Beyond mean field limits: Local dynamics for large sparse networks of interacting processes

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March 8, 2019

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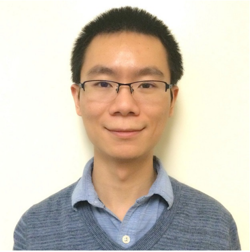

Ruoyu Wu

Networks of interacting Markov chains

Inputs:

- Graph $G = (V, E)$
- Independent noises $\xi_v(t)$, $v \in V$, $t = 0, 1, \ldots$
- Transition rule F,

Particles labeled by $v \in V$ evolve/interact according to

$$X_v(t+1) = F\Big(X_v(t), (X_u(t))_{u \sim v}, \xi_v(t+1)\Big).$$

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See also: probabilistic cellular automaton, synchronous Markov chain, simultaneous updating

Examples: Contact process, voter model, exclusion processes, spin systems...

Example: Voter model

State space $S = \{0, 1\}$.
Let d_v = degree of vertex v.

Transition rule: At time t, if particle v is at...

- state $X_v(t) = 0$, it switches to $X_v(t+1) = 1$ w.p.

$$\frac{1}{d_v} \sum_{u \sim v} 1_{\{X_u(t)=1\}},$$

- state $X_v(t) = 1$, it switches to $X_v(t+1) = 0$ w.p.

$$\frac{1}{d_v} \sum_{u \sim v} 1_{\{X_u(t)=0\}}.$$

Tendency to follow the majority of neighboring particles.

Example: Voter model

State space $S = \{0, 1\}$. Parameters $p, q \in [0, 1]$.
Let d_v = degree of vertex v.

Transition rule: At time t, if particle v is at...

- state $X_v(t) = 0$, it switches to $X_v(t+1) = 1$ w.p.

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$$q \frac{1}{d_v} \sum_{u \sim v} 1_{\{X_u(t)=0\}}.$$

Tendency to follow the majority of neighboring particles.

Networks of interacting diffusions

Particles labeled by $v \in V$ interact according to

$$dX_v(t) = b(X_v(t), (X_u(t))_{u \sim v})dt + dW_v(t),$$

where $(W_v)_{v \in V}$ are independent Brownian motions.

This talk focuses on discrete time, but there is a parallel story for these continuous-time models, with completely different proofs!

Aside: systemic risk models

Most systemic risk models can be grouped in two camps:

(A) Dynamic particle system models.
$\rightsquigarrow$ Mean field analysis is very tractable but works only for complete networks.

(B) Static network models.
$\rightsquigarrow$ Capture realistic network structure but devoid of dynamics.

Bridge this gap by incorporating networks into particle systems?

Networks of interacting Markov chains, more precisely

Inputs:

- Arbitrary (Polish) state space S.
- Independent noises $\xi_v(t)$, $v \in V$, $t = 0, 1, \ldots$, values in Ξ.
- Continuous transition rule $F : S \times \bigcup_{k=0}^{\infty} S^k \times \Xi \to S$, symmetric in second argument.
- Initial distribution for i.i.d. initial states

On any finite/countable locally finite graph $G = (V, E)$, define:

$$X_v^G(t+1) = F\Big(X_v^G(t), (X_u^G(t))_{u \sim v}, \xi_v(t+1)\Big), \quad v \in V,\ t \in \mathbb{N}_0.$$

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Example: $F(x, (y_i)_{i=1,\ldots,k}, \xi) = \widehat{F}\left(x, \frac{1}{k}\sum_{i=1}^{k} \delta_{y_i}, \xi\right)$ depends on empirical distribution of neighbors, $\widehat{F} : S \times \mathcal{P}(S) \times \Xi \to S$.

Large n behavior?

$$X_v^G(t+1) = F\Big(X_v^G(t), (X_u^G(t))_{u \sim v}, \xi_v(t+1)\Big).$$

Key question

Given a sequence of graphs $G_n = (V_n, E_n)$ with $|V_n| \to \infty$, how can we approximate the system or describe the limiting behavior?

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Prior literature: Plenty of work on long-time/stationary behavior, connections with Gibbs measures.

Our work: Large-scale behavior.

Large n behavior?

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Mean field as a special case

If G_n is the complete graph on n vertices, and F depends on neighbors through empirical distribution, then $X_v^{G_n} \Rightarrow X$, where

$$X(t+1) = \widehat{F}(X(t), \mathrm{Law}(X(t)), \xi(t+1)).$$

Moreover, the empirical measure process $\frac{1}{|G_n|}\sum_{v\in G_n}\delta_{X_v^{G_n}(t)}$ converges in probability to $\mathrm{Law}(X(t))$.

$\rightsquigarrow$ asymptotically i.i.d. particles

Large n behavior

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Given a sequence of graphs $G_n = (V_n, E_n)$ with $|V_n| \to \infty$, how can we describe the limiting behavior of...

- a "typical" or tagged particle $X_v^{G_n}(t)$?
- the empirical distribution of particles $\frac{1}{|V_n|}\sum_{v\in V_n} \delta_{X_v^{G_n}(t)}$?

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Theorem (Bhamidi-Budhiraja-Wu '16, for diffusions)

Suppose $G_n = G(n, p_n)$ is Erdős-Rényi, with $np_n \to \infty$. Then everything behaves like in the mean field case.

See also Delattre-Giacomin-Luçon '16, Delarue '17, Coppini-Dietert-Giacomin '18, Oliveira-Reis '18, Luçon '18.

Large n behavior

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Observation: $np_n \approx$ average degree, so $np_n \to \infty$ means the graphs are dense.

Large n behavior, for sparse graphs?

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Our focus: The sparse regime, where degrees do not diverge. How does does the $n \to \infty$ limit reflect the graph structure?

Example: Erdős-Rényi $G(n, p_n)$ with $np_n \to p \in (0, \infty)$.

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Example: Detering-Fouque-Ichiba '18 treats directed cycle graph.

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Our approach:

1. Show that if $G_n \to G$ in a sense then also $X^{G_n} \to X^G$.

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Our approach:

1. Show that if $G_n \to G$ in a sense then also $X^{G_n} \to X^G$.
2. Show that if limiting G is a "nice tree" then X^G can be characterized by autonomous dynamics for a single particle and its neighborhood, the local dynamics.

Local convergence of graphs

Idea: Encode sparsity via local convergence of graphs.
(a.k.a. Benjamini-Schramm convergence, see Aldous-Steele '04)

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(a.k.a. Benjamini-Schramm convergence, see Aldous-Steele '04)

Definition: A graph $G = (V, E, \o)$ is assumed to be rooted, finite or countable, locally finite, and connected.

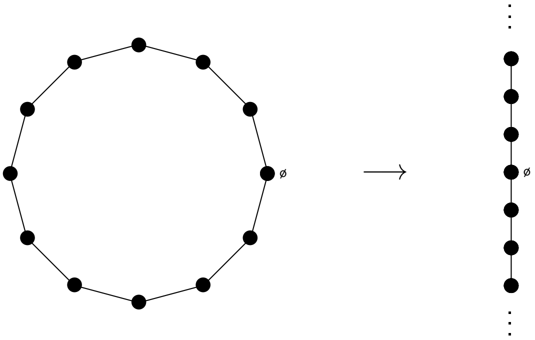
Definition: Rooted graphs G_n converge locally to G if:

$$\forall k \; \exists N \text{ s.t. } B_k(G) \cong B_k(G_n) \text{ for all } n \geq N,$$

where $B_k(\cdot)$ is ball of radius k at root, and $\cong$ means isomorphism.

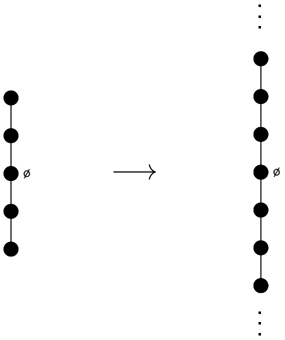
Examples of local convergence

1. Cycle graph converges to infinite line



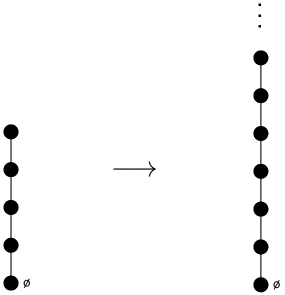
Examples of local convergence

2. Line graph converges to infinite line



Examples of local convergence

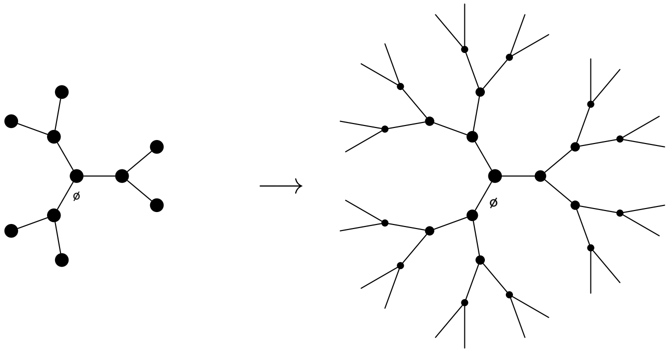
3. Line graph rooted at end converges to semi-infinite line



Examples of local convergence

4. Finite to infinite d-regular trees

(A graph is d-regular if ever vertex has degree d.)

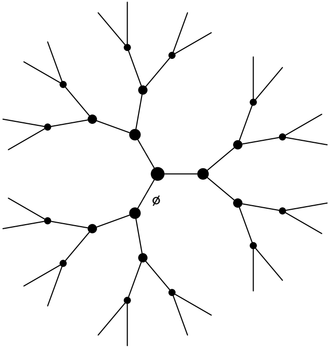


Examples of local convergence

5. Uniformly random regular graph to infinite regular tree

Fix d. Among all d-regular graphs on n vertices, select one uniformly at random. Place the root at a (uniformly) random vertex. When $n \to \infty$, this converges (in law) to the infinite d-regular tree. (Bollobás '80)

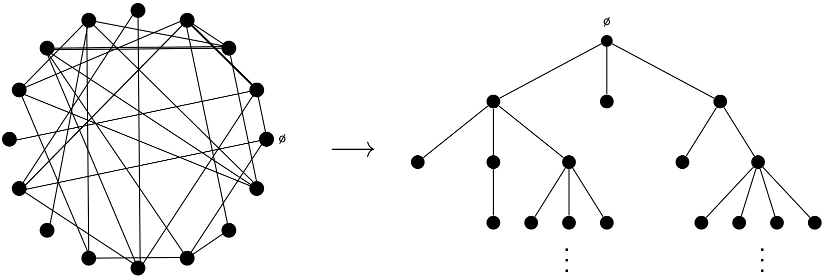
$\longrightarrow$



Examples of local convergence

6. Erdős-Rényi to Galton-Watson(Poisson)

If $G_n = G(n, p_n)$ with $np_n \to p \in (0, \infty)$, then G_n converges in law to the Galton-Watson tree with offspring distribution $\mathrm{Poisson}(p)$.



Examples of local convergence

7. Configuration model to unimodular Galton-Watson

If G_n is drawn from the configuration model on n vertices with degree distribution $\rho \in \mathcal{P}(\mathbb{N}_0)$, then G_n converges in law to the unimodular Galton-Watson tree $\mathrm{UGW}(\rho)$.

- Construct $\mathrm{UGW}(\rho)$ by letting root have ρ-many children, and each child thereafter has $\widehat{\rho}$-many children, where

$$\widehat{\rho}(n) = \frac{(n+1)\rho(n+1)}{\sum_k k\rho(k)}.$$

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- **Example 1**: $\rho = \mathrm{Poisson}(p) \implies \widehat{\rho} = \mathrm{Poisson}(p)$.

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- **Example 1**: $\rho = \mathrm{Poisson}(p) \implies \widehat{\rho} = \mathrm{Poisson}(p)$.
- **Example 2**: $\rho = \delta_d \implies \widehat{\rho} = \delta_{d-1}$, so $\mathrm{UGW}(\delta_d)$ is the (deterministic) infinite d-regular tree.

Intuition: Root is equally likely to be any vertex. Aldous-Lyons '07

Local convergence of marked graphs

Recall: $G_n = (V_n, E_n, \emptyset_n)$ converges locally to $G = (V, E, \emptyset)$ if

$\forall k\ \exists N$ s.t. $B_k(G) \cong B_k(G_n)$ for all $n \geq N$.

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Definition: With G_n, G as above: Given a metric space $(\mathcal{X}, d_{\mathcal{X}})$ and a sequence $\boldsymbol{x}^n = (x_v^n)_{v \in G_n} \in \mathcal{X}^{G_n}$, say that $(G_n, \boldsymbol{x}^n)$ converges locally to $(G, \boldsymbol{x})$ if

$$\forall k, \epsilon > 0 \ \exists N \text{ s.t. } \forall n \geq N \ \exists \varphi : B_k(G_n) \to B_k(G) \text{ isomorphism}$$
$$\text{s.t. } \max_{v \in B_k(G_n)} d_{\mathcal{X}}(x_v^n, x_{\varphi(v)}) < \epsilon.$$

Lemma

The set $\mathcal{G}_*[\mathcal{X}]$ of (isomorphism classes of) $(G, \boldsymbol{x})$ admits a Polish topology compatible with the above convergence.

Local convergence of marked graphs

Recall: Particle system on a rooted graph $G = (V, E, \varnothing)$:

$$X_v^G(t+1) = F(X_v^G(t), (X_u^G(t))_{u \sim v}, \xi_v(t+1)).$$

Theorem

If $G_n \to G$ locally, then (G_n, X^{G_n}) converges in law to (G, X^G) in $\mathcal{G}_*[S^\infty]$. Valid for random graphs too.

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In particular, root particle dynamics converge: $X_{\varnothing_n}^{G_n} \Rightarrow X_{\varnothing}^G$ in S^∞.

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Empirical measure convergence is harder. In general, if $G_n \to G$ with G infinite,

$$\frac{1}{|G_n|} \sum_{v \in G_n} \delta_{X_v^{G_n}} \not\Rightarrow \mathrm{Law}(X_\varnothing^G).$$

Example: G_n a d-regular tree of height n, $d \geq 3$.

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Empirical measure convergence is harder. If $G_n \sim G(n, p_n)$, $np_n \to p \in (0, \infty)$, then

$$\frac{1}{|G_n|} \sum_{v \in G_n} \delta_{X_v^{G_n}} \Rightarrow \mathrm{Law}(X_\o^T), \quad \text{in } \mathcal{P}(S^\infty),$$

where $T \sim \mathrm{GW}(\mathrm{Poisson}(p))$.

Local convergence of marked graphs

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Goal: For infinite regular trees (more generally UGW trees), find "local" dynamics for root particle and neighbors, $\{X_\o, X_v : v \sim \o\}$.

Key idea: Exploit conditional independence structure.

Markov random fields

Notation: For a set A of vertices in a graph $G = (V, E)$, define

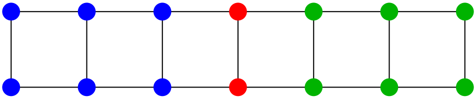
$$\text{Boundary: } \partial A = \{u \in V \backslash A : \exists u \in A \text{ s.t. } u \sim v\}.$$

Definition: A family of random variables $(Y_v)_{v \in G}$ is a Markov random field if

$$(Y_v)_{v \in A} \perp (Y_v)_{v \in B} \;\mid\; (Y_v)_{v \in \partial A},$$

for all finite sets $A, B \subset V$ with $B \cap (A \cup \partial A) = \emptyset$.

Example: A ∂A B



Markov random fields

$$X_v(t+1) = F_v\Big(X_v(t), (X_u(t))_{u\sim v}, \xi_v(t+1)\Big),$$

Assume the initial states $(X_v(0))_{v\in G}$ are i.i.d.

Question #1:

For each time t, do the particle positions $(X_v(t))_{v\in G}$ form a Markov random field?

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Question #2:

For each time t, do the particle trajectories $(X_v[t])_{v\in G}$ form a Markov random field? Here $x[t] = (x(0), \dots, x(t))$.

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Answer #2:

NO

Second-order Markov random fields

Notation: For a set A of vertices in a graph $G = (V, E)$, define

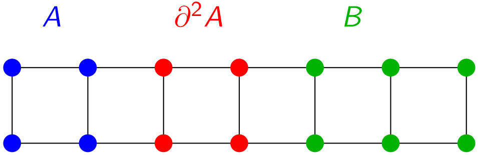
$$\text{Double-boundary: } \partial^2 A = \partial A \cup \partial(A \cup \partial A).$$

Definition: A family of random variables $(Y_v)_{v \in G}$ is a 2nd-order Markov random field if

$$(Y_v)_{v \in A} \perp (Y_v)_{v \in B} \mid (Y_v)_{v \in \partial^2 A},$$

for all finite sets $A, B \subset V$ with $B \cap (A \cup \partial^2 A) = \emptyset$.

Example:



Second-order Markov random fields

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Assume the initial states $(X_v(0))_{v\in G}$ are i.i.d.

Question #3:

For each time t, do the particle positions $(X_v(t))_{v\in G}$ form a second-order Markov random field?

Second-order Markov random fields

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Question #3:

For each time t, do the particle positions $(X_v(t))_{v\in G}$ form a second-order Markov random field?

Answer #3:

NO

Second-order Markov random fields

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Assume the initial states $(X_v(0))_{v\in G}$ are i.i.d.

Question #4:

For each time t, do the particle trajectories $(X_v[t])_{v\in G}$ form a second-order Markov random field? Here $x[t] = (x(0), \ldots, x(t))$.

Second-order Markov random fields

$$X_v(t+1) = F_v\Big(X_v(t), (X_u(t))_{u\sim v}, \xi_v(t+1)\Big),$$

Assume the initial states $(X_v(0))_{v\in G}$ are i.i.d.

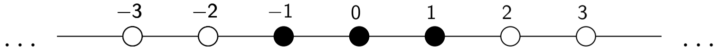
Question #4:

For each time t, do the particle trajectories $(X_v[t])_{v\in G}$ form a second-order Markov random field? Here $x[t] = (x(0), \ldots, x(t))$.

Answer #4:

Theorem: YES. Holds also if $(X_v(0))_{v\in G}$ is a second-order MRF.

Local dynamics for infinite 2-regular tree



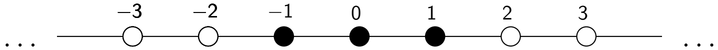
Particle system on infinite line graph, $i \in \mathbb{Z}$:

$$X_i(t+1) = F\Big(X_i(t), X_{i-1}(t), X_{i+1}(t), \xi_i(t+1)\Big)$$

Assume: F is symmetric in neighbors, $F(x, y, z, \xi) = F(x, z, y, \xi)$.

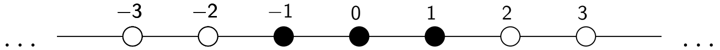
Goal: Find an autonomous stochastic process (Y_{-1}, Y_0, Y_1) which agrees in law with (X_{-1}, X_0, X_1).

Local dynamics for infinite 2-regular tree



1. Start with $(Y_{-1}, Y_0, Y_1)(0) = (X_{-1}, X_0, X_1)(0)$.

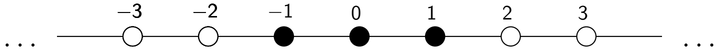
Local dynamics for infinite 2-regular tree



1. Start with $(Y_{-1}, Y_0, Y_1)(0) = (X_{-1}, X_0, X_1)(0)$.
2. **Inductively**: At time t, define transition kernel

$$\gamma_t(\cdot \mid y_0, y_1) = \mathrm{Law}\Big(Y_{-1}(t) \mid Y_0[t] = y_0[t], Y_1[t] = y_1[t] \Big).$$

Local dynamics for infinite 2-regular tree



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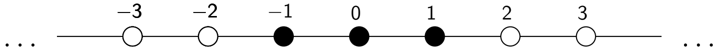
3. Sample ghost particles $Y_{-2}(t)$ and $Y_2(t)$ so that

$$\mathbb{P}\Big(Y_{-2}(t) \in dy_{-2}, Y_2(t) \in dy_2 \,\Big|\, Y_{-1}[t], Y_0[t], Y_1[t]\Big) = \gamma_t(dy_{-2} \mid Y_{-1}, Y_0) \times \gamma_t(dy_2 \mid Y_1, Y_0)$$

Local dynamics for infinite 2-regular tree



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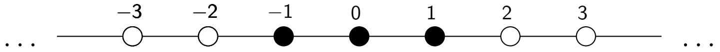
4. Sample new noises $(\xi_{-1}, \xi_0, \xi_1)(t+1)$ independently, and update:

$$Y_i(t+1) = F\Big(Y_i(t), Y_{i-1}(t), Y_{i+1}(t), \xi_i(t+1) \Big), \quad i = -1, 0, 1$$

Symmetries, or why "ghost particles" work

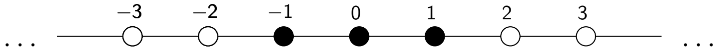


Particle system on infinite line graph, $i \in \mathbb{Z}$:

$$X_i(t+1) = F\Big(X_i(t), X_{i-1}(t), X_{i+1}(t), \xi_i(t+1)\Big),$$

$$\gamma_t(\cdot \mid x_0, x_1) = \mathrm{Law}\Big(X_{-1}(t) \mid X_0[t] = x_0[t], X_1[t] = x_1[t]\Big).$$

Symmetries, or why “ghost particles” work



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$$\gamma_t(\cdot \mid x_0, x_1) = \mathrm{Law}\Big(X_{-1}(t) \mid X_0[t] = x_0[t], X_1[t] = x_1[t]\Big).$$

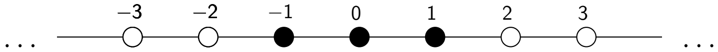
Symmetry 1: Shifts.

$$(X_1, \dots, X_k) \stackrel{d}{=} (X_{i+1}, \dots, X_{i+k}), \quad \forall i \in \mathbb{Z}, k \in \mathbb{N}$$

Implication:

$$\gamma_t(X_{-1}, X_0) = \mathrm{Law}\big(X_{-2}(t) \mid X_{-1}[t],\ X_0[t]\big)$$

Symmetries, or why “ghost particles” work



Particle system on infinite line graph, $i \in \mathbb{Z}$:

$$X_i(t+1) = F\Big(X_i(t), X_{i-1}(t), X_{i+1}(t), \xi_i(t+1)\Big),$$

$$\gamma_t(\cdot \mid x_0, x_1) = \mathrm{Law}\Big(X_{-1}(t) \mid X_0[t] = x_0[t], X_1[t] = x_1[t]\Big).$$

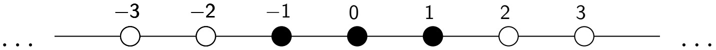
Symmetry 2: Reflection.

$$(X_1, X_2, \ldots, X_k) \stackrel{d}{=} (X_k, X_{k-1}, \ldots, X_1), \quad \forall i \in \mathbb{Z}, k \in \mathbb{N}$$

Implication:

$$\gamma_t(X_1, X_0) = \mathrm{Law}\big(X_2(t) \mid X_1[t],\ X_0[t]\big)$$

Symmetries, or why "ghost particles" work



Particle system on infinite line graph, $i \in \mathbb{Z}$:

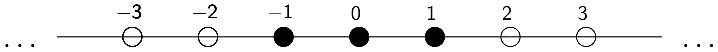
$$X_i(t+1) = F\Big(X_i(t), X_{i-1}(t), X_{i+1}(t), \xi_i(t+1)\Big),$$

$$\gamma_t(\cdot \mid x_0, x_1) = \mathrm{Law}\Big(X_{-1}(t) \mid X_0[t] = x_0[t], X_1[t] = x_1[t]\Big).$$

Combining two symmetries & conditional independence:

$$\mathbb{P}\Big(X_{-2}(t) \in dx_{-2}, X_2(t) \in dx_2 \;\Big|\; X_{-1}[t], X_0[t], X_1[t]\Big)$$
$$= \gamma_t(dx_{-2} \mid X_{-1}, X_0) \times \gamma_t(dx_2 \mid X_1, X_0)$$

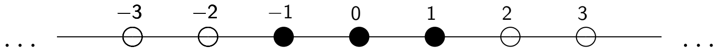
Continuous-time case



Particle system on infinite line graph, $i \in \mathbb{Z}$:

$$dX_t^i = b(X_t^i, X_t^{i-1}, X_t^{i+1})dt + dW_t^i$$

Continuous-time case



Particle system on infinite line graph, $i \in \mathbb{Z}$:

$$dX_t^i = b(X_t^i, X_t^{i-1}, X_t^{i+1})dt + dW_t^i$$

Local dynamics:

$$\begin{aligned}
dY_t^1 &= \Big\langle \gamma_t(Y^1, Y^0),\ b(Y_t^1, Y_t^0, \cdot) \Big\rangle dt + dW_t^1 \\
dY_t^0 &= b(Y_t^0, Y_t^{-1}, Y_t^1)dt + dW_t^0 \\
dY_t^{-1} &= \Big\langle \gamma_t(Y^{-1}, Y^0),\ b(Y_t^{-1}, \cdot, Y_t^0) \Big\rangle dt + dW_t^{-1} \\
\gamma_t(y^0, y^{-1}) &= \mathrm{Law}\big(Y_t^1 \mid Y^0[t] = y^0[t],\ Y^{-1}[t] = y^{-1}[t]\big)
\end{aligned}$$

Thm: Exist/unique in law, and $(Y^{-1}, Y^0, Y^1) \stackrel{d}{=} (X^{-1}, X^0, X^1)$.

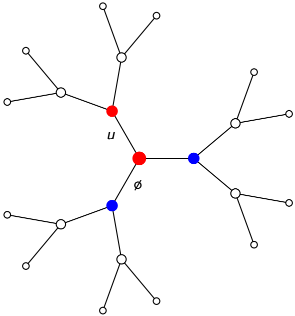
Infinite d-regular trees

Autonomous dynamics for root particle and its neighbors,

$$X_\varnothing(t),\ (X_v(t))_{v\sim\varnothing},$$

involving conditional law of $d-1$ children given root and one other child u:

$$\mathrm{Law}\big((X_v)_{v\sim\varnothing,\,v\neq u}\,\big|\,X_\varnothing,\ X_u\big)$$



$d = 3$

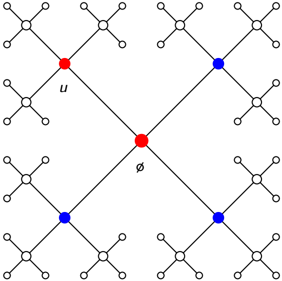
Infinite d-regular trees

Autonomous dynamics for root particle and its neighbors,

$$X_{\varnothing}(t),\ (X_v(t))_{v\sim\varnothing},$$

involving conditional law of $d-1$ children given root and one other child u:

$$\mathrm{Law}\big((X_v)_{v\sim\varnothing,\,v\neq u}\,\big|\,X_{\varnothing},\ X_u\big)$$



$d = 4$

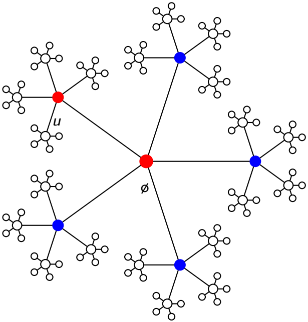
Infinite d-regular trees

Autonomous dynamics for root particle and its neighbors,

$$X_\varnothing(t),\ (X_v(t))_{v\sim\varnothing},$$

involving conditional law of $d-1$ children given root and one other child u:

$$\mathrm{Law}\big((X_v)_{v\sim\varnothing,\,v\neq u}\,\big|\,X_\varnothing,\ X_u\big)$$

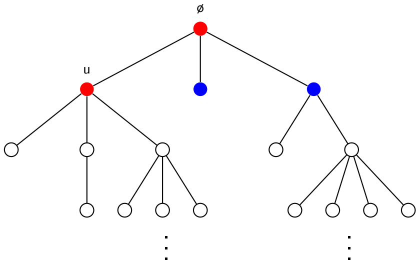


$d = 5$

Unimodular Galton-Watson trees

Autonomous dynamics for root & first generation involving conditional law of 1st-generation given root and one child.

Condition on tree structure as well!



Summary

Theorem 1

If graph sequence converges locally, then particle systems converge locally as well. (Similar result in independent work Oliveira-Reis-Stolerman '19.)

Theorem 2

Root neighborhood particles in a unimodular Galton-Watson tree admits well-posed local dynamics.

Corollary: If finite graph sequence converges locally to a unimodular Galton-Watson tree, then root neighborhood particles converges to the local dynamics.

Example: Linear Gaussian dynamics

State space $\mathbb{R}$, noises $\xi_v(t)$ are independent standard Gaussian.

$$X_v(t+1) = aX_v(t) + b\sum_{u\sim v} X_u(t) + c + \xi_v(t+1)$$
$$X_v(0) = \xi_v(0), \qquad a, b, c \in \mathbb{R}$$

$\rightsquigarrow$ conditional laws are all Gaussian

Proposition

Suppose the graph G is an infinite d-regular tree, $d > 2$.
Simulating local dynamics for one particle up to time t is $O(t^2 d^2)$.

Compare: Naive simulation using infinite tree is $O((d-1)^{t+1})$.

Example: Contact process

Each particle is either 1 or 0. Parameters $p, q \in [0, 1]$.

Transition rule: At time t, if particle v...

- is at state $X_v(t) = 1$, it switches to $X_v(t+1) = 0$ w.p. q,
- is at state $X_v(t) = 0$, it switches to $X_v(t+1) = 1$ w.p.

$$\frac{p}{d_v} \sum_{u \sim v} X_u(t),$$

where $d_v =$ degree of vertex v.

Example: Contact process

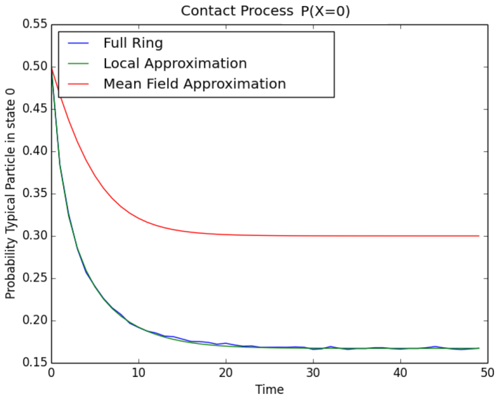


Figure: Infinite 2-regular tree (line), $p = 2/3$, $q = 0.1$

Credit: Ankan Ganguly & Mitchell Wortsman, Brown University