Geometry and Optimization of Relative Arbitrage

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Introduction

Main themes

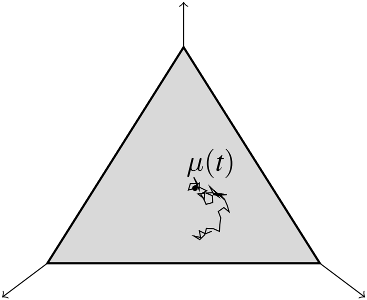
- Model-free and robust investment strategies
- Depend only on directly observable market quantities
- Stochastic portfolio theory (SPT), volatility pumping
- Connections with universal portfolio theory and other approaches

Set up

- n stocks, discrete time
- Market weight

$$\mu_i(t) = \frac{\text{market value of stock } i \text{ at time } t}{\text{total market value at time } t} \in (0, 1)$$

- $\mu(t) = (\mu_1(t), \ldots, \mu_n(t)) \in \Delta_n$ (open unit simplex)



Portfolio

- For each t, pick weights

$$\pi(t) = (\pi_1(t), \ldots, \pi_n(t)) \in \overline{\Delta}_n$$

Market portfolio: $\mu(t) = (\mu_1(t), \ldots, \mu_n(t))$

- Relative value w.r.t. market portfolio μ:

$$V_\pi(t) = \frac{\text{growth of \$1 of portfolio } \pi \text{ at time } t}{\text{growth of \$1 of market portfolio } \mu \text{ at time } t}$$

- $V_\pi(0) = 1$ and

$$\frac{V_\pi(t+1)}{V_\pi(t)} = \sum_{i=1}^{n} \pi_i(t) \frac{\mu_i(t+1)}{\mu_i(t)}$$

Relative arbitrage

Definition

For $t_0 > 0$, a relative arbitrage over the horizon $[0, t_0]$ (w.r.t. μ) is a portfolio π such that $V_\pi(t_0) > 1$ for "all" possible realizations of $\{\mu(t)\}_{t=0}^{\infty}$.

Interpretations:

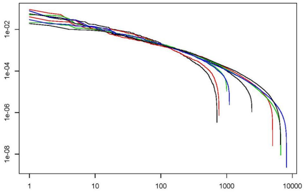
(i) (Probabilistic) If $\{\mu(t)\}_{t=0}^{\infty}$ is a stochastic process, we mean

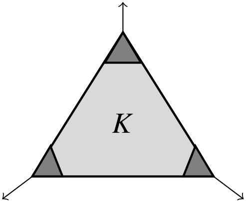
$$\mathbb{P}\left(V_\pi(t_0) > 1\right) = 1.$$

(ii) (Pathwise) Let $\mathcal{P}$ be a path property. We require $V_\pi(t_0) > 1$ for all sequences $\{\mu(t)\}_{t=0}^{\infty} \subset \Delta_n$ satisfying property $\mathcal{P}$.

Stochastic portfolio theory

- Relative arbitrage exists (for t_0 sufficiently large) under realistic conditions:
- Stability of capital distribution; diversity $\mu(t) \in K$





- Sufficient volatility
- Functionally generated portfolios (Fernholz)

$$\text{e.g. } \pi_i(t) = \frac{-\mu_i(t)\log\mu_i(t)}{\sum_{j=1}^{n} -\mu_j(t)\log\mu_j(t)} \quad \text{(entropy-weighted portfolio)}$$

Geometry

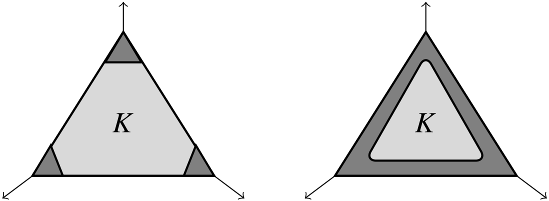
Geometry of relative arbitrage

- Consider deterministic portfolios:

$$\pi(t) = \pi(\mu(t)),$$

where $\pi : \Delta_n \to \overline{\Delta}_n$ is a portfolio map.

- Let $K \subset \Delta_n$ be open and convex:



- Ask: If the portfolio map $\pi : \Delta_n \to \overline{\Delta}_n$ is a relative arbitrage given the market is volatile and satisfies the generalized diversity condition $\mu(t) \in K$, what does π look like?

Relative arbitrage as FGP

Theorem (Pal and W. (2014))

$K \subset \Delta_n$ open convex, $\pi : \Delta_n \to \overline{\Delta}_n$ portfolio map. TFAE:

(i) *π is a pseudo-arbitrage over the market on K: there exists $\epsilon > 0$ such that $\inf_{t \geq 0} V_\pi(t) \geq \epsilon$ for all sequences $\{\mu(t)\}_{t=0}^\infty \subset K$, and there exists a sequence $\{\mu(t)\}_{t=0}^\infty \subset K$ along which $\lim_{t \to \infty} V_\pi(t) = \infty$.*

(ii) *π is functionally generated: there exists a non-affine, concave function $\Phi : \Delta_n \to (0, \infty)$ such that $\log \Phi$ is bounded below on K and*

$$\sum_{i=1}^n \pi_i(p) \frac{q_i}{p_i} \geq \frac{\Phi(q)}{\Phi(p)}, \quad \text{for all } p, q \in K.$$

- The ratios π_i / μ_i are given by supergradients of the exponential concave function $\log \Phi$.

Examples

	$\pi_i(\mu)$	$\Phi(\mu)$
Market	μ_i	1
Diversity-weighted	$\frac{\mu_i^\lambda}{\sum_{j=1}^n \mu_j^\lambda}$	$\left(\sum_{j=1}^n \mu_j^\lambda\right)^{\frac{1}{\lambda}}$
Constant-weighted	π_i	$\mu_1^{\pi_1} \cdots \mu_n^{\pi_n}$
Entropy-weighted	$\frac{-\mu_i \log \mu_i}{\sum_{j=1}^n -\mu_j \log \mu_j}$	$\sum_{j=1}^n -\mu_j \log \mu_j$

`R` package `RelValAnalysis` on `CRAN`

FGP leads to relative arbitrage

- Recall

$$\frac{V_\pi(t+1)}{V_\pi(t)} = \sum_{i=1}^{n} \pi_i(\mu(t)) \frac{\mu_i(t+1)}{\mu_i(t)} \geq \frac{\Phi(\mu(t+1))}{\Phi(\mu(t))}$$

- Write

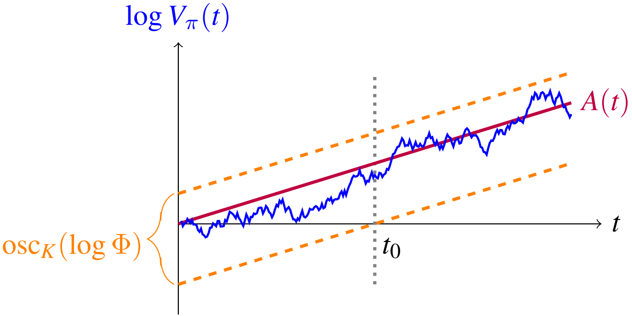
$$\log \frac{V_\pi(t+1)}{V_\pi(t)} = \log \frac{\Phi(\mu(t+1))}{\Phi(\mu(t))} + T\left(\mu(t+1) \mid \mu(t)\right)$$

where

$$T\left(\mu(t+1) \mid \mu(t)\right) \geq 0$$

is a measure of volatility

FGP leads to relative arbitrage



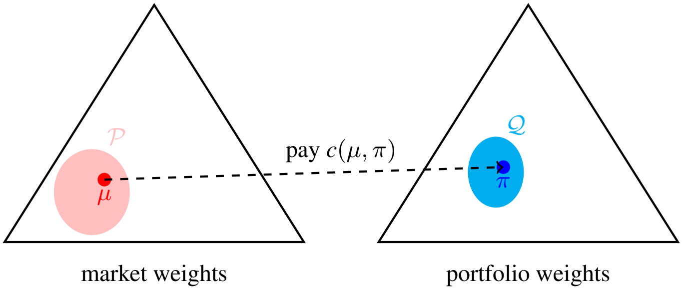
$$\log V_\pi(t) = \underbrace{\log \frac{\Phi(\mu(t))}{\Phi(\mu(0))}}_{\text{diversity}} + \underbrace{\sum_{s=0}^{t-1} T\left(\mu(s+1) \mid \mu(s)\right)}_{=A(t) \text{ sufficient volatility}}$$

Relative arbitrage as optimal transport maps

- $\mathcal{X}, \mathcal{Y}$: Polish spaces
- c: cost function
- $\mathcal{P}$: Borel probability measure on $\mathcal{X}$
- $\mathcal{Q}$: Borel probability measure on $\mathcal{Y}$
- Coupling: $\mathcal{R}$ p.m. on $\mathcal{X} \times \mathcal{Y}$ with marginals $\mathcal{P}$ and $\mathcal{Q}$
- Monge-Kantorovich optimal transport problem:

$$\min_{\mathcal{R}} \mathbb{E}_{\mathcal{R}}[c(X, Y)], \quad (X, Y) \sim \mathcal{R}$$

- Intuition: $\pi : \mu \mapsto \pi(\mu)$



Optimal transport and FGP

Take

- $\mathcal{X} = \overline{\Delta}_n$, $\mathcal{Y} = [-\infty, \infty)^n$
- Cost:

$$c(\mu, h) = \log\left(\sum_{i=1}^{n} e^{h_i}\mu_i\right)$$

- Portfolio at μ given h:

$$\pi_i(\mu) = \frac{\mu_i e^{h_i}}{\sum_{j=1}^{n} \mu_j e^{h_j}} \tag{1}$$

Theorem (Pal and W. (2014))

Let $\mathcal{P}$ be any Borel probability measure on Δ_n, and let π be an FGP. Define $h = h(\mu)$ via (1), and let $\mathcal{Q}$ be the law of h when $\mu \sim \mathcal{P}$. Then the coupling (μ, h) minimizes $\mathbb{E}_{\mathcal{R}}[c(\mu, h)]$ where $\mathcal{R}$ has marginals $\mathcal{P}$ and $\mathcal{Q}$.

Optimization

Point estimation of FGP

- Assume $\frac{1}{M} \leq \frac{\mu_i(t+1)}{\mu_i(t)} \leq M$ (M is unknown to the investor)
- $\mathcal{FG} := \left\{\pi : \Delta_n \to \overline{\Delta}_n \mid \text{functionally generated}\right\}$
- For $\pi \in \mathcal{FG}$,

$$\log \frac{V_\pi(t+1)}{V_\pi(t)} = \log \left(\sum_{i=1}^{n} \pi_i(\mu(t)) \frac{\mu_i(t+1)}{\mu_i(t)} \right)$$
$$=: \ell_\pi(\mu(t), \mu(t+1))$$

- Logarithmic growth rate:

$$\frac{1}{t} \log V_\pi(t) = \int_{\Delta_n \times \Delta_n} \ell_\pi d\mathbb{P}_t$$

where

$$\mathbb{P}_t := \frac{1}{t} \sum_{s=0}^{t-1} \delta_{(\mu(s), \mu(s+1))}$$

is the empirical measure

Point estimation of FGP

- $\mathbb{P}$: Borel probability measure on

$$\mathcal{S} := \left\{ (p, q) \in \Delta_n \times \Delta_n : \frac{1}{M} \leq \frac{q_i}{p_i} \leq M \right\}$$

- Given $\mathbb{P}$, consider

$$\sup_{\pi \in \mathcal{FG}} \int_{\mathcal{S}} \ell_\pi d\mathbb{P}$$

  Solution $\widehat{\pi}$ is analogous to MLE

- Nonparametric MLE: given a random sample $X_1, ..., X_N \sim f_0$ in $\mathbb{R}^d$, consider

$$\sup_{f \in \mathcal{F}} \sum_{k=1}^{N} \log f(X_k)$$

  where $\mathcal{F}$ is a class of densities

Point estimation of FGP

Theorem (W. 2015)

Under suitable regularity conditions on $\mathbb{P}$*:*

- *(Solvability) The problem has an optimal solution which is a.e. unique.*
- *(Consistency) If* $\widehat{\pi}^{(N)}$ *is optimal for* $\mathbb{P}_N$*,* $\mathbb{P}_N \to \mathbb{P}$ *weakly, and* $\widehat{\pi}$ *is optimal for* $\mathbb{P}$*, then*

$$\widehat{\pi}^{(N)} \to \widehat{\pi} \quad a.e.$$

Point estimation of FGP

Theorem (continued)

- *(Finite-dimensional reduction) Suppose $\mathbb{P}$ is discrete:*

$$\mathbb{P} = \frac{1}{N} \sum_{j=1}^{N} \delta_{(p(j),q(j))}.$$

Then the optimal solution $(\widehat{\pi}, \widehat{\Phi})$ can be chosen such that $\widehat{\Phi}$ is polyhedral over the data points. In particular, $\widehat{\Phi}$ is piecewise affine over a triangulation of the data points.

Universal portfolio for $\mathcal{FG}$

- Aim: achieve the asymptotic growth rate of

$$V^*(t) = \sup_{\pi \in \mathcal{FG}} V_\pi(t)$$

- Cover (1991): wealth-weighted average (for constant-weighted portfolios)
- We will construct a "market portfolio" $\widehat{\pi}$ whose basic assets are the portfolios in $\mathcal{FG}$

Wealth distribution

- Equip $\mathcal{FG}$ with topology of uniform convergence
- ν_0: Borel probability measure on $\mathcal{FG}$ (initial distribution)
- At time 0, distribute $\nu_0(d\pi)$ wealth to $\pi \in \mathcal{FG}$
- Total wealth at time t is

$$\widehat{V}(t) := \int_{\mathcal{FG}} V_\pi(t) d\nu_0(\pi)$$

- Wealth distribution at time t:

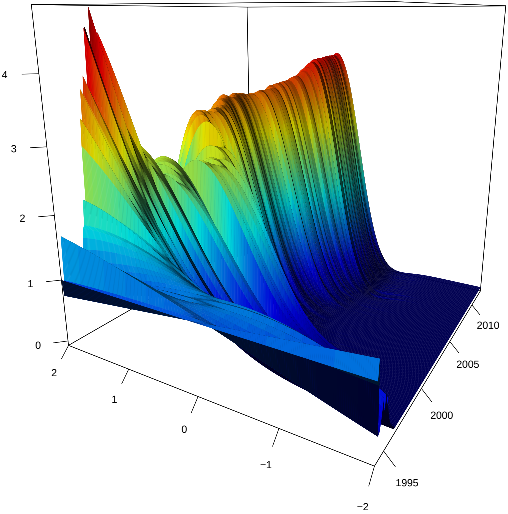
$$\nu_t(B) := \frac{1}{\widehat{V}(t)} \int_B V_\pi(t) d\nu_0(\pi), \quad B \subset \mathcal{FG}$$

- Bayesian interpretation:

$$\nu_0 : \text{prior}, \quad V_\pi(t) : \text{likelihood}, \quad \nu_t : \text{posterior}$$

Example

Wealth density for a family of constant-weighted portfolios:



Universal portfolio for $\mathcal{FG}$

- In this context, define Cover's portfolio by the posterior mean

$$\widehat{\pi}(t) := \int_{\mathcal{FG}} \pi(\mu(t)) d\nu_t(\pi)$$

- Then

$$V_{\widehat{\pi}}(t) \equiv \widehat{V}(t)$$

Suppose an asymptotically optimal portfolio π^* exists. Hope:

- Posterior ν_t concentrates around π^*
- $\widehat{V}(t)$ is "close" to $V^*(t)$

Universal portfolio for $\mathcal{FG}$

Theorem (W. (2015))

Let $\{\mu(t)\}_{t=0}^{\infty}$ be a sequence in Δ_n such that the empirical measure $\mathbb{P}_t = \frac{1}{t}\sum_{s=0}^{t-1}\delta_{(\mu(s),\mu(s+1))}$ converges weakly to an absolutely continuous Borel probability measure on $\mathcal{S}$. Here, the asymptotic growth rate $W(\pi) := \lim_{t\to\infty}\frac{1}{t}\log V_\pi(t)$ exists for all $\pi \in \mathcal{FG}$.

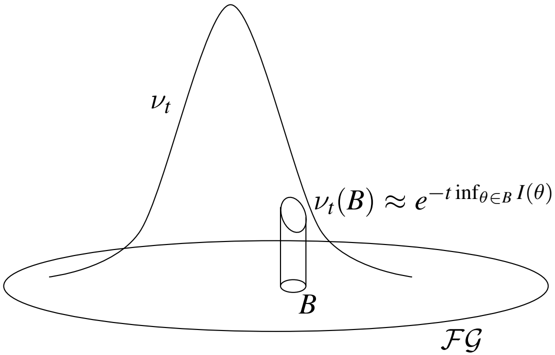
(i) *For any initial distribution ν_0, the sequence $\{\nu_t\}_{t=0}^{\infty}$ of wealth distributions satisfies the large deviation principle (LDP) with rate*

$$I(\pi) = \begin{cases} W^* - W(\pi) & \text{if } \pi \in \operatorname{supp}(\nu_0), \\ \infty & \text{otherwise}, \end{cases}$$

where $W^* := \sup_{\pi\in\operatorname{supp}(\nu_0)} W(\pi)$.

Universal portfolio for $\mathcal{FG}$

Wealth distribution of $\mathcal{FG}$ satisfies LDP:



Universal portfolio for $\mathcal{FG}$

Theorem (Continued)

(ii) *There exists an initial distribution ν_0 on $\mathcal{FG}$ such that $W^* = \sup_{\pi \in \mathcal{FG}} W(\pi)$ for any absolutely continuous $\mathbb{P}$. For this initial distribution, Cover's portfolio satisfies the asymptotic universality property*

$$\lim_{t \to \infty} \frac{1}{t} \log \frac{\widehat{V}(t)}{V^*(t)} = 0.$$

Future research

- Connections between SPT, universal portfolio theory and online portfolio selection (statistical learning)
- Risk-adjusted universal portfolios
- Relative arbitrage under price impacts and transaction costs
- Economic models inspired by SPT

References

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