A Stochastic Portfolio Optimization Model with Bounded Memory

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1 Introduction.

- Stochastic optimal control problems:

$$\begin{aligned} dX(s) &= f(s, X(s), u(s))ds + g(s, X(s), u(s))dW(s), \quad \forall s \in [t, T] \\ X(t) &= x. \end{aligned}$$

Value function:

$$V(t, x) = \sup_u \mathbf{E}_{t,x} \left[\int_t^T L(s, X(s), u(s))ds + \Psi(T, X(T), u(T)) \right].$$

  - Typically, we can derive a HJB equation for the value function. By solving the associated HJB equation (explicitly or numerically), we can obtain the value function and optimal control policies.
  - The HJB equation is usually a second order partial differential equation of parabolic type.
  - In many cases, the value function is just a viscosity solution of the HJB equation.

- In many real world applications, some physical systems can only be modeled by stochastic dynamical systems whose evolutions depend on the past history of the states.

- General form:

$$
\begin{aligned}
dX(s) &= f(s, X_s, u(s))ds + g(s, X_s, u(s))dW(s), \quad s \in [t, T], && (1) \\
X(s) &= \psi(s), \quad s \in [t-h, t], && (2)
\end{aligned}
$$

  where $X_s : [-h, 0] \to \Re^n$ is defined by

$$X_s(\theta) = X(s+\theta).$$

  - We will derive the associated HJB equation.
  - The HJB equation is in infinite dimensional space.
  - The HJB equation involves Fréchet derivatives.

- A model with memory:

$$
\begin{aligned}
dX(s) &= \alpha(s, X(s), Y(s), Z(s), u(s))ds \\
&\quad + \beta(s, X(s), Y(s), Z(s), u(s))dW(s), \quad s \in [t, T], \qquad (3) \\
X(s) &= \psi(s), \quad \forall s \in [t-h, t], \qquad (4)
\end{aligned}
$$

where

$$
Y(s) = \int_{-h}^{0} e^{\lambda\theta} X(s+\theta)d\theta, \quad Z(s) = X(s-h).
$$

  - The value function $V(t, \psi)$ will be defined on $[0, T] \times C[-h, 0]$, which is an infinite dimensional space.
  - Under certain conditions, the associated HJB equation can be turned into a PDE in a finite dimensional space.

- Another model

$$
\begin{aligned}
dX(t) &= b(t, X(t), Y(t), Z(t)u(t))dt \\
&\quad + \sigma(t, X(t), Y(t), Z(t), u(t))dW(t), \quad t \in (s, T] \qquad (5) \\
X(t) &= \eta(t-s), \qquad t \in (s-\delta, s], \quad \eta \in C((-\delta, 0]; \mathbb{R}), \qquad (6)
\end{aligned}
$$

where $W(t)$ is a standard 1-dimensional Brownian motion, $u(t)$ is the control variable, and $Y(t)$ is given by

$$
Y(t) = \int_{-\delta}^{0} \psi(r)X(t+r)dr, \quad Z(t) = X(t-h), \quad t \in (s, T], \qquad (7)
$$

where $\psi(r)$ is a function of the form

$$
\psi(r) = a_0 + a_1 r + a_2 r^2 + \cdots + a_n r^n. \qquad (8)
$$

2 A Portfolio Optimization Model with Memory

2.1 Problem Formulation

- One risky asset and one riskless asset with interest rate r.
- $K(t)$: the amount invested on the risky asset; $L(t)$: the amount invested on the riskless asset. Total wealth: $X(t) = K(t) + L(t)$.
- We consider the situation in which the performance of the risky asset depends on the history (memory) through the following delay variables $Y(t)$ and $Z(t)$:

$$Y(t) \equiv \int_{-h}^{0} e^{\lambda s} X(t+s)ds, \quad Z(t) \equiv X(t-h). \tag{9}$$

- Assume that $K(t) > 0$ almost surely. Instead of $Y(t)$, $Z(t)$, we first consider

$$\tilde{Y}(t) \equiv \frac{Y(t)}{K(t)} = \frac{1}{K(t)} \int_{-h}^{0} e^{\lambda s} X(t+s) ds, \tag{10}$$

$$\tilde{Z}(t) \equiv \frac{Z(t)}{K(t)} = \frac{X(t-h)}{K(t)}. \tag{11}$$

- We model that $K(t)$ and $L(t)$ with the stochastic differential equations:

$$\begin{aligned} dK(t) &= [(\mu_1 + \mu_2 \tilde{Y}(t) + \mu_3 \tilde{Z}(t))K(t) + I(t)]dt \\ &\quad + \sigma K(t) dB(t), \end{aligned} \tag{12}$$

$$dL(t) = [rL(t) - C(t) - I(t)]dt, \tag{13}$$

where $I(t)$ is the investment rate and $C(t)$ is the consumption rate.

- Using the definition of $\tilde{Y}(t), \tilde{Z}(t)$, we can get that $K(t)$ and $L(t)$ follow the stochastic differential equations:

$$\begin{aligned} dK(t) &= [\mu_1 K(t) + \mu_2 Y(t) + \mu_3 Z(t) + I(t)]dt \\ &\quad + \sigma K(t) dB(t), \qquad (14) \\ dL(t) &= [rL(t) - C(t) - I(t)]dt. \qquad (15) \end{aligned}$$

- Assume that $X(t) > 0$ almost surely. Then we can use $c(t) \equiv \frac{C(t)}{X(t)}, k(t) = \frac{K(t)}{X(t)}$ as our controls. It is easy to see that $L(t) = X(t)(1 - k(t))$. Now we can get the equation for $X(t)$ as

$$\begin{aligned} dX(t) &= \left[((\mu_1 - r)k(t) - c(t) + r)X(t) + \mu_2 Y(t) + \mu_3 Z(t)\right] dt \\ &\quad + \sigma k(t) X(t) dB(t), \quad \forall t \in [s, T]. \qquad (16) \end{aligned}$$

- Remark: It can showed that $X(t) > 0$ almost surely.

- The initial condition is the information about $X(s)$ for $s \in [-h, 0]$:

$$X(s+t) = \varphi(t), \quad \forall t \in [-h, 0], \tag{17}$$

where $\varphi \in \mathbb{J}$ where $\mathbb{J} \equiv C[-h, 0]$ is the space for all continuous function defined on $[-h, 0]$ equipped with sup-norm:

$$\|\varphi\| = \sup_{s \in [-h,0]} |\varphi(s)|. \tag{18}$$

- Let $U(C)$ be the utility function and Ψ be the terminal utility function. The objective function is

$$J(s, \varphi, k, c) = \mathbf{E}_{s,\varphi}\Big[\int_s^T e^{-\beta(t-s)} U(c(t)X(t))dt + e^{-\beta(T-s)}\Psi(X(T), Y(T))\Big] \tag{19}$$

- The value function is given by

$$\begin{aligned} V(s, \varphi) \; = \; \sup_{k,c \geq 0} \mathbf{E}_{s,\varphi}\Big[\int_s^T & e^{-\beta(t-s)} U(c(t)X(t))dt \\ & + e^{-\beta(T-s)}\Psi(X(T), Y(T))\Big]. \end{aligned} \tag{20}$$

• Under certain conditions, we have

$$V(s,\varphi) = V(s,x,y,z), \tag{21}$$

where

$$x = x(\varphi) \equiv \varphi(0), \tag{22}$$

$$y = y(\varphi) \equiv \int_{-h}^{0} e^{\lambda s}\varphi(s)ds, \tag{23}$$

$$z = z(\varphi) \equiv \varphi(-h). \tag{24}$$

• Further we will give the conditions that V only depends on s, x, y, i.e.,

$$V(s,\varphi) = V(s,x,y,z) = V(x,y). \tag{25}$$

2.2 Hamilton-Jacobi-Bellman Equation.

Let $f \in C^{1,2,2}([0,T] \times \mathbb{R}^2)$ and define

$$G(t) = f(t, X^\varphi(t), y(X_t^\varphi)), \tag{26}$$

where

$$y(\eta) = \int_{-h}^{0} e^{\lambda u}\eta(u)du, \quad \forall \eta \in \mathbb{J}, \quad X_t(u) \equiv X(t+u), \quad \forall u \in [-h, 0]. \tag{27}$$

Then we have the following Ito's formula:

Lemma 2.1 (Ito's formula) *Let the system be given by (16)-(17), and $Y(t), Z(t)$ be given by (11). The Ito's formula is*

$$dG(t) = \mathcal{L}f dt + (\sigma k x) f_x dB(t), \tag{28}$$

where

$$\begin{aligned} \mathcal{L}f &= \mathcal{L}^{k,c} f(t,x,y,z) \\ &= f_t + (((\mu_1 - r)k - c + r)x + \mu_2 y + \mu_3 z)\, f_x \\ &\quad + \frac{1}{2}(\sigma k x)^2 f_{xx} + (x - \lambda y - e^{-\lambda h} z) f_y. \end{aligned} \tag{29}$$

We assume that the value function V depends on the initial path φ only through the functionals $x(\varphi), y(\varphi)$ defined by (22-23). That is,

$$V(s,\varphi) = V(s, x(\varphi), y(\varphi)) = V(s,x,y). \tag{30}$$

Then we can obtain the following HJB equation:

Lemma 2.2 (HJB equation) *Assume that* (30) *holds and* $V(s,x,y) \in C^2(\mathbb{R}^2)$*. Then the HJB equation for* $V(s,x,y)$ *is given by*

$$\begin{aligned} \beta V - V_s \; = \; & \max_k \left[\frac{1}{2}(\sigma k x)^2 V_{xx} + ((\mu_1 - r)k)xV_x \right] + (rx + \mu_2 y + \mu_3 z)V_x \\ & + \max_{c\geq 0}[-cxV_x + U(cx)] + (x - \lambda y - e^{-\lambda h} z)V_y, \quad \forall z \in \mathbb{R}, \end{aligned} \tag{31}$$

with the boundary condition

$$V(T,x,y) = \Psi(x,y). \tag{32}$$

2.3 The Solution of the HJB Equation.

Assume that the utility function is of the HARA type:

$$U(cX) = \frac{1}{\gamma}(cX)^{\gamma}, \tag{33}$$

where $\gamma \in (-\infty, 1], \gamma \neq 0$ is a constant. Then we can get

$$\begin{aligned} \beta V - V_s \;=\; & \max_{k}\left[\frac{1}{2}(\sigma k x)^2 V_{xx} + (\mu_1 - r)kxV_x\right] + (rx + \mu_2 y + \mu_3 z)V_x \\ & + \max_{c\geq 0}\left[-cxV_x + \frac{1}{\gamma}(cx)^{\gamma}\right] + (x - \lambda y - e^{-\lambda h} z)V_y. \end{aligned} \tag{34}$$

The candidate for the optimal control policy is

$$k^* \;=\; -\frac{(\mu_1 - r)V_x}{\sigma^2 x V_{xx}}, \tag{35}$$

$$c^* \;=\; \frac{1}{x} V_x^{\frac{1}{\gamma - 1}}. \tag{36}$$

Plug k^*, c^* into the HJB equation, and we can get

$$\begin{aligned}\beta V - V_s \; = \; & -\frac{1}{2}\frac{(\mu_1 - r)^2 V_x^2}{\sigma^2 V_{xx}} + \left(\frac{1}{\gamma} - 1\right) V_x^{\frac{\gamma}{\gamma-1}} + (x - \lambda y - e^{-\lambda h} z) V_y \\ & + (rx + \mu_2 y + \mu_3 z)\, V_x. \end{aligned} \tag{37}$$

It can be rewritten as

$$\begin{aligned}\beta V - V_s \; = \; & -\frac{1}{2}\frac{(\mu_1 - r)^2 V_x^2}{\sigma^2 V_{xx}} + \left(\frac{1}{\gamma} - 1\right) V_x^{\frac{\gamma}{\gamma-1}} + (rx + \mu_2 y)\, V_x \\ & + (x - \lambda y) V_y + \left(\mu_3 V_x - e^{-\lambda h} V_y\right) z. \end{aligned} \tag{38}$$

Suppose the terminal utility function $\Psi(x, y)$ is given in a form

$$\Psi(x, y) = \frac{1}{\gamma}(x + \mu_3 e^{\lambda h} y)^{\gamma}. \tag{39}$$

We look for a solution of the form

$$V(s,x,y) = Q(s)\psi(x,y). \tag{40}$$

Now we can get

$$\begin{aligned} & [\beta Q(s) - Q'(s)]\psi(x,y) \\ = \; & -\frac{1}{2}\frac{(\mu_1 - r)^2 Q(s)\psi_x^2}{\sigma^2 \psi_{xx}} + \left(\frac{1}{\gamma} - 1\right)[Q(s)\psi_x]^{\frac{\gamma}{\gamma-1}} + (rx + \mu_2 y)\, Q(s)\psi_x \\ & + (x - \lambda y)Q(s)\psi_y + (\mu_3 \psi_x - e^{-\lambda h}\psi_y)Q(s)z. \end{aligned} \tag{41}$$

Apparently, equation (41) has a solution which does not depend on z if we have the following condition:

$$(\mu_3\psi_x - e^{-\lambda h}\psi_y)Q(s)z = 0, \quad \forall z \in \mathbb{R}. \tag{42}$$

Define $u \equiv x + \mu_3 e^{\lambda h} y$ and we look for a solution of the form

$$\psi(x,y) = \frac{1}{\gamma}(x + \mu_3 e^{\lambda h} y)^\gamma = \frac{1}{\gamma}u^\gamma, \tag{43}$$

Plug them into (41), and we can get

$$\begin{aligned}
& \frac{1}{\gamma}[\beta Q(s) - Q'(s)]u^{\gamma} \\
= & -\frac{1}{2}\frac{(\mu_1 - r)^2}{\sigma^2(\gamma-1)}Q(s)u^{\gamma} + \left(\frac{1}{\gamma} - 1\right)[Q(s)]^{\frac{\gamma}{\gamma-1}}u^{\gamma} \\
& + \left[(r + \mu_3 e^{\lambda h})x + \left(\mu_2 - \lambda\mu_3 e^{\lambda h}\right)y\right]Q(s)u^{\gamma-1}. \qquad (44)
\end{aligned}$$

Assume that

$$\mu_2 - \lambda\mu_3 e^{\lambda h} = (r + \mu_3 e^{\lambda h})\mu_3 e^{\lambda h}. \qquad (45)$$

(Some discussions on this assumption are given in Section 2.4.) Then it is easy to verify that

$$\begin{aligned}
& \left[(r + \mu_3 e^{\lambda h})x + \left(\mu_2 - \lambda\mu_3 e^{\lambda h}\right)y\right]Q(s)u^{\gamma-1} \\
= & (r + \mu_3 e^{\lambda h})Q(s)(x + \mu_3 e^{\lambda h}y)u^{\gamma-1} \qquad (46) \\
= & (r + \mu_3 e^{\lambda h})Q(s)u^{\gamma} \qquad (47)
\end{aligned}$$

Canceling the term u^γ on both sides, we can get

$$Q'(s) \;=\; (\gamma-1)\,[Q(s)]^{\frac{\gamma}{\gamma-1}} + \Lambda Q(s), \tag{48}$$

where

$$\Lambda \equiv \beta + \frac{(\mu_1 - r)^2\gamma}{2\sigma^2(\gamma-1)} - \gamma(r + \mu_3 e^{\lambda h}). \tag{49}$$

We assume that the all parameters involved here satisfy

$$\Lambda > 0, \tag{50}$$

to guarantee that we have a well-defined solution.

At the point $t = T$, we have

$$V(T, x, y) = Q(T)\frac{1}{\gamma}(x + \mu_3 e^{\lambda h} y)^{\gamma} = \Psi(x, y) = \frac{1}{\gamma}(x + \mu_3 e^{\lambda h} y)^{\gamma}. \tag{51}$$

Therefore, the boundary condition for $Q(s)$ at $s = T$ is given by

$$Q(T) = 1. \tag{52}$$

By solving $(48) - (52)$, we can get the solution

$$Q(s) = \left[\left(1 - \frac{1-\gamma}{\Lambda} \right) e^{-\frac{\Lambda}{1-\gamma}(T-s)} + \frac{1-\gamma}{\Lambda} \right]^{1-\gamma}. \tag{53}$$

It is easy to verify that, if $\Lambda > 0$, we have

$$Q(s) > 0, \quad \forall s \in [0, T]. \tag{54}$$

Therefore, the solution of the HJB equation $(31) - (32)$ is given by

$$V(s,x,y) = \frac{1}{\gamma} Q(s)(x + \mu_3 e^{\lambda h} y)^\gamma. \tag{55}$$

and the optimal investment ratio and the optimal consumption rate control are

$$k^*(s) = \frac{(\mu_1 - r)(x + \mu_3 e^{\lambda h} y)}{(1-\gamma)\sigma^2 x}, \tag{56}$$

$$c^*(s) = \frac{x + \mu_3 e^{\lambda h} y}{x} Q(s)^{\frac{1}{\gamma - 1}}, \tag{57}$$

where $Q(s)$ is given by (53) and x, y are estimated at time s as the following:

$$x = X(s), \quad y = Y(s) = \int_{-h}^{0} e^{\lambda\theta} X(s+\theta) d\theta. \tag{58}$$

A verification theorem is needed to ensure that the solution is actually equal to the value function defined by (20).

Theorem 2.1 (Verification Theorem) *Let* $V(s,x,y) \in C^{1,2,2}([0,T] \times \mathbb{R} \times \mathbb{R})$ *be a solution of the HJB equation* $(31)-(32)$ *such that*

$$\mathbf{E}\left[\int_0^T [k(t)X(t)V_x(t,X(t),Y(t))]^2 dt\right] < \infty, \quad \forall k \in \Pi. \tag{59}$$

Then we have

$$V(s,x,y) = \sup_{k,c\geq 0} \mathbf{E}_{x,\phi}\Bigg[\int_s^T e^{-\beta(t-s)}U(c(t)X(t))dt + e^{-\beta(T-s)}\Psi(X(T),Y(T))\Bigg]. \tag{60}$$

In addtion, if the utility function is given by

$$U(x) = \frac{1}{\gamma}x^\gamma, \quad \gamma \in (-\infty,1] \text{ and } \gamma \neq 0, \tag{61}$$

then the optimal control policy is given by

$$k^* = -\frac{(\mu_1 - r)V_x}{\sigma^2 x V_{xx}}, \quad c^* = \frac{1}{x}V_x^{\frac{1}{\gamma-1}}. \tag{62}$$

2.4 Some Examples.

In this section, we discuss some examples. For convenience, we rewrite the dynamic equation for the wealth process $X(t)$ here:

$$\begin{aligned} dX(t) &= [((\mu_1 - r)k(t) - c(t) + r)X(t) + \mu_2 Y(t) + \mu_3 Z(t)]\, dt \\ &\quad + [\sigma k(t)X(t)(t)]dB(t), \quad \forall t \in [s, T]. \end{aligned} \tag{63}$$

The initial condition is given by

$$X(s+t) = \varphi(t), \quad \forall t \in [-h, 0]. \tag{64}$$

In last section, we have obtained an explicit solution given the assumption (equation (45)):

$$\mu_2 - \lambda\mu_3 e^{\lambda h} = (\mu_3 e^{\lambda h} + r)\mu_3 e^{\lambda h}. \tag{65}$$

We discuss some interest cases here.

Case 1. Let $\mu_3 = 0$ Then we must have $\mu_2 = 0$

$$\begin{aligned} dX(t) &= [((\mu_1 - r)k(t) - c(t) + r)X(t)]\,dt \\ &\quad + \sigma k(t)X(t)dB(t), \quad \forall t \in [s,T], \end{aligned} \tag{66}$$

$$X(s+\theta) = \varphi(\theta), \qquad \forall \theta \in [-h, 0]. \tag{67}$$

The optimal control policy is

$$k_s^* = \frac{(\mu_1 - r)}{\sigma^2(1-\gamma)}, \tag{68}$$

$$c_s^* = [Q(s)]^{\frac{1}{\gamma-1}}, \tag{69}$$

where $Q(s)$ is given by

$$Q(s) = \left[\left(1 - \frac{1-\gamma}{\Lambda}\right) e^{-\frac{\Lambda(T-s)}{1-\gamma}} + \frac{1-\gamma}{\Lambda}\right]^{1-\gamma} \tag{70}$$

and Λ is given by

$$\Lambda = \beta + \frac{(\mu_1 - r)^2\gamma}{2\sigma^2(\gamma - 1)} - \gamma r. \tag{71}$$

The value function is given by

$$V(s, x, y) = \frac{1}{\gamma} Q(s) x^{\gamma}. \tag{72}$$

The value function and the optimal consumption control policy is the same with the optimal consumption control policy of the classical Merton's problem on a finite time horizon with objective function

$$J(s, x) = \max_{k,c} \mathbf{E}_{s,x} \left[\int_s^T e^{-\beta(t-s)} \frac{1}{\gamma} (c(t)X(t))^{\gamma} dt + e^{-\beta(T-s)} \frac{1}{\gamma} [X(T)]^{\gamma} \right], \tag{73}$$

with dynamic equations for $X(t)$ being

$$dX(t) = [(\mu_1 - r)k(t) - c(t) + r]X(t)dt + \sigma k(t)X(t)dB(t), \tag{74}$$

$$X(0) = x. \tag{75}$$

where $x = x(\varphi) = \varphi(0)$.

Case 2. Let $\mu_3 = \nu e^{-\lambda h}$ for a constant $\nu > 0$ and let $\mu_2 = \nu^2 + \nu(r+\lambda)$.Now the equations of $X(t)$ become

$$\begin{aligned} dX(t) &= [((\mu_1 - r)k(t) - c(t) + r)X(t) + (\nu^2 + \nu(r+\lambda))Y(t) \\ &\quad + \nu e^{-\lambda h} Z(t)]dt + \sigma k(t) X(t) dB(t), \qquad (76) \\ X(s+\theta) &= \varphi(\theta), \qquad \forall \theta \in [-h, 0]. \qquad (77) \end{aligned}$$

The optimal control is now given by

$$\begin{aligned} k_s^* &= \frac{(\mu_1 - r)(x + \nu y)}{(1-\gamma)\sigma^2 x} = \frac{\mu_1 - r}{\sigma^2(1-\gamma)} + \left[\frac{(\mu_1 - r)\nu}{\sigma^2(1-\gamma)}\right]\frac{y}{x}; \qquad (78) \\ c_s^* &= \frac{x + \nu y}{x} Q(s)^{\frac{1}{\gamma-1}} = \left(1 + \nu \frac{y}{x}\right) Q(s)^{\frac{1}{\gamma-1}}, \qquad (79) \end{aligned}$$

where $Q(s)$ is given by

$$Q(s) = \left[\left(1 - \frac{1-\gamma}{\Lambda}\right) e^{-\frac{\Lambda(T-s)}{1-\gamma}} + \frac{1-\gamma}{\Lambda}\right]^{1-\gamma} \qquad (80)$$

and Λ is given by

$$\Lambda = \beta + \frac{(\mu_1 - r)^2 \gamma}{2\sigma^2(\gamma - 1)} - \gamma(\nu + r). \tag{81}$$

The value function is given by

$$V(s, x, y) = \frac{1}{\gamma} Q(s)(x + \nu y)^{\gamma}. \tag{82}$$

As we can see, both the control policy and the value function now depend on the parameter ν.

Case 3. Now we assume $\mu_3 = e^{-\lambda h}$. Then we have

$$r + \mu_3 e^{\lambda h} = r + 1.$$

If μ_2 satisfies

$$\mu_2 = r + 1 + \lambda, \tag{83}$$

then it is easy to verify that (65) holds. Then the dynamic equation for $X(t)$ is now given by

$$\begin{aligned} dX(t) &= \left[((\mu_1 - r)k(t) - c(t) + r)X(t) + (\lambda + r + 1)Y(t) + e^{-\lambda h} Z(t)\right] dt \\ &\quad + \sigma k(t)X(t)dB(t), \quad \forall t \in [s, T], \end{aligned} \tag{84}$$

$$X(s+\theta) = \varphi(\theta), \qquad \forall \theta \in [-h, 0]. \tag{85}$$

The optimal control policy is

$$k_s^* = \frac{(\mu_1 - r)(x+y)}{\sigma^2(1-\gamma)x} = \frac{\mu_1 - r}{\sigma^2(1-\gamma)} + \left[\frac{(\mu_1 - r)}{\sigma^2(1-\gamma)}\right] \frac{y}{x}, \tag{86}$$

$$c_s^* = \frac{x+y}{x} Q(s)^{\frac{1}{\gamma-1}} = \left(1 + \frac{y}{x}\right) Q(s)^{\frac{1}{\gamma-1}}, \tag{87}$$

where $Q(s)$ is given by

$$Q(s) = \left[\left(1 - \frac{1-\gamma}{\Lambda} \right) e^{-\frac{\Lambda(T-s)}{1-\gamma}} + \frac{1-\gamma}{\Lambda} \right]^{1-\gamma} \tag{88}$$

and Λ is given by

$$\Lambda = \beta + \frac{(\mu_1 - r)^2 \gamma}{2\sigma^2(\gamma - 1)} - \gamma(1 + r). \tag{89}$$

The value function is given by

$$V(s, x, y) = \frac{1}{\gamma} Q(s)(x + y)^{\gamma}. \tag{90}$$

As we can see, both the control policy and the value function now depend on the delay variable y. Actually, this is a special case of Case 2 with $\nu = 1$.

Final Remark. From the condition (65), we can get

$$\begin{aligned}\mu_2 &= \lambda\mu_3 e^{\lambda h} + (r + \mu_3 e^{\lambda h})\mu_3 e^{\lambda h} \\ &= \mu_3 e^{\lambda h}(\lambda + r + \mu_3 e^{\lambda h}).\end{aligned} \tag{91}$$

So it is easy to see that $\mu_2 = 0$ if and only if $\mu_3 = 0$, provided that $\mu_3 \geq 0$, and

$$\lim_{\mu_3 \to \infty} \mu_2 = \infty.$$

In other words, the price change of $X(t)$ must depend on both $Y(t)$ and $Z(t)$ at the same time with similar manner in order to obtain a explicit solution $V(s, x, y)$.

3 Stochastic Control Problems with Memory: General Framework

3.1 Problem Formulation

- We study the finite time horizon optimal control problem for a general system of stochastic functional differential equations on the interval $[t, T]$.
- Let $h > 0$ be a fixed constant, and let $\mathbb{J} = [-h, 0]$ denote the duration of the bounded memory of the equations considered in this paper. For the sake of simplicity, we denote $C(\mathbb{J}; \Re^n)$, the space of continuous functions $\phi : \mathbb{J} \to \Re^n$, by $\mathbf{C}$. Note that $\mathbf{C}$ is a real separable Banach space under the supremum norm defined by

$$\|\phi\| = \sup_{t \in \mathbb{J}} |\phi(t)|, \quad \phi \in \mathbf{C},$$

where $| \cdot |$ is the Euclidean norm in $\Re^n$.

- Denote by $(\cdot \mid \cdot)$ the inner product in $L^2(\mathbb{J}, \Re^n)$ as the following

$$(\phi|\psi) = \int_{-r}^{0} \langle \phi(s), \psi(s) \rangle ds, \qquad \text{and } \|\phi\|_2 = (\phi|\phi)^{\frac{1}{2}}, \quad \forall \phi, \psi \in \mathbf{C},$$

  where $\langle \cdot , \cdot \rangle$ is the inner product in $\Re^n$.

- Notation: If $\psi \in C([-r, \infty); \Re^n)$ and $t \in \Re_+$, let $\psi_t \in \mathbf{C}$ be defined by $\psi_t(\theta) = \psi(t + \theta)$, $\theta \in \mathbb{J}$.

- Let $\{W(t), t \geq 0\}$ be a certain m-dimensional standard Brownian motion defined on a complete filtered probability space $(\Omega, \mathcal{F}, P; \mathbf{F})$.

- Let $L^2(\Omega, \mathbf{C})$ be the space of $\mathbf{C}$-valued random variables $\Xi : \Omega \to \mathbf{C}$ such that

$$\|\Xi\|_{L^2} = \left\{ \int_{\Omega} \|\Xi(\omega)\|^2 dP(\omega) \right\}^{\frac{1}{2}} < \infty.$$

  In addition, let $L^2(\Omega, \mathbf{C}; \mathcal{F}(t))$ be those $\Xi \in L^2(\Omega, \mathbf{C})$ which are $\mathcal{F}(t)$-measurable.

- We consider the following system of controlled stochastic functional differential equations with a bounded memory:

$$dX(s) = f(s, X_s, u(s))ds + g(s, X_s, u(s))dW(s), \quad s \in [t, T], \tag{92}$$

with the initial condition

$$X_t = \psi_t, \quad \forall \psi_t \in L^2(\Omega, \mathbf{C}; \mathcal{F}(t))$$

- The functions, $f : [0, T] \times \mathbf{C} \times U \to \Re^n$ and $g : [0, T] \times \mathbf{C} \times U \to \Re^{n \times m}$ are given deterministic functions and they satisfy the following linear growth and Lipschitz conditions (See also Mohammed [**?**, **?**]).

Assumption 1 *There exists a constant* $\Lambda > 0$ *such that*

$$|f(t, \varphi, u) - f(t, \phi, u)| + |g(t, \varphi, u) - g(t, \phi, u)| \leq \Lambda \|\varphi - \phi\|,$$
$$\forall (t, \varphi, u), (t, \phi, u) \in [0, T] \times \mathbf{C} \times U.$$

Assumption 2 *There exists a constant* $K > 0$ *such that*

$$|f(t, \phi, u)| + |g(t, \phi, u)| \leq K(1 + \|\phi\|), \ \forall (t, \phi, u) \in [0, T] \times \mathbf{C} \times U.$$

- Let L and Ψ be two continuous real-valued functions on $[0,T]\times\mathbf{C}\times U$ and $[0,T]\times\mathbf{C}$, with at most polynomial growth in $L^2(\mathbb{J};\Re)$. In other words, there exist a constant $\Lambda>0$ and an integer $k>0$ such that

$$|L(t,\phi,u)|\le\Lambda(1+\|\phi\|_2)^k,\qquad \text{and}\quad |\Psi(t,\phi)|\le\Lambda(1+\|\phi\|_2)^k.$$

- The objective function is

$$J(t,\psi;u(\cdot))\ \equiv\ \mathbb{E}\Big[\int_t^T e^{-\rho(s-t)}L(s,X_s(t,\psi,u(\cdot)),u(s))ds \\ +e^{-\rho(T-t)}\Psi(X_T(t,\psi,u(\cdot)))\Big], \tag{93}$$

where $\rho>0$ denotes a discount factor.

- The value function $V:[0,T]\times\mathbf{C}\to\Re$ is defined as

$$V(t,\psi)=\sup_{u(\cdot)\in\mathcal{U}[t,T]} J(t,\psi;u(\cdot)). \tag{94}$$

3.2 The Hamilton-Jacobi-Bellman Equation

- Let $\mathbf{C}^*$ and $\mathbf{C}^\dagger$ be the space of bounded linear functionals $\Phi : \mathbf{C} \to \Re$ and bounded bilinear functionals $\tilde{\Phi} : \mathbf{C} \times \mathbf{C} \to \Re$, of the space $\mathbf{C}$, respectively.

- Let $\mathbf{B} = \{v\mathbf{1}_{\{0\}}, v \in \Re^n\}$, where $\mathbf{1}_{\{0\}} : [-r, 0] \to \Re$ is defined by

$$\mathbf{1}_{\{0\}}(\theta) = \begin{cases} 0 & \text{for } \theta \in [-r, 0), \\ 1 & \text{for } \theta = 0. \end{cases}$$

  We form the direct sum

$$\mathbf{C} \oplus \mathbf{B} = \{\phi + v\mathbf{1}_{\{0\}} \mid \phi \in \mathbf{C},\ v \in \Re^n\}$$

  and equip it with the norm $\|\cdot\|$ defined by

$$\|\phi + v\mathbf{1}_{\{0\}}\| = \sup_{\theta \in [-r,0]} |\phi(\theta)| + |v|, \quad \phi \in \mathbf{C},\ v \in \Re^n.$$

- Frèchet derivative: $D\Phi(\varphi) \in \mathbf{C}^*$.
  It has a unique and continuous linear extension $\overline{D\Phi(\varphi)} \in (\mathbf{C} \oplus \mathbf{B})^*$.

- The Second order Fréchet derivative, $D^2\Phi(\varphi) \in \mathbf{C}^\dagger$, has a unique and continuous linear extension $\overline{D^2\Phi(\varphi)} \in (\mathbf{C} \oplus \mathbf{B})^\dagger$.

- $\mathcal{S}$-operator: For a Borel measurable function $\Phi : \mathbf{C} \to \Re$, we also define

$$\mathcal{S}(\Phi)(\phi) = \lim_{h \to 0+} \frac{1}{h}\left[\Phi(\tilde{\phi}_h) - \Phi(\phi)\right] \tag{95}$$

  for all $\phi \in \mathbf{C}$, where $\tilde{\phi} : [-r, T] \to \Re^n$ is an extension of ϕ defined by

$$\tilde{\phi}(t) = \begin{cases} \phi(t) & \text{if } t \in [-r, 0) \\ \phi(0) & \text{if } t \geq 0, \end{cases}$$

  and $\tilde{\phi}_t \in \mathbf{C}$ is defined by

$$\tilde{\phi}_t(\theta) = \tilde{\phi}(t + \theta), \quad \theta \in [-r, 0].$$

- Let $C^{1,2}_{lip}([0, T] \times \mathbf{C})$ be the space of functions $\Phi : [0, T] \times \mathbf{C} \to \Re$ such that $\frac{\partial \Phi}{\partial t} : [0, T] \times \mathbf{C} \to \Re$ and $D^2\Phi : [0, T] \times \mathbf{C} \to \mathbf{C}^\dagger$ exist and are continuous and satisfy the following Lipschitz condition:

$$\|D^2\Phi(t, \phi) - D^2\Phi(t, \varphi)\|^\dagger \leq K\|\phi - \varphi\| \quad \forall t \in [0, T],\ \phi, \varphi \in \mathbf{C}.$$

Theorem 3.2 *Suppose that $\Phi \in C^{1,2}_{lip}([0,T]\times \mathbf{C}) \cap \mathcal{D}(\mathcal{S})$. Let $u(\cdot) \in \mathcal{U}[t,T]$ and $\{X_s, s \in [t,T]\}$ be the $\mathbf{C}$-valued Markov solution process of equation (92) with the initial data $(t,\varphi_t) \in [0,T]\times \mathbf{C}$. Then*

$$\begin{aligned} &\lim_{\epsilon \downarrow 0} \frac{E[\Phi(t+\epsilon, X_{t+\epsilon})] - \Phi(t,\varphi_t)}{\epsilon} \qquad (96) \\ = &\ \frac{\partial}{\partial t}\Phi(t,\varphi_t) + \mathcal{S}(\Phi)(t,\varphi_t) + \overline{D\Phi(t,\varphi_t)}(f(t,\varphi_t,u(t))\mathbf{1}_{\{0\}}) \\ &+\frac{1}{2}\sum_{j=1}^{m} \overline{D^2\Phi(t,\varphi_t)}(g(t,\varphi_t,u(t))\mathbf{e}_j\mathbf{1}_{\{0\}}, g(t,\varphi_t,u(t))\mathbf{e}_j\mathbf{1}_{\{0\}}), \end{aligned}$$

where $\mathbf{e}_j, j = 1,2,\cdots,m,$ is the jth unit vector of the standard basis in $\Re^m$.

Theorem (Larssen) *Let Assumptions 1-2 hold. Then for any* $(t, \psi) \in [0, T] \times \mathbf{C}$ *and* $\mathbf{F}(t)$*-stopping time* $\tau \in [t, T]$,

$$V(t, \psi) = \sup_{u(\cdot) \in \mathcal{U}[t,T]} \mathbf{E}\bigg[\int_t^\tau e^{-\rho(s-t)} L(s, X_s(t, \psi, u(\cdot)), u(s)) ds + e^{-\rho(\tau - t)} V(\tau, X_\tau(t, \psi, u(\cdot))) \bigg]. \tag{97}$$

Let $v \in U$*. We define:*

$$\mathcal{A}^v V(t, \psi) \equiv \mathcal{S}(V)(t, \psi) + \overline{DV(t, \psi)}(f(t, \psi, v)\mathbf{1}_{\{0\}}) + \frac{1}{2} \sum_{i=1}^{m} \overline{D^2 V(t, \psi)}(g(t, \psi, v)\mathbf{e}_i \mathbf{1}_{\{0\}}, g(t, \psi, v)\mathbf{e}_i \mathbf{1}_{\{0\}}).$$

Theorem 3.3 *Suppose V is the value function defined by (94) and atisfies $V \in C^{1,2}_{lip}([0,T] \times \mathbf{C}) \cap \mathcal{D}(\mathcal{S})$. Then the value function V satisfies the following HJB equation:*

$$\rho V(t,\psi) - \frac{\partial V}{\partial t}(t,\psi) - \max_{v \in U} \left[\mathcal{A}^v V(t,\psi) + L(t,\psi,v)\right] = 0 \tag{98}$$

on $[0,T] \times \mathbf{C}$, and $V(T,\psi) = \Psi(\psi)$, $\forall \psi \in \mathbf{C}$.

- The value function V satisfies the necessary smoothness condition $V \in C^{1,2}_{lip}([0,T] \times \mathbf{C}) \cap \mathcal{D}(\mathcal{S})$.
- In general we need to consider viscosity solution instead of a classical solution for HJB equation (98).
- Actually, the value function is a unique viscosity solution of the HJB equation (98).

3.3 Viscosity Solution of the HJB Equation

Definition 1 *Let $w \in C([0,T] \times \mathbf{C})$. We say that w is a viscosity subsolution of (98) if, for every $\Gamma \in C^{1,2}_{lip}([0,T] \times \mathbf{C}) \cap \mathcal{D}(\mathcal{S})$, for $(t,\psi) \in [0,T] \times \mathbf{C}$ satisfying $\Gamma \geq w$ on $[0,T] \times \mathbf{C}$ and $\Gamma(t,\psi) = w(t,\psi)$, we have*

$$\rho\Gamma(t,\psi) - \frac{\partial \Gamma}{\partial t}(t,\psi) - \max_{v \in U} \left[\mathcal{A}^v \Gamma(t,\psi) + L(t,\psi,v)\right] \leq 0.$$

We say that w is a viscosity super solution of (98) if, for every $\Gamma \in C^{1,2}_{lip}([0,T] \times \mathbf{C}) \cap \mathcal{D}(\mathcal{S})$, and for $(t,\psi) \in [0,T] \times \mathbf{C}$ satisfying $\Gamma \leq w$ on $[0,T] \times \mathbf{C}$ and $\Gamma(t,\psi) = w(t,\psi)$, we have

$$\rho\Gamma(t,\psi) - \frac{\partial \Gamma}{\partial t}(t,\psi) - \max_{v \in U} \left[\mathcal{A}^v \Gamma(t,\psi) + L(t,\psi,v)\right] \geq 0.$$

We say that w is a viscosity solution of (98) if it is both a viscosity supersolution and a viscosity subsolution of (98).

For our value function V defined by (94), we now show that it has the following property.

Lemma 3.3 *The value function V defined in (94) is continuous and there exists a constant $\Lambda > 0$ and a positive integer k such that, for every $(t, \phi) \in [0, T] \times \mathbf{C}$,*

$$|V(t, \phi)| \leq \Lambda(1 + \|\phi\|_2)^k. \tag{99}$$

and there exists a constant $K > 0$ such that

$$|V(s, \phi) - V(s, \varphi)| \leq K\|\phi - \varphi\|, \ \forall (s, \phi), (s, \varphi) \in [0, T] \times \mathbf{C}. \tag{100}$$

We have the following result:

Theorem 3.4 *The value function V is a viscosity solution of the HJB equation*

$$\rho V(t, \psi) - \frac{\partial V}{\partial t}(t, \psi) - \max_{v \in U} \ [\mathcal{A}^v V(t, \psi) + L(t, \psi, v)] = 0 \tag{101}$$

on $[0, T] \times \mathbf{C}$, and $V(T, \psi) = \Psi(\psi), \ \forall \psi \in \mathbf{C}$.

3.4 Uniqueness

Since a viscosity solution is both a subsolution and a supersolution, the uniqueness result will follow immediately after we establish the following comparison principle:

Theorem 3.5 (Comparison Principle) *Assume that* $V_1(t, \psi)$ *and* $V_2(t, \psi)$ *are both continuous with respect to the argument* (t, ψ) *and are respectively viscosity subsolution and supersolution of (98) with at most a polynomial growth. In other terms, there exists a real number* $\Lambda > 0$ *and a positive integer* $k > 0$ *such that,*

$$|V_i(t, \psi)| \leq \Lambda(1 + \|\psi\|_2)^k, \ \ \textit{for} \ \ (t, \psi) \in [0, T] \times \mathbf{C}, \ \ i = 1, 2.$$

Then

$$V_1(t, \psi) \leq V_2(t, \psi) \quad \textit{for all } (t, \psi) \in [0, T] \times \mathbf{C}. \tag{102}$$

THANK YOU!