Solvable Stochastic Control and Stochastic Differential Games

Tyrone E. Duncan

University of Kansas

February 2016

Outline

1. Some general methods for solving control problems
2. Linear-quadratic control and generalizations
3. Fractional Brownian motions and other noise processes for controlled linear systems
4. Linear-exponential-quadratic Gaussian control and games
5. Nash equilibria for some stochastic differential games
6. Discrete time LQ control with correlated noise
7. Control and differential games for nonlinear stochastic systems in spheres, projective spaces and hyperbolic spaces
8. Infinite time horizon linear-quadratic control for distributed parameter (SPDEs) systems with fractional Brownian motions
9. Linear-quadratic control for SPDEs with multiplicative Gaussian noise
10. Some potential generalizations

Some of this is joint work with B. Maslowski and B. Pasik-Duncan.

Solution Methods for Stochastic Control Problems

1. Hamilton-Jacobi-Bellman equations
2. Stochastic maximum principle or dynamic programming and backward stochastic differential equations

Hamilton-Jacobi-Bellman Equation

Hamilton-Jacobi Equation for a deterministic problem

$$\frac{\partial V(t,x(t))}{\partial t} + min_{u \in \mathbb{R}^m} H(x(t), u(t), \nabla V, t) = 0$$

$H(x(t), u(t), p(t), t)$ is the Hamiltonian.

Hamilton-Jacobi-Bellman Equation for a stochastic problem

$$\frac{\partial V}{\partial t} + min_{u \in \mathbb{R}^m} [\frac{1}{2} tr(D^2 V) + H(x(t), u(t), p(t), t)] = 0$$

Stochastic Maximum Principle

Maximize the Hamiltonian for the problem.

This method provides a necessary condition for optimality. With some convexity conditions, the necessary condition is also sufficient. Since a condition is at the final time it is necessary to solve a backward stochastic differential equation which means solving backward in time but having a forward measurability for the solution.

Linear-Quadratic and Linear-Quadratic Gaussian Control

$$
\begin{aligned}
\frac{dx}{dt} &= Ax + Bu \qquad (1)\\
x(0) &= x_0\\
J(u) &= \frac{1}{2}[\int_0^T (< Qx(t), x(t) > + < Ru(t), u(t) >)dt\\
&+ < Mx(T), x(T) >]
\end{aligned}
$$

Admissible controls $\quad \mathcal{U} = \{u : u \in L^2([0, T])\}$

$$
\begin{aligned}
dX(t) &= (AX(t) + BU(t))dt + dW(t) \qquad (2)\\
X(0) &= X_0\\
J(U) &= \frac{1}{2}\mathbb{E}[\int_0^T (< QX(t), X(t) > + < RU(t), U(t) >)dt\\
&+ < MX(T), X(T) >]
\end{aligned}
$$

Solution Methods

Hamilton-Jacobi Equation for the deterministic problem

$$\frac{\partial V(t,x(t))}{\partial t} + min_{u \in \mathbb{R}^m} H(x(t),u(t),\nabla V,t) = 0$$

$$H(x(t),u(t),p(t),t) = \frac{1}{2} < Qx(t),x(t) > + \frac{1}{2} < Ru(t),u(t) >$$
$$+ < p(t),Ax(t) > + < p(t),Bu(t) >$$

Hamilton-Jacobi-Bellman Equation for the stochastic problem

$$\frac{\partial V}{\partial t} + min_{u \in \mathbb{R}^m}[\frac{1}{2}tr(D^2V) + < DV,Ax+Bu > + \frac{1}{2} < Qx,x >$$
$$+ \frac{1}{2} < Ru,u >] = 0$$

Optimal Control

An optimal control for both problems is

$$U^*(t) = -R^{-1}BP(t)X(t)$$

where P is the unique, symmetric, positive solution of the following Riccati equation

$$\frac{dP}{dt} = -PA - A^T P + PBR^{-1}B^T P - Q$$
$$P(T) = M$$

The value function for the deterministic problem is $V(s,y) = \frac{1}{2} < P(s)y, y >$ and the value function for the stochastic problem is

$$V(s,y) = \frac{1}{2}(< P(s)y, y > + q(s))$$
$$q(s) = \int_s^T tr(P(r))dr$$

Fractional Brownian Motions

Let $H \in (0,1)$ be fixed. The process $(B(t), t \geq 0)$ is a real-valued standard fractional Brownian motion with the Hurst parameter index $H \in (0,1)$ if it is a Gaussian process with continuous sample paths that satisfies

$$\mathbb{E}[B(t)] = 0$$
$$\mathbb{E}[B(s)B(t)] = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right)$$

for all $s, t \in \mathbb{R}_+$.

The formal derivative $\frac{dB}{dt}$ is called fractional Gaussian noise.

Some properties

1. Self-similarity

$$(B^H(\alpha t), t \geq 0) \overset{L}{\sim} (\alpha^H B^H(t), t \geq 0)$$

   for $\alpha > 0$

2. Long range dependence for $H \in (\frac{1}{2}, 1)$

$$r(n) = \mathbb{E}[B^H(1)(B^H(n+1) - B^H(n))]$$

$$\Sigma_{n=0}^{\infty} r(n) = \infty$$

3. A sample path property
$(B^H(t), t \geq 0)$ is of unbounded variation so the sample paths are not differentiable a.s.

$$\Sigma_i |B^H(t_{i+1}^{(n)}) - B^H(t_i^{(n)})|^p \to \begin{cases} 0 & pH > 1 \\ c(p) & pH = 1 \\ +\infty & pH < 1 \end{cases}$$

$c(p) = \mathbb{E}|B^H(1)|^p$
$(t_i^{(n)}, i = 0, 1, \cdots, n; n \in \mathbb{N})$ is a sequence of nested partitions of $[0, 1]$ such that $t_0^{(n)} = 0$ and $t_n^{(n)} = 1$ for all $n \in \mathbb{N}$ and the sequence of partitions becomes dense in $[0, 1]$.
4. For $H \neq \frac{1}{2}$ a FBM is neither Markov nor semimartingale.

Some Applications of FBMs

1. Turbulence
2. Hydrology
3. Economic Data
4. Telecommunications
5. Earthquakes
6. Epilepsy
7. Cognition
8. Biology

A Generalization of Linear-Quadratic Control

Theorem. For the control problem given above where W is an arbitrary square integrable process with continuous sample paths and filtration $(\mathcal{F}(t), t \in [0, T])$ and the family of admissible controls, $\mathcal{U}$, there is an optimal control U^* that can be expressed as

$$U^*(t) = -R^{-1}B^T(P(t)X(t) + V(t))$$

where $(P(t), t \in [0, T])$ is the unique symmetric positive definite solution of the Riccati equation

$$\frac{dP}{dt} = -PA - A^T P + PBR^{-1}B^T P - Q$$
$$P(T) = M$$

$$V(t) = \mathbb{E}[\int_t^T \Phi_P(s,t)P(s)dW(s)|\mathcal{F}(t)]$$

and Φ_P is the fundamental solution for the matrix equation

$$\begin{aligned} \frac{d\Phi_P(s,t)}{dt} &= -(A^T - P(t)BR^{-1}B^T)\Phi_P(s,t) \\ \Phi_P(s,s) &= I \end{aligned}$$

Corollary. If W is an arbitrary standard fractional Brownian motion then

$$V(t) = \int_0^t u_{1/2-H} I_{t-}^{1/2-H} (I_{T-}^{H-1/2} u_{H-1/2} \Phi_P(., t) P) dW$$

and I_b^a is a fractional integral if $a > 0$ and a fractional derivative if $a < 0$.

A Method of Proof

$$
\begin{aligned}
J_n^0(U) \quad - \quad & \frac{1}{2} < P(0)X_0, X_0 > - < \phi_n(0), X_0 > \\
= \quad & \frac{1}{2}\int_0^T [(|R^{-1/2}[RU + B^T P X_n + B^T \phi_n]|^2 - |R^{-1/2} B^T \phi_n|^2)dt \\
& +2 < \phi_n, dB_n >]
\end{aligned}
$$

$$
U_n^*(t) = -R^{-1}(B^T P(t) X_n(t) + B^T \phi_n(t))
$$

A Hilbert Space for a FBM

Let L_H^2 be the Hilbert space whose inner product $< \cdot, \cdot >_H$ is given by

$$< f, g >_H = \rho(H) \int_0^T u_{\frac{1}{2}-H}^2(r)(I_{T-}^{H-\frac{1}{2}} u_{H-\frac{1}{2}} f)(r)(I_{T-}^{H-\frac{1}{2}} u_{H-\frac{1}{2}} g)(r) dr$$

where $\rho(H) = \frac{2H\Gamma(H+\frac{1}{2})\Gamma(\frac{3}{2}-H)}{\Gamma(2-2H)}$. The term $I_{T-}^{H-\frac{1}{2}}$ is a fractional integral for $H \in (\frac{1}{2}.1)$ and a fractional derivative for $H \in (0, \frac{1}{2})$. This Hilbert space is naturally associated with a fractional Brownian motion with Hurst parameter H by the covariance factorization.

$$\begin{aligned} \left(I_{T-}^{\alpha}\varphi\right)(t) &= \frac{1}{\Gamma(\alpha)} \int_t^T (s-t)^{\alpha-1}\varphi(s)\, ds \\ \left(D_{T-}^{\alpha}\psi\right)(t) &= \frac{1}{\Gamma(1-\alpha)} \left(\frac{\psi(t)}{(T-t)^{\alpha}} + \alpha \int_t^T \frac{\psi(s)-\psi(t)}{(s-t)^{\alpha+1}}\, ds \right) \end{aligned}$$

A General Linear Stochastic Control System

$$X(t-) = x + \int_0^t (A(s)X(s) + B(s)U(s))ds$$
$$+ \int_0^t \Sigma_{i=1}^d (C(s)^i X(s) + D(s)^i U(s)) dW^i(s)$$
$$+ \int_0^t F(s)dW(s) + Y(t-)$$

Linear Exponential Quadratic Gaussian Control

Stochastic system

$$\begin{aligned} dX(t) &= AX(t)dt + BU(t)dt + FdW(t) \\ X(0) &= X_0 \end{aligned}$$

where $X_0 \in \mathbb{R}^n$ is not random, $A \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$,
$B \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n), F \in \mathcal{L}(\mathbb{R}^p, \mathbb{R}^n), U(t) \in \mathbb{R}^m, U \in \mathcal{U}$,
$(W(t), t \in [0, T])$ is an $\mathbb{R}^p$-valued standard Brownian motion.
The family of admissible controls is
$\mathcal{U} = \{U : U$ is an $\mathbb{R}^m$-valued process adapted to
$(\mathcal{F}(t), t \in [0, T])$ such that $U \in L^2([0, T])$ a.s.$\}$.
Risk sensitive cost functional

$$\begin{aligned} J(U) &= \mu \mathbb{E} \exp[\frac{\mu}{2} \int_0^T (< QX(s), X(s) > + < RU(s), U(s) >) ds \\ &\quad + \frac{\mu}{2} < MX(T), X(T) >] \end{aligned}$$

where $\mu > 0$ is fixed.

Theorem. For the LEQG control problem given above there is an optimal control $(U^*(t), t \in [0, T])$ in $\mathcal{U}$ given by

$$U^*(t) = -R^{-1}B^T P(t)X(t)$$

where $(P(t), t \in [0, T])$ is assumed to be the unique, symmetric, positive solution of the following Riccati equation

$$\begin{aligned} -\frac{dP}{dt} &= PA + A^T P - P(BR^{-1}B^T - \mu FF^T)P + Q \\ P(T) &= M \end{aligned}$$

and the optimal cost is

$$J(U^*) = \mu G(0) exp[\frac{\mu}{2} < P(0)X_0, X_0 >]$$

and $(G(t), t \in [0, T])$ satisfies

$$\begin{aligned} -\frac{dG}{dt} &= \frac{\mu}{2} G \ tr(PFF^T) \\ G(T) &= 1 \end{aligned}$$

Sketch of Proof

$$J(U) = \mu \mathbb{E} exp[L(U)]$$

$$L(U) - \frac{\mu}{2} < P(0)X_0, X_0 >$$

$$= \frac{\mu}{2}[\int_0^T (< RU, U > + < PCR^{-1}B^T PX, X > +2 < B^T PX, U >)dt$$

$$+2 \int_0^T < PX, FdW > -\mu \int_0^T < PFF^T PX, X > dt + \int_0^T tr(PFF^T$$

$$= \frac{\mu}{2} \int_0^T |R^{-\frac{1}{2}}[RU + B^T PX]|^2 dt$$

$$+\mu \int_0^T < PX, FdW > -\frac{\mu^2}{2} \int_0^T < PFF^T PX, X > dt$$

$$+\frac{\mu}{2} \int_0^T tr(PFF^T)dt$$

Linear Exponential Quadratic Stochastic Differential Games

$$\begin{aligned} dX(t) &= AX(t)dt + BU(t)dt + CV(t)dt + FdW(t) \\ X(0) &= X_0 \end{aligned}$$

where $X_0 \in \mathbb{R}^n$ is not random,
$X(t) \in \mathbb{R}^n, A \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n), B \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n),\ U(t) \in \mathbb{R}^m,\ U \in \mathcal{U}$, $C \in \mathcal{L}(\mathbb{R}^p, \mathbb{R}^n),\ V(t) \in \mathbb{R}^p,\ V \in \mathcal{V}$, and $F \in \mathcal{L}(\mathbb{R}^q, \mathbb{R}^n)$. The positive integers (m, n, p, q) are arbitrary. The process $(W(t), t \geq 0)$ is an $\mathbb{R}^q$ valued standard Brownian motion

Payoff

$$
\begin{aligned}
J^0_\mu(U,V) &= \mu exp[\frac{\mu}{2}\int_0^T (< QX(s), X(s) > \\
&+ < RU(s), U(s) > - < SV(s), V(s) >)ds \\
&+ \frac{\mu}{2} < MX(T), X(T) >] \\
J_\mu(U,V) &= \mathbb{E}[J^0_\mu(U,V)]
\end{aligned}
$$

where
$Q \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n), R \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^m), S \in \mathcal{L}(\mathbb{R}^p, \mathbb{R}^p), M \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$, and $Q > 0$, $R > 0$, $S > 0$, and $M \geq 0$ are symmetric linear transformations and $\mu \neq 0$ is fixed. An assumption on the possible values for μ is given in the following theorem. The player with control U seeks to minimize the payoff J_μ while the player with control V seeks to maximize the payoff J_μ.

Theorem. The two person zero sum stochastic differential game described above has a Nash equilibrium using the optimal admissible control strategies for the two players, denoted U^* and V^*, given by

$$\begin{aligned} U^*(t) &= -R^{-1}B^TP(t)X(t) \\ V^*(t) &= S^{-1}C^TP(t)X(t) \end{aligned}$$

where $(P(t), t \in [0, T])$ is the unique positive symmetric solution of the following Riccati equation

$$\begin{aligned} -\frac{dP}{dt} &= Q + PA + A^TP \\ &- P(BR^{-1}B^T - CS^{-1}C^T - \mu FF^T)P \\ P(T) &= M \end{aligned}$$

and it is assumed that $BR^{-1}B^T - CS^{-1}C^T - \mu FF^T > 0$. The optimal payoff is

$$J_\mu(U^*, V^*) = \mu exp[\frac{\mu}{2}(< P(0)X_0, X_0 > + \int_0^T tr(PFF^T)dt)]$$

$$L_\mu(U,V) - \frac{\mu}{2} < P(0)X_0, X_0 >$$

$$= \frac{\mu}{2}[\int_0^T (< RU, U > - < SV, V >$$

$$+2 < PBU, X > +2 < PCV, X >$$

$$+ < PBR^{-1}B^T PX, X > - < PCS^{-1}C^T PX, X >$$

$$+2 < FdW, PX > -\mu < PFF^T PX, X > +tr(PFF^T))dt]$$

$$= \frac{\mu}{2} \int_0^T (|R^{-\frac{1}{2}}[RU + B^T PX]|^2 - |S^{-\frac{1}{2}}[SV - C^T PX]|^2)dt$$

$$+\mu \int_0^T < PX, FdW > -\frac{\mu^2}{2} \int_0^T < PFF^T PX, X > dt$$

$$+\frac{\mu}{2} \int_0^T tr(PFF^T)dt$$

$$
\begin{aligned}
dX(t) &= AX(t)dt + BU(t)dt + CV(t)dt + FdW(t) \\
X(0) &= X_0
\end{aligned}
$$

where $X_0 \in \mathbb{R}^n$ is not random,
$X(t) \in \mathbb{R}^n, A \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n), B \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n),\ U(t) \in \mathbb{R}^m,\ U \in \mathcal{U}$,
$C \in \mathcal{L}(\mathbb{R}^p, \mathbb{R}^n),\ V(t) \in \mathbb{R}^p,\ V \in \mathcal{V}$, and $F \in \mathcal{L}(\mathbb{R}^q, \mathbb{R}^n)$. The process W is square integrable with continuous sample paths with the filtration $(\mathcal{F}(t), t \in [0, T])$. The terms U and V are the strategies of the two players.
A Nash equilibrium occurs if the optimal strategy of one player is not influenced by knowledge of the strategy of the other player.

Payoff for Game

The payoff, J, is

$$\begin{aligned} J^0(U,V) &= \frac{1}{2}[\int_0^T (< QX(s), X(s) > + < RU(s), U(s) > \\ &\quad - < SV(s), V(s) >)ds + < MX(T), X(T) >] \\ J(U,V) &= \mathbb{E}[J^0(U,V)] \end{aligned}$$

where
$Q \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n), R \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^m), S \in \mathcal{L}(\mathbb{R}^p, \mathbb{R}^p), M \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$, and $Q > 0$, $R > 0$, $S > 0$, and $M \geq 0$ are symmetric linear transformations.

Admissible Strategies

The family of admissible strategies for U is $\mathcal{U}$ and for V is $\mathcal{V}$ where

$\mathcal{U} = \{U : U$ is an $\mathbb{R}^m$-valued process that is progressively measurable with respect to $(\mathcal{F}(t), t \in [0, T])$ such that $U \in L^2([0, T])$ a.s.$\}$
and
$\mathcal{V} = \{V : V$ is an $\mathbb{R}^p$-valued process that is progressively measurable with respect to $(\mathcal{F}(t), t \in [0, T])$ such that $V \in L^2([0, T])$ a.s.$\}$

$J^+ = inf_{U \in \mathcal{U}} sup_{V \in \mathcal{V}} J(U, V)$ is the upper value of the game.
$J^- = sup_{V \in \mathcal{V}} inf_{U \in \mathcal{U}} J(U, V)$ is the lower value of the game.
If these two values are equal then the game is said to have a value.

Theorem. The two person zero sum stochastic differential game has optimal admissible strategies for the two players, denoted U^* and V^*, given by

$$\begin{aligned} U^*(t) &= -R^{-1}(B^T P(t)X(t) + B^T \hat{\phi}(t)) \\ V^*(t) &= S^{-1}(C^T P(t)X(t) + C^T \hat{\phi}(t)) \end{aligned}$$

where $(P(t), t \in [0, T])$ is the unique positive solution of the following equation

$$\begin{aligned} -\frac{dP}{dt} &= Q + PA + A^T P - P(BR^{-1}B^T - CS^{-1}C^T)P \\ P(T) &= M \end{aligned}$$

and it is assumed that $BR^{-1}B^T - CS^{-1}C^T > 0$. The optimal strategies form a Nash equilibrium. $\hat{\phi}$ is the best estimate (conditional expectation) of the response of the optimal system to the future noise.

Discrete Time Linear-Quadratic Control with Correlated Noise

The discrete time controlled linear stochastic system is described as follows:

$$\begin{aligned} X(k+1) &= A(k)X(k) + B(k)U(k) + W(k) \\ X(0) &= x_0 \end{aligned} \tag{3}$$

where $x_0 \in \mathbb{R}^n$,
$A(k) \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n), B(k) \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n), k \in \{0, 1, ..., T-1\} = \mathbb{T}$, and $(W(k), k \in \mathbb{T})$ is an $\mathbb{R}^n$-valued family of square integrable, zero mean random variables that are defined on the complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The term $\mathcal{L}(\mathbb{R}^k, \mathbb{R}^l)$ denotes the family of linear transformations from $\mathbb{R}^k$ to $\mathbb{R}^l$. The filtration for $(X(k), k \in \mathbb{T})$ is denoted $(\mathcal{G}(k), k \in \mathbb{T})$.

Quadratic Cost Functional

$$J_T^0(U) = \Sigma_{k=0}^{T-1}[< Q(k)X(k), X(k) > + < R(k)U(k), U(k) >] + < Q(T)X(T), X(T) > \quad (4)$$

$$J_T(U) = \mathbb{E}[J_T^0(U)] \quad (5)$$

where $(Q(k), k \in \{0, ..., T\})$ and $(R(k), k \in \mathbb{T})$ are two families of symmetric, nonnegative definite linear transformations and $< \cdot, \cdot >$ is the standard inner product in the appropriate Euclidean space.

Theorem. For the optimal control problem with the linear system (3), the cost functional (5) and the family of admissible controls, $\mathcal{U}$, there is an optimal control $(U^*(k), k \in \mathbb{T})$ that is given by

$$\begin{aligned} U^*(k) = & - & P^{-1}(k+1)B^T(k)S(k+1)A(k)X(k) \\ & - & \mathbb{E}[P^{-1}(k+1)B^T(k)\phi(k+1)|\mathcal{G}(k)] \\ & - & \mathbb{E}[P^{-1}(k+1)B^T(k)S(k+1)W(k)|\mathcal{G}(k)] \end{aligned}$$

$$
\begin{aligned}
S(k) &= A^T(k)S(k+1)A(k) \\
&- A^T(k)S(k+1)B(k)P^{-1}(k+1)B^T(k)S(k+1)A(k) \\
&+ Q(k) \\
S(T) &= Q(T) \\
\phi(k) &= (A^T(k) - A^T(k)S(k+1)B(k)P^{-1}(k+1)B^T(k))\phi(k+1 \\
&+ A^T(k)S(k+1)W(k) \\
&+ A^T(k)S(k+1)B(k)P^{-1}(k+1)B^T(k)S(k+1)W(k) \\
\phi(T) &= 0 \\
P(k+1) &= B^T(k)S(k+1)B(k) + R(k)
\end{aligned}
$$

It is assumed that $(P(k), k \in \{1, ..., T\})$ is a family of positive definite linear transformations.

Linear-quadratic control of SPDEs with Fractional Brownian Motions

$$
\begin{aligned}
dX(t) &= (AX(t) + Bu(t))dt + dB_H(t) \\
X(0) &= x
\end{aligned}
$$

where $x \in V$, $X(t) \in V$, V is an infinite dimensional real separable Hilbert space with inner product $< \cdot, \cdot >$ and norm $|\cdot|$. The process $(B_H(t), t \geq 0)$ is a V-valued fractional Brownian motion with the Hurst parameter $H \in (\frac{1}{2}, 1)$ and having the incremental covariance $\widetilde{Q}$ where $\widetilde{Q}$ is trace class ($Tr(\widetilde{Q}) < \infty$) so that

$$\mathbb{E} < B_H(t), x >< B_H(s), y >= \frac{1}{2} < \widetilde{Q}x, y > (t^{2H} + s^{2H} - |t-s|^{2H}).$$

for $x, y \in V$. The operator $A : Dom(A) \to V$ with $Dom(A) \subset V$ is a linear, densely defined operator on V which is the infinitesimal generator of a strongly continuous semigroup $(S(t), t \geq 0)$.

Admissible Controls

$$\mathcal{U} = \{u : \mathbb{R}_+ \times \Omega \to U, u \text{ is progressively measurable,} \quad \mathbb{E}\int_0^T |u(t)|_U^2 dt < \infty \text{ for all } T > 0\}$$

Ergodic Quadratic Cost Functional

$$J_T(x,u) := \frac{1}{2}\int_0^T (|LX(s)|^2 + < Ru(s), u(s) >_U) ds$$

where $L \in \mathcal{L}(V)$, $R \in \mathcal{L}(U)$, R is self-adjoint and invertible. The control problem is to minimize the following ergodic cost

$$lim\ sup_{T \to \infty} \frac{1}{T} \mathbb{E} J_T(x,u).$$

Assumptions

(A1) There are $K \in \mathcal{L}(V), M_K > 0,$ and $\omega_K > 0$ such that

$$|e^{(A+KL)t}|_{\mathcal{L}(V)} \leq M_K e^{-\omega_K t}$$

for all $t > 0$ (detectability).

(A2) There are $F \in \mathcal{L}(V, U), M_F > 0,$ and $\omega_F > 0$ such that

$$|e^{(A+BF)t}|_{\mathcal{L}(V)} \leq M_F e^{-\omega_F t}$$

for all $t > 0$ (stabilizability).

Stationary Riccati Equation

The stationary Riccati equation has a weak solution as follows

$$< Px, Ay > + < Ax, Py > + < L^* Lx, y > - < R^{-1} B^* Px, B^* Py >= 0$$

for all $x, y \in Dom(A)$. Moreover the strongly continuous semigroup $(\Phi(t), t \geq 0)$ generated by $A_P = A - BR^{-1}B^*P$ is exponentially stable, that is

$$|\Phi(t)|_{\mathcal{L}(V)} \leq M_P e^{-\widetilde{\omega} t}$$

for some constants $M_P > 0$ and $\widetilde{\omega} > 0$.

Let (A1)-(A2) be satisfied and let $u \in \mathcal{U}$ be a control satisfying

$$lim_{T \to \infty} \frac{1}{T} \mathbb{E} < PX^u(T), X^u(T) >= 0 \tag{6}$$

where $(X^u(T), T \in [0, \infty))$ is the solution to the system equation with the control $u \in \mathcal{U}$. Then

$$lim\ sup_{T \to \infty} \frac{1}{T} \mathbb{E} J_T(x, u) \geq J_\infty$$

where

$$J_\infty := lim\ sup_{T \to \infty} \frac{-1}{2T} \mathbb{E} \int_0^T |R^{\frac{1}{2}} B^* W(s)|_U^2 ds + \int_0^\infty Tr(\widetilde{Q} P \Phi(t)) \phi_H(r) dr$$

for each $x \in V$ where $\phi_H(r) = H(2H-1)|r|^{2H-2}, r \in \mathbb{R}$, $W(t) = \mathbb{E}[\varphi(t)|\mathcal{F}(t)]$. Moreover, the feedback control $\hat{u}(t) = -R^{-1} B^* (PX^{\hat{u}}(s) + V(s))$ is admissible, satisfies the condition (6).

Finite Time Horizon Optimal Control

An optimal control $\hat{u}$ with the measurability condition is
$\hat{u}(t) = -R^{-1}B^*P(t)X(t) + \psi(t)$
$\psi(t) = \mathbb{E}[\phi(t)|\mathcal{F}(t)]$
$= \int_0^t s^{-(H-\frac{1}{2})}(I_{t-}(I_{T-}^{(H-\frac{1}{2})} u_{H-\frac{1}{2}} U_P(\cdot, t)P(\cdot)C))(s)dB_H(s)$

$$
\begin{aligned}
(I_{b-}^{\alpha} f)(x) &= \frac{1}{\Gamma(\alpha)} \int_x^b \frac{f(t)}{(t-x)^{1-\alpha}} dt \\
u_a(s) &= s^a
\end{aligned}
$$

$$\frac{\partial y}{\partial t}(t,\xi) = (L_{2m}y)(t,\xi) + (Bu_t)(\xi) + \eta^H(t,\xi)$$

for $(t,\xi) \in \mathbb{R}_+ \times D$ with the initial condition

$$y(0,\xi) = x(\xi)$$

for $\xi \in D$ and the Neumann boundary conditions

$$\frac{\partial^k}{\partial \nu^k} y(t,\xi) = 0$$

for $(t,\xi) \in \mathbb{R}_+ \times \partial D$, $k = 0, 1, \ldots, m-1$, where $D \subset \mathbb{R}^d$ is a bounded domain with a smooth boundary, $\frac{\partial}{\partial \nu}$ stands for conormal derivative, $x \in L^2(D)$, η^H is a space dependent fractional noise and L_{2m} is a $2m$th order uniformly elliptic operator of the form

$$L_{2m} = \sum_{|\alpha| \leq 2m} a_\alpha(\xi) D^\alpha$$

with $a_\alpha \in C_b^\infty(D)$.

For stochastic heat equation $Dom(A) = H_0^1(D) \cap H^2(D)$ for Dirichlet boundary conditions
$Dom(A) = \{\varphi \in H^2(D) : \frac{\partial \varphi}{\partial \nu} = 0 \text{ on } \partial D\}$ for Neumann boundary conditions

Stochastic Wave Equation

$$\frac{\partial^2 w}{\partial t^2}(t,\xi) = \frac{\partial^2 w}{\partial \xi^2}(t,\xi) + u_t(\xi) + \eta^H(t,\xi)$$

for $(t,\xi) \in \mathbb{R}_+ \times (0,1)$ with the boundary condition

$$w(t,0) = w(t,1) = 0, \;\; t > 0,$$

and initial condition

$$w(0,\xi) = x_1(\xi), \;\; \frac{\partial w}{\partial t}(0,\xi) = x_2(\xi), \;\; \xi \in (0,1),$$

where $u_t \in L^2(0,1)$ and η^H is a fractional noise on $L^2(0,1)$.

Stochastic Evolution Equations with a Multiplicative Gaussian Noise

$$
\begin{aligned}
dX(t) &= (A(t)X(t) + B(t)K(t)X(t))dt + \sigma(t)X(t)dB(t) \\
X(0) &= x_0
\end{aligned}
$$

where $X(t) \in V$ a real, separable Hilbert space, $(B(t), t \geq 0)$ is a real-valued Volterra-type Gaussian process, $(A(t), t \geq 0)$ is a family of closed, unbounded operators on V such that $Dom(A(t)) = Dom(A(0))$ and $Dom(A^*(t)) = Dom(A^*(0))$ for each $t \in \mathbb{R}_+$ and the family generates a strongly continuous evolution operator, $B \in C_s(\mathbb{R}_+, \mathcal{L}(U, V))$ and $K \in C_s(\mathbb{R}_+, \mathcal{L}(V, U))$, σ is a real-valued continuous function. The control is

$$u(t) = K(t)X(t)$$

where K is to be determined. This can be described as a Markov type control.

Noise Process

The noise B is generated from a Wiener process W as follows

(R2) $$B(t) = \int_0^t K(t,r)dW(r) \qquad t \in \mathbb{R}_+$$

There is a continuous version of the process B.

Assume that $K(\cdot, s)$ has bounded variation on (s, T) and

(R3) $$\int_0^T |K|^2((s,T],s)ds < \infty$$

This family of noise processes includes the family of FBMs for $H \in (\frac{1}{2}, 1)$.

The V-valued process $(X(t), t \geq 0)$ is a strong solution to the equation if

$$X(t) = x + \int_0^t (A(s)X(s) + B(s)K(s)X(s))ds + \int_0^t \sigma(s)X(s)dB(s)$$

and a weak solution to the equation exists if for each $z \in \mathcal{D}, z \in Dom(A^*(0))$ the following equality is satisfied

$$< X(t), z > = < x, z > + \int_0^t < X(s), A^*(s)z > ds$$
$$+ < B(s)K(s)X(s), z > ds + \int_0^t \sigma(s) < X(s), z > dB(s)$$

$$\widetilde{A}(t) = A(t) + B(t)K(t) - \alpha(t)I \qquad t \geq 0$$

$$\widetilde{A}_\lambda(t) = A(t) + B_\lambda(t)K(t) - \alpha(t)I \qquad t \geq 0$$

$\widetilde{A}$ and $\widetilde{A}_\lambda$ generate mild and strong evolution operators respectively on V denoted $(U(t,s))$ and $(U_\lambda(t,s))$, $B_\lambda = \lambda(\lambda I - A)^{-1}B$ and

$$U(t,s) = exp[-\int_s^t \alpha(r)dr]U_K(t,s)$$

$$U_\lambda(t,s) = exp[-\int_s^t \alpha(r)dr]U_K^\lambda(t,0)$$

$$X_\lambda(t) = exp[Z(t)]U_\lambda(t,s) \qquad t \geq 0$$

$$X(t) = exp[Z(t)]U(t,0) \qquad t \geq 0$$

where

$$Z(t) = \int_0^t \sigma(r)dB(r)$$

$$\alpha(t) = \sigma^2 \frac{\partial}{\partial t}(\int_0^t K(t,r)dr)^2 = \sigma^2 \frac{\partial}{\partial t} R(t,t) = \sigma^2 \frac{\partial}{\partial t}(\mathbb{E}B^2(t))$$

For fractional Brownian motion or Liouville fractional Brownian motion $\alpha(t) = c_H t^{2H-1}$ where the constant depends on whether it is FBM or LFBM.

For FBM if σ is not a constant

$\alpha(t) = \sigma(t) \int_0^t \sigma(s)\phi_H(t-s)ds$

where $\phi(t) = H(2H-1)t^{2H-2}$ so that for continuous σ the condition (K3) is satisfied.

Finite Time Horizon Control

The cost functional, J_T, is the following

$$J_T(K) = \mathbb{E}\int_0^T (|L(t)X(t)|^2 + < R(t)K(t)X(t), K(t)X(t) >_U)dt$$
$$+\mathbb{E} < GX(T), X(T) >$$

where $L \in C_s([0,T], \mathcal{L}(V))$, $G = G^*$, $G \in \mathcal{L}(V)$, $G > 0$, $R \in C_s([0,T], \mathcal{L}(U))$, $R(t) = R^*(t)$, $< R(t)u, u > \geq \lambda_0 |u|_U^2$, $u \in U$, $t \in [0,T]$.

The family of admissible controls is $K \in C_s([0,T], \mathcal{L}(V,U))$.

Riccati Equation

The Riccati differential equation associated with the control problem is

$$\frac{dP}{dt} + A^* P + PA - PBR^{-1}B^* P + L^* L - 2\alpha(t)P = 0 \qquad t \in [0, T]$$
$$P(T) = G$$

Lemma. With the assumptions given above, there is a unique weak solution $(P(t), t \in [0, T])$ to the Riccati equation that satisfies $P \in C_s([0, T], \mathcal{L}(V)), P(t) \geq 0, P(t) = P^*(t) \; t \in [0, T]$ such that

$$\frac{d{<}P(t)x, y{>}}{dt} + < A(t)x, P(t)y > + < P(t)x, A(t)y >$$
$$- < R^{-1}(t)B^*(t)P(t)x, B^* P(t)y >$$
$$+ < L(t)x, L(t)y > -2\alpha(t) < P(t)x, y >= 0 \qquad P(T) = G$$

for $t \in [0, T], x, y \in D$.

Theorem. Let (A1), (K1)-(K3) be satisfied. The feedback control

$$u(t) = -R^{-1}(t)B^*(t)P(t)X(t)$$
$$K(t) = -R^{-1}(t)B^*(t)P(t)$$

is an optimal control for the control problem.
The optimal cost is

$$J_T(K) =< P(0)x_0, x_0 >$$

Proof. Apply an Ito formula to $(< P(t)X_\lambda(t), X_\lambda(t) >, t \in [0, T])$ and then let $\lambda \to \infty$.

Stochastic Control and Differential Games for Some Nonlinear Systems

1. Some structure for the nonlinear stochastic systems–symmetric spaces (quotients of Lie groups)
2. Some symmetries for these nonlinear systems to facilitate explicit solutions
3. Some rank one symmetric spaces (compact–spheres and projective spaces; noncompact–hyperbolic spaces)

Control of Brownian Motion in $\mathbb{H}^2(\mathbb{R})$

$\mathbb{H}^2(\mathbb{R})$ is the real hyperbolic space of dimension two that is the (noncompact) symmetric space

$$SO_o(2,1)/SO(2) \times SO(1) = G/K$$

Let $o \in G/K$ be chosen and denoted as the origin. G/K can be modeled as the open unit disk in $T_o G/K$. with the metric
$ds^2 = 4(1 - |y|^2)^{-2}(dy_1^2 + dy_2^2)$.
The controlled stochastic system for the distance from the origin o is

$$dX(t) = \frac{1}{2}\ coth \frac{X(t)}{2} dt + U(t)dt + dB(t)$$

where $(B(t), t \in [0, T])$ is a standard Brownian motion.

$$\begin{aligned} J^0(U) &= \int_0^T (a\ sinh^2 \frac{X(t)}{4} + U^2(t) cosh^2 \frac{X(t)}{4}) dt \\ J(U) &= \mathbb{E} J^0(U) \end{aligned}$$

Theorem. An optimal control, U^*, is given by

$$U^*(t) = -\frac{1}{2} g(t) tanh \frac{X(t)}{4}$$

where $t \in [0, T]$ and g satisfies the Riccati equation given below. The optimal cost is

$$J(U^*) = g(0) sinh^2 \frac{X(0)}{4} + h(0)$$

$$\begin{aligned} \frac{dg(t)}{dt} &= -\frac{3}{8} g + \frac{1}{4} g^2 - a \\ g(T) &= 0 \\ \frac{dh(t)}{dt} &= -\frac{3}{16} g \\ h(T) &= 0 \end{aligned}$$

$f(t,x) = g(t) sinh^2 \frac{x}{2} + h(t)$ and $Y(t) = f(t, X(t))$ where X is the solution of the system equation. Apply the Ito formula to $(Y(t), t \in [0, T])$.

Eigenfunctions for the Radial Part of the Laplacian for Noncompact Rank One Symmetric Spaces

Some generalizations of the above example of $\mathbb{H}^2(\mathbb{R})$. Let $R(\Delta_{G/K})$ be the radial part of the Laplacian.

$$R(\Delta_{G/K}) = \frac{d^2}{dr^2} + (\gamma p \; coth\gamma r + 2\gamma q \; coth2\gamma r)\frac{d}{dr}$$

Eigenvalue-eigenfunction problem

$$z(z-1)\frac{d^2\phi_\lambda}{dz^2} + [(a+b+1)z - c]\frac{d\phi_\lambda}{dz} + ab\phi_\lambda = 0$$

A solution is given in terms of a hypergeometric function, $F(-m, b, c, z)$. If m is a positive integer, then F is a polynomial in z.

Let $z = -(sinh\ \gamma r)^2$ so $F(-m, b, c, -(sinh\ \gamma r)^2)$ is an eigenfunction for the radial part of the Laplacian.

Two-Sphere

The sphere S^2 is diffeomorphic to the rank one symmetric space $SO(3)/SO(2)$ and is a simply connected compact Riemannian manifold of constant positive sectional curvature.
A metric is obtained by restricting the standard metric in $\mathbb{R}^3$. The maximal distance between any two points in S^2 using this metric is $L = \pi$. The mapping $exp_o : T_o S^2 \to S^2$ is a diffeomorphism of the open ball $B_L(o) = \{x \in T_o S^2 : |x| < L\}$ onto the open set $S^2 \setminus A_o$.

$$exp_o Y \to (r, \theta)$$

where $Y \in B_L(o), r = |Y|$ and θ is the local coordinate of the unit vector $Y/|Y|$. In these coordinates the Laplace-Beltrami operator Δ_{S^2} is

$$\Delta_{S^2} = \frac{\partial^2}{\partial r^2} + cot(\frac{r}{2})\frac{\partial}{\partial r} + \Delta_{S_r}$$

$$
\begin{aligned}
dX(t) &= \frac{1}{2}\,cot(\frac{X(t)}{2})dt + U(t)dt + dB(t) \\
X(0) &= X_0
\end{aligned}
$$

where $X(t) \in S^2 \setminus A_o$, $(B(t), t \in [0, T])$ is a real-valued standard Brownian motion for a fixed $T > 0$, and $X_0 \in (0, L)$ is a constant. The Brownian motion is defined on the complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and $(\mathcal{F}(t), t \in [0, T])$ is the filtration for the Brownian motion B. If $U(t)$ is a smooth function of $X(t)$ then $(X(t), t \in [0, T])$ is a Markov process with the infinitesimal generator

$$
\frac{1}{2}\frac{\partial^2}{\partial r^2} + \frac{1}{2}cot(\frac{r}{2})\frac{\partial}{\partial r} + U(r)\frac{\partial}{\partial r}
$$

$$J^0(U) = \int_0^T (a\ sin^2 \frac{X(t)}{4} + U^2(t) cos^2 \frac{X(t)}{4}) dt \quad (7)$$

$$J(U) = \mathbb{E} J^0(U) \quad (8)$$

The cost functional only depends on the radial distance from the origin o, that is, $X(t) = |Y(t)|$ for $Y(t) \in S^2$, so the control only appears in the radial component of the process. Note that $sin^2 \frac{x}{4}$ is an increasing function for $x \in (0, \pi)$. The family of admissible controls, $\mathcal{U}$ is

$\mathcal{U} = \{U | U : [0, T] \times \Omega \to \mathbb{R}$ is jointly measurable, $(U(t), t \in [0, T])$ is adapted to $(\mathcal{F}(t), t \in [0, T])$ and $\int_0^T |U(t)|^2 dt < \infty$ a.s. $\}$

Two Ordinary Differential Equations

$$\frac{dg(t)}{dt} = \frac{3}{8}g + \frac{1}{4}g^2 - a \tag{9}$$
$$g(T) = 0$$

$$\frac{dh(t)}{dt} = -\frac{3}{16}g \tag{10}$$
$$h(T) = 0$$

Theorem

The stochastic control problem described above has an optimal admissible control, U^*, that is

$$U^*(t) = -\frac{1}{2} g(t) tan \frac{X(t)}{4}$$

where $t \in [0, T]$ and g satisfies (9). The optimal cost is

$$J(U^*) = g(0) sin^2 \frac{X(0)}{4} + h(0)$$

where h satisfies (10).

Some Generalizations

1. Stochastic control and differential games in spheres of higher dimension and in projective spaces
2. Stochastic control and differential games in hyperbolic spaces of higher dimension
3. Future work:

a) Control and differential games in higher rank symmetric spaces, e.g. positive definite matrices, Grassmannians

b) Random matrices, e.g. asymptotic behavior of eigenvalues

References

- T. E. Duncan, B. Maslowski, and B. Pasik-Duncan, Linear-quadratic control for stochastic equations in a Hilbert space with a fractional Brownian motion, *SIAM J. Control Optim.*, 50 (2012), 507-531.
- T. E. Duncan and B. Pasik-Duncan, Linear quadratic fractional Gaussian control, *SIAM J. Control. Optim.* 51 (2013), 4604-4619.
- T. E. Duncan and B. Pasik-Duncan, Discrete time linear quadratic control with arbitrary correlated noise, *IEEE Trans. Autom. Control*, 58 (2013), 1290-1293.
- T. E. Duncan, Linear-quadratic stochastic differential games with general noise processes, *Models and Methods in Economics and Management Science: Essays in Honor of Charles S. Tapiero*, (eds. F. El Ouardighi and K. Kogan), Operations Research and Management Series, Springer Intern. Publishing

- T. E. Duncan and B. Pasik-Duncan, A control problem in the two-sphere, Proc. IEEE Multiconf. Systems Control, 2012, Dubrovnik, 1441-1444.
- S. Tang, Dynamic programming for general linear quadratic optimal stochastic control with random coefficients, *SIAM J. Control Optim.* 2015.
- D. H. Jacobson, Optimal stochastic linear systems with exponential performance criteria and their relation to deterministic differential games, *IEEE Trans. Autom. Control* AC-18 (1973), 124-131.

Thank You