$$

$$\partial_{z_0} U = -2E_0(t)(m_0 - z_0) + 2\int_{-\tau}^{0} E_1(t,-\tau-s)(m_1 - z_1)ds,$$

$$\partial_{z_1} U = 2E_1(t,-\tau-s)(m_0 - z_0) - 2\int_{-\tau}^{0} E_2(t,-\tau-s,-\tau-r)(m_1 - z_1)dr,$$

$$D_{\mu_0} U = 2E_0(t)(m_0 - z_0) - 2\int_{-\tau}^{0} E_1(t,-\tau-s)(m_1 - z_1)ds,$$

$$D_{\mu_1} U = -2E_1(t,-\tau-s)(m_0 - z_0) + 2\int_{-\tau}^{0} E_2(t,-\tau-s,-\tau-r)(m_1 - z_1)dr,$$

$$\partial_{z_0 z_0} U = 2E_0(t),$$

Collecting $(m_0 - z_0)^2$ terms, $(m_0 - z_0)(m_1 - z_1)$ terms, $(m_1 - z_1)^2$ terms, and constant terms, we obtain That the function $E_i, i = 0, \cdots, 3$, satisfy the system of PDEs:

$$
\begin{aligned}
&\frac{dE_0(t)}{dt} - 2(E_0(t) + E_1(t,0))^2 + \frac{\epsilon}{2} = 0, \\
&\frac{\partial E_1(t,s)}{\partial t} - \frac{\partial E_1(t,s)}{\partial s} - 2(E_0(t) + E_1(t,0))(E_1(t,s) + E_2(t,0,r)) = 0, \\
&\frac{\partial E_2(t,s,r)}{\partial t} - \frac{\partial E_2(t,s,r)}{\partial s} - \frac{\partial E_2(t,s,r)}{\partial r} \\
&\qquad\qquad - 2(E_1(t,s) + E_2(t,s,0))(E_1(t,r) + E_2(t,r,0)) = 0, \\
&\frac{dE_3(t)}{dt} + E_0(t)\sigma^2 = 0,
\end{aligned}
$$

with boundary conditions

$$
\begin{aligned}
&E_0(T) = \frac{c}{2}, \quad E_1(T,s) = 0, \quad E_2(T,s,r) = 0, \quad E_2(t,s,r) = E_2(t,r,s), \\
&E_1(t,-\tau) = -E_0(t), \quad E_2(t,s,-\tau) = -E_1(t,s), \quad E_3(T) = 0.
\end{aligned}
$$

Finite Dimensional Projection

- Set $u^i(t, z_0, z_1) := U(t, z_0^i, z_1^i, \nu^i)$, where $\nu^i = \frac{1}{N-1}\sum_{k\neq i}\delta_{(z_0^k, z_1^k)}$, denotes the joint empirical measure of z_0 and z_1. The empirical measure of z_0 is given by $\mu_0^i = \frac{1}{N-1}\sum_{k\neq i}\delta_{z_0^k}$, and the empirical measure of z_1 is given by $\mu_1^i = \frac{1}{N-1}\sum_{k\neq i}\delta_{z_1^k}$.
- By direct computation, for $k \neq i$, and any $N \geq 2$,

$$
\begin{aligned}
\partial_{z_0^k} u^i(t, z_0, z_1) =& \frac{1}{N-1} D_{\mu_0^i} U(t, z_0^i, z_1^i, \nu^i, z_0^k), \\
\partial_{z_1^k} u^i(t, z_0, z_1) =& \frac{1}{N-1} D_{\mu_1^i} U(t, z_0^i, z_1^i, \nu^i, z_1^k), \\
\partial_{z_0^k z_0^k} u^i(t, z_0, z_1) =& \frac{1}{N-1} \partial_{z_0^k} [D_{\mu_0^i} U](t, z_0^i, z_1^i, \nu^i, z_0^k) \\
&+ \frac{1}{(N-1)^2} D_{\mu_0^i \mu_0^i} U(t, z_0^i, z_1^i, \nu^i, z_0^k, z_0^k).
\end{aligned}
$$

Convergence of the Nash System

Proposition

For any $i \in \{1, \cdots, N\}$, $u^i(t, z_0, z_1)$ satisfies

$$\partial_t u^i + \sum_{k=1}^{N} \frac{1}{2}\sigma^2 \partial_{z_0^k z_0^k} u^i + \sum_{k=1}^{N} \int_{-\tau}^{0} z_1^k \frac{d}{ds}(\partial_{z_1^k} u^i) ds$$
$$- \sum_{k \neq i}^{N} \left(\partial_{z_0^k} u^k - [\partial_{z_1^k} u^k](-\tau)\right)\left(\partial_{z_0^k} u^i - [\partial_{z_1^k} u^i](-\tau)\right)$$
$$- \frac{1}{2}\left(\partial_{z_0^i} u^i - [\partial_{z_1^i} u^i](-\tau)\right)^2 + \frac{\epsilon}{2}(\bar{z}_0 - z_0^i)^2 + e^i(t, z) = 0,$$

where $\|e^i(t, z)\| < \frac{C}{N}$, with terminal condition $u^i(T, z) = \frac{c}{2}(\bar{z}_0 - z_0^i)^2$.

This shows that $(u^i)_{i \in \{1, \ldots, N\}}$ is **"almost"** a solution to the Nash system.

Convergence of the Nash System

Let V^i be the solution to the HJB equation of the N-player system, where $N \geq 1$ fixed, and U be the solution to the master equation. Fix any $(t_0, \nu_0) \in [0, T] \times \boldsymbol{P}(\boldsymbol{R} \times \boldsymbol{H})$. For any $z \in \boldsymbol{R}^N \times \boldsymbol{H}^N$, let $\nu^i = \frac{1}{N-1} \sum_{j \neq i}^{N} \delta_{(z_0^j, z_1^j)}$, then we have

$$\frac{1}{N} \sum_{i=1}^{N} |V^i(t_0, z) - U(t_0, z^i, \nu^i)| \leq CN^{-1}.$$

The End

THANKS FOR YOUR ATTENTION