Optimal Contracting under Mean-Volatility Ambiguity Uncertainties

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November 5, 2018

Mathematical Finance Colloquium at
University of Southern California

1 Introduction

- We examine a principal-agent model under moral hazard in the presence of mean-volatility joint ambiguity uncertainties.
- **Uncertainties** = risks + ambiguities.
  - risk: the probability distribution of the outcome is known.
    E.g., $\tilde{x} \sim N(\mu, \sigma^2)$, where μ and σ are known constants.
  - ambiguity: the distribution is unknown.
    E.g., $\tilde{x} \sim N(\mu, \sigma^2)$, where μ and σ are unknown.
- Preferences under ambiguity uncertainties:
  maxmin, Choquet, and smooth-ambiguity expected utilities.

- In our agency economy,
  - Both the principal and agent are endowed with Gilboa-Schmeidler's maxmin (multiple-priors) utilities.
  - Ambiguity uncertainties arise in terms of both the mean and volatility of the continuous-time outcome.

- Literature on contracting under ambiguity
  - Weinschenk (2010, WP): a discrete-time model, agree-to-disagree.
  - Szydlowski (2012, WP): ambiguity leads to excessive incentives.
  - Miao and Rivera (2015, WP): a tradeoff between incentives and ambiguity sharing.
  - Mastrolia and Possamäi (2018): similar to this paper, but no joint ambiguity.
  - **Remark:** Unlike these studies, we
    (1) distinguish between ex-ante perceived and ex-post realized volatilities,
    (2) show that the worst priors of the two contracting parties are symmetrized at optimum,
    (3) ...., and a number of other results.

The road map

Section 2: Three major issues in contracting under ambiguity

Section 3: The general mean-volatility control problem

Section 4: The contracting model

- The first-best contracting
- The second-best contracting

Section 5: A linear-quadratic case: an example

Section 6: Conclusion

2 Three major issues in contracting under ambiguity

i. Agree-to-disagree possibilities.

ii. Two different dynamics for the same outcome:
ex-ante perceived vs. ex-post realized.

iii. Volatility control method in weak formulation.

i. Agree-to-disagree issues

Two individuals with maxmin utilities under ambiguity.
Given an uncertain payoff ξ with an ambiguity set $\mathcal{P}$,
two different individuals, A and B would maximize, resp.,

$$\min_{P \in \mathcal{P}} E^P[U_A(\xi^A)], \quad \text{and} \quad \min_{P \in \mathcal{P}} E^P[U_B(\xi^B)].$$

Let

$$P^A \in \arg\min_{P \in \mathcal{P}} E^P[U_A(\xi^A)], \quad \text{and} \quad P^B \in \arg\min_{P \in \mathcal{P}} E^P[U_B(\xi^B)].$$

- P^A and $P^B \neq$ **the true probability measure.**
  They are merely *perceptions* about ambiguity.
- Different perceptions, $P^A \neq P^B$, $\sim$ *info. asymm. without learning.*
  Then, *individuals A and B agree to disagree*.

ii. The Two Different Dynamics for the same outcome, because of ambiguity parameter $\widetilde{v}$.

$$\begin{aligned} \text{ex-post realized:} \qquad & dY_t = f(t, Y, \widetilde{v})dt + \sigma(t, Y, \widetilde{v})dB_t^{\widetilde{v}}, \\ \text{ex-ante perceived:} \qquad & dY_t = f(t, Y, v)dt + \sigma(t, Y, v)dB_t^{v}. \end{aligned}$$

Ex-post realized process: ex-ante unknown,
yet ex-post verifiable and thus contractable.

Ex-ante perceived process: ex-ante known,
yet privately perceived and thus noncontractable.

Note: Three different volatilities for the same outcome.

- $\sigma(t, Y, v)$: ex-ante perceived volatility,
- $\widetilde{\sigma}$: realized volatility, i.e., $\widetilde{\sigma}^2 dt \equiv d\langle Y_t \rangle$,
- $\boldsymbol{\sigma}$: $\mathcal{P}$-aggregator of all admissible σ's.

iii. Mean-Volatility Control Method in Weak Formulation

$\Rightarrow$ Need to introduce singular measures.

- Our problem requires weak formulation.
  - Maxmin (multiple-priors) utilities in weak formulation.
  - Contracting problems in weak formulation.
- Formulation
  - Strong formulation: fix a probability measure, and choose a process.
  - Weak formulation: fix a process, and choose a probability measure.

For weak formulation, we need to find a fixed process Y_t whose QV varies with probability measure.

We want to find a fixed process Y_t with the property that under each P^v,

$$dY_t = \sigma(t, Y, v) dW_t^v.$$

- Let $\Omega \equiv \{\omega \in C[0,1] \,|\, \omega(0) = 0\}$.
- Define Y_t, pathwise, for each $\omega \in \Omega$,

$$dY_t(\omega) \equiv d\omega_t, \quad Y_0 \in \mathcal{R}.$$

- **Partition Ω by QV, σ^2, of each ω.**

$$\Omega^\sigma := \{\omega \in \Omega \,|\, d\langle Y_t \rangle = \sigma_t^2 dt,\, t \in [0,1]\},$$

- $\Omega^\sigma \cap \Omega^{\sigma'} = \emptyset$, if $\sigma \neq \sigma'$.
- **Then P^σ and $P^{\sigma'}$ have to be singular.**
- Let P^σ be a Wiener measure on partition Ω^σ, and W_t^σ be a P^σ-standard BM.

- $\boldsymbol{\sigma}^2(\omega) :=$ the QV density of $\omega(\in \Omega)$.

  Then $\boldsymbol{\sigma} = \sigma$ under P^σ a.s..

$$\text{Namely,} \quad \boldsymbol{\sigma}(\omega) \quad = \quad \begin{cases} \sigma(\omega) & \omega \in \Omega^\sigma \\ \sigma'(\omega) & \omega \in \Omega^{\sigma'} \\ \sigma''(\omega) & \omega \in \Omega^{\sigma''} \\ .... & .... \end{cases}$$

- For $\boldsymbol{\sigma} > 0$, Let $W_t := \int_0^t \frac{1}{\boldsymbol{\sigma}_s} dY_s.$

  Then, W_t is a universal std BM such that $W_t = W_t^\sigma$ under P^σ.

$$\text{Namely,} \quad W_t \quad = \quad \begin{cases} W_t^\sigma(\omega) & \text{under } P^\sigma \text{ a.s..} \\ W_t^{\sigma'}(\omega) & \text{under } P^{\sigma'} \text{ a.s..} \\ W_t^{\sigma''}(\omega) & \text{under } P^{\sigma''} \text{ a.s..} \\ .... & .... \end{cases}$$

- The family $\mathcal{P}$ of admissible singular measures:

$$\mathcal{P} := \{\text{singular Wiener measures } P^{\sigma}\text{'s, } \sigma \in \boldsymbol{\Sigma}\}, \tag{1}$$

where $\boldsymbol{\Sigma}$ = the class of admissible volatilities.

- Then,

$$dY_t \equiv d\omega_t = \boldsymbol{\sigma}_t dW_t,\ \mathcal{P}\text{-q.s.} = \begin{cases} \sigma_t dW_t^{\sigma} & \text{under } P^{\sigma} \text{ a.s..} \\ \sigma_t' dW_t^{\sigma'} & \text{under } P^{\sigma'} \text{ a.s..} \\ \sigma_t'' dW_t^{\sigma''} & \text{under } P^{\sigma''} \text{ a.s..} \\ \ldots\ldots & \ldots\ldots \end{cases}$$

- Given $\mathcal{P}$, we have the volatility control in weak formulation as follows:

$$\begin{aligned} \text{optimize}_{P^{\sigma} \in \mathcal{P}} \quad & E^{\sigma}[\xi(Y)] \\ \text{s.t.} \quad & dY_t = \boldsymbol{\sigma}_t dW_t = \sigma dW_t^{\sigma}. \end{aligned}$$

Assumption 1. *The class* $\mathbf{\Sigma}$ *consists of diffusion coefficients,* $\sigma : U \times D \times [0,1] \times \Omega \to \mathcal{R}_+$.

(1) *All* σ*'s in* $\mathbf{\Sigma}$ *are* $\hat{\mathcal{F}}^{\mathcal{P}}$*-progressively measurable, and uniformly bounded away from zero.*

(2) $\mathbf{\Sigma}$ *is closed under concatenation, i.e., if* $\sigma, \sigma' \in \mathbf{\Sigma}$*, then* $\sigma \mathbf{1}_{[0,t]} + \sigma' \mathbf{1}_{(t,1]} \in \mathbf{\Sigma}$ *for* $t \in [0,1]$.

(3) *The SDE,* $dY_t = \sigma_t dW_t$*, has a unique strong solution under* P^σ*,* $\sigma \in \mathbf{\Sigma}$.

Assumption 2. *The class* Φ *is the collection of functionals,* $f : U \times D \times [0,1] \times \Omega \to \mathcal{R}$*, with the following properties: for each* $(u,v) \in \mathcal{U} \times \mathcal{D}$ *and* $\sigma(u,v,t,Y) \in \mathbf{\Sigma}$*,* $f(u,v,t,Y)$ *is* $\hat{\mathcal{F}}^{\mathcal{P}}$*-progressively measurable,* $\int_0^t \frac{f^2(u,v,t,Y)}{\sigma^2(u,v,t,Y)} ds < \infty$ *path by path for all* $t \in [0,1]$*, and there exists a constant* K *such that*

$$\left| \frac{f(u,v,t,Y)}{\sigma(u,v,t,Y)} \right| \leq K \left(1 + \max_{0 \leq s \leq t} |Y_s| \right), \quad \forall (u,v,t,Y).$$

Mean-Volatility Control in Weak Formulation

- Enlarge $\mathcal{P}$ to $\overline{\mathcal{P}}$ with abs. cont. measures.
  - Introduce Φ and $\vartheta^{\sigma,f}$, where

$$\Phi = \text{ the class of admissible drifts, } f\text{'s;}$$

$$\vartheta^{\sigma,f} = \frac{dP^{\sigma,f}}{dP^{\sigma}} = \exp\left(\int_0^1 \frac{f}{\sigma^2} dY_t - \frac{1}{2}\int_0^1 \frac{f^2}{\sigma_s^2} ds\right).$$

  - For each $P^{\sigma} \in \mathcal{P}$ and $f \in \Phi$, let $P^{\sigma,f}$ satisfy $dP^{\sigma,f} = \vartheta^{\sigma,f} dP^{\sigma}$. Then, under $\mathcal{P}^{\sigma,f}$,

$$dY_t = f dt + \sigma dB_t^{\sigma,f}.$$

Let

$$\overline{\mathcal{P}} := \left\{ P^{\sigma,f} \,|\, dP^{\sigma,f} = \vartheta^{\sigma,f} dP^{\sigma}, \quad (f,\sigma) \in \Phi \times \Sigma \right\}$$

- Then, the following mean-volatility control problem in weak formulation:

$$\begin{aligned} \text{optimize}_{P^{\sigma,f} \in \overline{\mathcal{P}}} \qquad & E^{\sigma,f}[\xi(Y)] \\ \text{s.t.} \qquad & dY_t = \boldsymbol{\sigma}_t dW_t = \sigma(v,t,Y) dW_t^{\sigma} \\ & \qquad\qquad = f dt + \sigma dW_t^{\sigma,f}, \end{aligned}$$

becomes equivalent to

$$\begin{aligned} \text{optimize}_{P^{\sigma} \in \mathcal{P}} \qquad & E^{\sigma}[\xi(Y)\vartheta^{\sigma,f}] \\ \text{s.t.} \qquad & dY_t = \boldsymbol{\sigma}_t dW_t. \end{aligned}$$

3 The General Mean-Volatility Control Problem

We consider the following general problem:

$$\sup_{u\in\mathcal{U}} \inf_{v\in\mathcal{D}} \quad E^{u,v}\left[-\exp\left\{-\gamma\left(\xi(Y)+\int_0^1 g(.)ds+\int_0^1 q(.)d\langle Y_s\rangle+\int_0^1 h(.)dY_s\right)\right\}\right] \tag{2}$$

$$\text{s.t.} \quad dY_t = f(u,v,t,Y)dt + \sigma(u,v,t,Y)dB_t^{u,v},$$

where $B_t^{u,v}(= W_t - \int_0^t \frac{f_s}{\sigma_s}ds)$ is the standard BM under $P^{u,v}$. We assume $\gamma > 0$.

Let

$$\hat{\vartheta}_1 = \frac{dP^{u,v}}{d\hat{P}^{u,v}} = \exp\left(\int_0^1 \frac{f}{\sigma^2}dY_t - \frac{1}{2}\int_0^1 \frac{f^2}{\sigma_s^2}ds\right). \tag{3}$$

Then the problem can be stated as follows:

$$\sup_{u\in\mathcal{U}} \inf_{v\in\mathcal{D}} \quad \hat{E}^{u,v}\left[-\exp\left\{-\gamma\xi(Y) - \int_0^1 \hat{G}_s ds - \int_0^1 \Gamma_s dY_s\right\}\right] \tag{4}$$
$$\text{s.t.} \quad dY_t = \boldsymbol{\sigma}_t dW_t,$$

where

$$\hat{G}_t = \gamma g(.) + \gamma q(.)\sigma^2 + \frac{1}{2}\frac{f^2}{\sigma^2},$$
$$\Gamma_t = \gamma h(.) - \frac{f}{\sigma^2}.$$

Let

$$\phi_t(u,v,Y) := -\exp\left\{-\gamma\xi(Y) - \int_t^1 \hat{G}(u,v,s,Y)ds - \int_t^1 \Gamma(u,v,s,Y)dY_s\right\}, \tag{5}$$

We define the CEQ wealth $\mathcal{Q}$ and value functions $(V, \mathcal{V})$ as follows:

$$\mathcal{Q}_t^{u,v} := -\frac{1}{\gamma}\ln\left(-\hat{E}_t^{u,v}\left[\phi_t(u,v,Y)\right]\right), \tag{6}$$

$$V_t(u,v^*(u)) = \operatorname*{ess\,inf}_{v\in\mathcal{D}_t^1} -\exp\left(-\gamma\mathcal{Q}_t^{u,v}\right), \tag{7}$$

$$\mathcal{V}_t = \operatorname*{ess\,sup}_{u\in\mathcal{U}_t^1} V_t(u,v^*(u)). \tag{8}$$

Assumption 3. *There exists a saddle point process* $(u^*, v^*) \in \mathcal{U}\times\mathcal{D}$ *such that for* $u\in\mathcal{U}_t^1$ *and* $v\in\mathcal{D}_t^1$*,* $\mathcal{P}$*-q.s.,*

$$-\exp(-\gamma Q_t^{u,v^*}) \le -\exp(-\gamma Q_t^{u,v^*})(=\mathcal{V}_t) \le -\exp(-\gamma Q_t^{u^*,v}).$$

Let us define the Hamiltonian H^o as follows:
for $(u_t, v_t, p, t, Y) \in U_t(Y) \times D_t(Y) \times \mathcal{R} \times [0,1] \times \Omega$,

$$H^o(u_t, v_t, p, t, Y) \equiv pK(.) + G(.) - \frac{\gamma}{2}\left(p\sigma(.)\right)^2, \tag{9}$$

where (.) is short for (u_t, v_t, t, Y), and

$$G := g + hf + \left[q - \frac{\gamma}{2}h^2\right]\sigma^2,$$
$$K := f - \gamma h \sigma^2.$$

Lemma 1. *Let Assumptions 1 to 3 hold. Also assume that* $\phi_t(u, v, Y) \in \mathcal{L}^2_{\mathcal{P}}$ *for all* $(t, u, v) \in [0, 1] \times \mathcal{U} \times \mathcal{D}$*. Then, there exists a unique* $\mathcal{P}$*-q.s. square integrable process* Z^*_t *such that* $H^o(u_t, v_t, Z^*_t, t, Y)$ *has a saddle point* (u^*_t, v^*_t)*, i.e., for all* $u_t \in U_t(Y)$ *and* $v_t \in D_t(Y)$*,*

$$H^o(u_t, v^*_t, Z^*_t, t, Y) \leq H^o(u^*_t, v^*_t, Z^*_t, t, Y) \leq H^o(u^*_t, v_t, Z^*_t, t, Y). \quad (10)$$

Under $\hat{P}^{u^*, v^*}$*,*

$$d\mathcal{Q}^*_t = -H^o(u^*_t, v^*_t, Z^*_t, t, Y)dt + Z^*_t dY_t, \quad \mathcal{Q}^*_1 = \xi(Y), \quad (11)$$

and $\mathcal{V}_t = -\exp(-\gamma \mathcal{Q}^*_t) = \operatorname{ess\,sup}_{u \in \mathcal{U}^1_t} \operatorname{ess\,inf}_{v \in \mathcal{D}^1_t} [-\exp(-\gamma \mathcal{Q}^{u,v}_t)]$.

A PREVIEW of the economics side of the paper

- The optimal contracts
  - The structure of the 1st-best contract under ambiguity uncertainties is similar to that of risk uncertainties.
  - The 2nd-best contract under ambiguity consists of two sharing rules:
    (1) one for realized outcome and (2) the other for realized volatility.
- 2nd-best vol. sharing rule $\Rightarrow$ compensation and realized volatility are positively related.
  Consistent with stock option granting practices in managerial compensation.
- Ambiguity decreases the 2nd-best pay-for-performance sensitivity.
- Agree-to-disagree issues arise neither in the 1st-best nor in 2nd-best cases.

4 The Model

- The time horizon is the unit interval $[0, 1]$.
- One principal and one agent with CARA preferences: coefficients are γ_P and γ_A, respectively.
- Before time 0, the principal owns an asset with a cashflow prospect Y. The agent has an employment opportunity with his reservation utility $= -\exp(-\gamma_A \mathcal{W}_0)$.
- At time 0, both the principal and agent sign a compensation scheme $S(Y)$, and then the agent manages the asset to improve its cashflow prospect.
- The agent's cumulative cost of effort up to time t: $\int_0^t c(e_t, t, Y)dt$.

- The (universal) filtered probability space (see STZ (2011)):

$$(\Omega, \hat{\mathcal{F}}, \{\hat{\mathcal{F}}_t\}, P \in \mathcal{P}),$$

where

  - $\Omega := \{\omega \in C([0,1]) \,|\, \omega_0 = 0\}$;
  - $\{\hat{\mathcal{F}}_t\}$ is the universal filtration for the family $\mathcal{P}$.
  - $\mathcal{P} := \{P^\sigma, \,|\, \sigma \in \mathbf{\Sigma}\}$, a family of singular Wiener measures.
  - $\overline{\mathcal{P}} := \{P^{e,\mu,\nu} \,|\, P^{e,\mu,\nu} = \vartheta_1(e,\mu,\nu)P^\nu, \quad P^\nu \in \mathcal{P}\}$.

- Given a contract S, the agent chooses $P^{e,\mu,\nu} \in \overline{\mathcal{P}}$ to solve the following problem:

$$\begin{aligned} \sup_{e \in \mathcal{U}} \inf_{(\mu,\nu) \in \mathcal{D}} \quad & E^{e,\mu,\nu}\left[-e^{-\gamma_A\left(S(Y)-\int_0^1 c(e,t,Y)dt\right)}\right] \\ \text{s.t.} \quad & dY_t = f(e,\mu,\nu,t,Y)dt + \sigma(\nu,t,Y)dB_t^{e,\mu,\nu}. \end{aligned}$$

- The agent's problem is equivalently transformed into a volatility control problem as follows.

$$\boxed{\begin{aligned} \sup_{e \in \mathcal{U}} \inf_{(\mu,\nu) \in \mathcal{D}} \quad & E^{\nu}\left[-e^{-\gamma_A\left(S(Y)-\int_0^1 c(e,t,Y)dt\right)}\vartheta_1(e,\mu,\nu)\right] \\ \text{s.t.} \quad & dY_t = \boldsymbol{\sigma}_t dW_t. \end{aligned}}$$

- The admissible class Ψ of contracts:

$$\Psi := \left\{ S \;\middle|\; \begin{array}{l} S \text{ is } \hat{\mathcal{F}}_1^{\mathcal{P}}\text{-measurable, } S \cdot \vartheta_1(e,\mu,\nu) \in \mathcal{L}^2_{\mathcal{P}}, \text{ and} \\ \exp\left\{ -\gamma_A \left(S - \int_0^1 c(e_t,t,Y)dt \right) \right\} \vartheta_1(e,\mu,\nu) \in \mathcal{L}^2_{\mathcal{P}} \end{array} \right\}. \tag{12}$$

Remark: Given each admissible S, the conditional expectation of the agent's expected utility satisfies the q.s. version of the MRT of STZ (2011).

4.1 Representation of Admissible Contracts

Given $(S, e) \in \Psi \times \mathcal{U}$, let

$$\mathcal{W}_0^{S,e} = \text{the CEQ wealth level of the agent's most pessimistic utility,}$$
$$v = (\mu, \nu), \text{ ambiguity parameter pair.}$$

Then,

$$-\exp\left(-\gamma_A \mathcal{W}_0^{S,e}\right) = \inf_v E^{e,v}\left[-\exp\left\{-\gamma_A\left(S(Y) - \int_0^1 c(e_t, t, Y))dt\right)\right\}\right]$$
$$\text{s.t.} \qquad dY_t = f(e_t, \mu_t, \nu_t, t, Y)dt + \sigma(\nu_t, t, Y)dB_t^{u,v}.$$

Proposition 1. *There exist unique $\mathcal{P}$-q.s. $\{\hat{\mathcal{F}}_t\}$-progressively measurable and square integrable processes, (β_t, θ_t, K_t), i.e., $\beta, \theta \in \mathcal{H}^2_{\mathcal{P}}$ and $K \in \mathbb{I}^2_{\mathcal{P}}$, such that S can be represented in the following form: $\mathcal{P}$-q.s.,*

$$S = \mathcal{W}_0^{S,e} + \int_0^1 \left\{ c(e_t, t, Y) - \beta_t f(e_t, \mu_t, \nu_t, t, Y) + \left[\frac{\gamma_A}{2} \beta_t^2 - \theta_t \right] \sigma^2(\nu_t, t, Y) \right\} dt + \int_0^1 \theta_t d\langle Y_t \rangle + \int_0^1 \beta_t dY_t + K_1, \qquad (13)$$

where $K_0 = 0$, under all $P \in \mathcal{P}$; and for all $t \in [0, 1]$, $K_t = 0$ under P^v, and K_t is nondecreasing over time under other $P^{v'}$'s in $\mathcal{P}$.

Remark: Two distinguishing features: volatility sharing rule θ and the process K.

Why the process K?

- To adjust for realized off-the-equilibrium singular events.
- The decision maker views, ex ante, ambiguity uncertainties through his most pessimistic prior, and thus he treats $K_t = 0$, ex ante.
- Singular deviations, if any, from the most pessimistic events lead to less pessimistic payoffs. Hence, adjustments by K have to be positive amounts.

4.2 First-Best Contracting

Problem 1. *(First-best contracting.) Choose a contract S by solving the following problem.*

$$\sup_{\substack{S \in \Psi,\, e \in \mathcal{U} \\ (\mu^A, \nu^A) \in \mathcal{D}}} \inf_{(\mu^P, \nu^P) \in \mathcal{D}} E^{e, \mu^P, \nu^P} \left[-\exp\left\{ -\gamma_P \left(Y_1 - S \right) \right\} \right]$$

$$\begin{aligned}
s.t. \quad & (i) \quad dY_t = f(e_t, \mu_t^P, \nu_t^P, t, Y)dt + \sigma(\nu_t^P, t, Y)dB_t^{e, \mu^P, \nu^P}, \\
& (ii) \quad (\mu^A, \nu^A) \in \arg \inf_{(\hat{\mu}, \hat{\nu}) \in \mathcal{D}} E^{e, \hat{\mu}, \hat{\nu}} \left[-\exp\left\{ -\gamma_A \left(S - \int_0^1 c(e, t, Y)dt \right) \right\} \right] \\
& \qquad\qquad s.t. \quad dY_t = f(e_t, \hat{\mu}, \hat{\nu}, t, Y)dt + \sigma(\hat{\nu}, t, Y)dB_t^{e, \hat{\mu}, \hat{\nu}}, \\
& (iii) \quad E^{e, \mu^A, \nu^A} \left[-\exp\left\{ -\gamma_A \left(S - \int_0^1 c(e_t, t, Y)dt \right) \right\} \right] \geq -\exp\left(-\gamma_A \mathcal{W}_0 \right).
\end{aligned}$$

Remark: Constraint (ii) is new: an incentive compatibility condition which arises even in the first best.

Theorem 1. *(First best.) Suppose that the optimizers,* $(e_t, \theta_t, \mu_t^P, \nu_t^P)$ *and* (μ_t^A, ν_t^A) *lie in the interiors of their respective domains, for all* $t \in [0,1]$*. Then, in the first best, the worst priors of the principal and agent are symmetrized such that* $(\mu_t^A, \nu_t^A) = (\mu_t^P, \nu_t^P) = (\mu_t^c, \nu_t^c) \in D_t(Y)$*. Moreover, there exists a unique* $\mathcal{P}$*-q.s. square integrable process* Z_t^{0P} *such that*

$$(\mu_t^c, \nu_t^c) \in \min_{(\bar{\mu},\bar{\nu})\in D_t(Y)} f(e_t, \bar{\mu}, \bar{\nu}, t, Y) - \frac{1}{2}\frac{\gamma_A\gamma_P}{\gamma_A+\gamma_P}(1+Z_t^{0P})\sigma^2(\bar{\nu}, t, Y), \tag{14}$$

and that the first-best optimal contract S *is:* $\mathcal{P}$*-q.s.,*

$$S = \mathcal{W}_0 + \int_0^1 \Big(c(e_t, t, Y) - \beta_t f(e_t, \mu_t^c, \nu_t^c, t, Y) + \frac{\gamma_A}{2}\beta_t^2\sigma^2(\nu_t^c, t, Y)\Big)\, dt + \int_0^1 \beta_t dY_t, \tag{15}$$

where $\beta_t = \frac{\gamma_P}{\gamma_A+\gamma_P}\left(1+Z_t^{0P}\right)$*, and* $1 + Z_t^{0P} = \frac{c_e(e_t,t,Y)}{f_e(e_t,\mu_t^c,\nu_t^c,t,Y)}$.

- The form of the 1st-best contract with ambiguity is the same as that of the classical 1st-best contract without ambiguity.
  - $\theta_t = 0$ for all t.
  - β_t is the same as that of the classical case without ambiguity.
  - Deviation from the (static) rule of marginal product of labor: $c_e \neq f_e$ whenever $Z_t^P \neq 0$, where Z_t^P represents the principal's outcome-share growth opportunities.
- At optimum, the worst priors of the two parties are symmetrized.

4.3 The Second-Best Contracting

Problem 2. *(Second-best contracting.) Choose a contract S by solving the following problem.*

$$\sup_{\substack{S \in \Psi,\, e \in \mathcal{U} \\ (\mu^A, \nu^A) \in \mathcal{D}}} \inf_{(\mu^P, \nu^P) \in \mathcal{D}} E^{e,\mu^P,\nu^P} \left[-\exp\left\{ -\gamma_P \left(Y_1 - S \right) \right\} \right]$$

$$\begin{aligned}
s.t. \quad (i) \quad & dY_t = f(e, \mu^P, \nu^P, t, Y)dt + \sigma(\nu^P, t, Y)dB_t^{e,\mu^P,\nu^P}, \\
(ii) \quad & (e, \mu^A, \nu^A) \in \arg\sup_{\hat{e} \in \mathcal{U}} \inf_{(\hat{\mu},\hat{\nu}) \in \mathcal{D}} E^{\hat{e},\hat{\mu},\hat{\nu}} \left[-\exp\left\{ -\gamma_A \left(S - \int_0^1 c(\hat{u}, t, Y)dt \right) \right\} \right] \\
& \qquad\qquad s.t. \quad dY_t = f(\hat{e}, \hat{\mu}, \hat{\nu}, t, Y)dt + \sigma(\hat{\nu}, t, Y)dB_t^{\hat{e},\hat{\mu},\hat{\nu}}, \\
(iii) \quad & E^{e,\mu^A,\nu^A} \left[-\exp\left\{ -\gamma_A \left(S - \int_0^1 c(e, t, Y)dt \right) \right\} \right] \geq -\exp\left(-\gamma_A \mathcal{W}_0 \right).
\end{aligned}$$

4.3.1 The Agent's Problem

The agent's Hamiltonian given a contract in the form (13):

$$H^A = -c(e,t,Y) + \beta_t f(e,\mu_t,\nu_t,t,Y) + \left(\theta_t - \frac{\gamma_A}{2}\beta_t\right)\sigma^2(\nu,t,Y).$$

Theorem 2. *(Incentive compatibility/implementability.) Given a contract $S \in \bar{\Psi}$ with admissible $(e_t^*, (\mu_t^*, \nu_t^*); (\beta_t, \theta_t)) \in U \times D_t(Y) \times \mathcal{R}^2$, for $t \in [0,1]$, the agent chooses $(e_t^*, \mu_t^*, \nu_t^*)$ if and only if*

$$(e_t^*, \mu_t^*, \nu_t^*) \in \arg\max_{\hat{e}} \min_{\hat{\mu},\hat{\nu}} H^A(\hat{e}, \hat{\mu}, \hat{\nu}; \beta_t, \theta_t, t, Y). \tag{16}$$

That is, the contract $S \in \bar{\Psi}$ with admissible $(e_t^*, (\mu_t^*, \nu_t^*); (\beta_t, \theta_t)) \in U \times D_t(Y) \times \mathcal{R}^2$ is implementable if and only if $(e_t^*, (\mu_t^*, \nu_t^*))$ is a saddle point of H^A given (β_t, θ_t), for $t \in [0,1]$.

Corollary 1. *If the optimal e_t lies in the interior of U, then $\beta_t = \frac{c_e(e,t,Y)}{f_e(e,\mu,\nu,t,Y)}$.*

- The process K does not affect the implementability condition.
- The agent is indifferent between contracts with and without the process K, because K matters for off-the-equilibrium events only.
- Hence, it is without loss of generality for the principal to consider contracts ignoring the process K as follows:

$$
\begin{aligned}
S = \mathcal{W}_0 - \int_0^1 \max_{\hat{e}} \min_{\hat{\mu},\hat{\nu}} \left[H^A(\hat{e}_t, \hat{\mu}_t, \hat{\nu}_t; \beta_t, \theta_t, t, Y) \right] dt \\
+ \int_0^1 \theta_t d\langle Y_t \rangle + \int_0^1 \beta_t dY_t.
\end{aligned} \tag{17}
$$

4.3.2 The Principal's Problem

Theorem 3. *Assume that the agent's effort e is in the interior of U. There exists a unique $\mathcal{P}$-q.s. square integrable process Z_t^P such that the principal's optimal decision $(e_t, \mu_t^A, \nu_t^A, \mu_t^P, \nu_t^P, \theta_t)$ solves the following problem for $t \in [0,1]$: $\mathcal{P}$-q.s.,*

$$\max_{\bar{e}_t, \bar{\theta}_t} \min_{(\bar{\mu}_t^P, \bar{\nu}_t^P) \in D_t(Y)} -c(\bar{e}_t, t, Y) + \left(1 - \bar{\beta}_t + Z_t^P\right) f(\bar{e}_t, \bar{\mu}_t^P, \bar{\nu}_t^P, t, Y)$$
$$- \left[\frac{\gamma_P}{2}\left(1 - \bar{\beta}_t + Z_t^P\right)^2 + \bar{\theta}_t\right] \sigma^2(\bar{\nu}_t^P, t, Y)$$
$$+ \bar{\beta}_t f(\bar{e}_t, \bar{\mu}_t^A, \bar{\nu}_t^A, t, Y) - \left(\frac{\gamma_A}{2}\bar{\beta}_t^2 - \bar{\theta}_t\right) \sigma^2(\bar{\nu}_t^A, t, Y), \tag{18}$$

$$s.t. \quad \bar{\beta}_t = \frac{c_e(\bar{e}_t, t, Y)}{f_e(\bar{e}_t, \bar{\mu}_t^A, \bar{\nu}_t^A, t, Y)}, \tag{19}$$

$$(\bar{\mu}_t^A, \bar{\nu}_t^A) \in \arg \min_{(\hat{\mu}, \hat{\nu}) \in D_t(Y)} \varphi_A(\bar{e}_t, \hat{\mu}, \hat{\nu}; \bar{\theta}_t, \bar{\beta}_t, t, Y), \tag{20}$$

where (μ_t^P, ν_t^P) and (μ_t^A, ν_t^A) are, respectively, the principal's and agent's worst ambiguity parameter pairs.

Principal's Hamiltonian (18)

$$
\begin{aligned}
H_t = -c(\bar{e}_t, t, Y) + \quad & \underbrace{\left(1-\bar{\beta}_t + Z_t^P\right) f(\bar{e}_t, \bar{\mu}_t^P, \bar{\nu}_t^P, t, Y)}_{\text{principal's perceived share of the drift}} \\
& - \underbrace{\left[\frac{\gamma_P}{2}\left(1-\bar{\beta}_t + Z_t^P\right)^2 + \bar{\theta}_t\right] \sigma^2(\bar{\nu}_t^P, t, Y)}_{\text{principal's perceived risk premium}} \\
& + \quad \underbrace{\bar{\beta}_t f(\bar{e}_t, \bar{\mu}_t^A, \bar{\nu}_t^A, t, Y)}_{\text{agent's perceived share of the drift}} \\
& - \quad \underbrace{\left(\frac{\gamma_A}{2}\bar{\beta}_t^2 - \bar{\theta}_t\right) \sigma^2(\bar{\nu}_t^A, t, Y)}_{\text{agent's perceived risk premium}} \quad .
\end{aligned}
$$

Hamiltonian (18) $\Longrightarrow$

- The principal behaves as if her outcome share were $1 - \beta_t + Z_t^P$, not $1 - \beta_t$.
- She can shift, to and from the agent, 'the perceived risk premia' by using volatility sharing rule θ_t.

- **Nominal shares** to the principal and agent, resp: $1 - \beta_t$ and β_t.
- **Effective shares** to the principal and agent, resp: $1 - \beta_t + Z_t^P$ and β_t.
- Z_t^P = **an imaginary extra share** given to the principal: growth opportunity over time.

Theorem 4. *Assume an interior optimum. The worst priors of the two contracting parties are symmetrized such that* $(\mu_t^A, \nu_t^A) = (\mu_t^P, \nu_t^P) = (\mu_t^c, \nu_t^c)$. *Under the symmetrized prior, there exists a unique* $\mathcal{P}$*-q.s. square integrable process* Z_t^P *such that the optimal outcome- and volatility-sharing sensitivities* (β_t, θ_t) *and the common prior* (μ_t^c, ν_t^c) *can be expressed as follows.*

$$\beta_t = \frac{f_e + \gamma_P \beta_e (\sigma_t^c)^2}{f_e + (\gamma_P + \gamma_A)\beta_e(\sigma_t^c)^2}(1 + Z_t^P), \tag{21}$$

$$\theta_t = \frac{1}{2(1 + Z_t^P)}\beta_t(1 - \beta_t + Z_t^P)\left(\gamma_A \beta_t - \gamma_P(1 - \beta_t + Z_t^P)\right) \geq 0, \tag{22}$$

where $\beta_e = \frac{\partial}{\partial e}\left(\frac{c_e(e_t, t, Y)}{f_e(e_t, \mu_t^c, \nu_t^c, t, Y)}\right)$, *and*

$$\begin{aligned}(\mu_t^c, \nu_t^c) \in \arg \min_{(\hat{\mu}, \hat{\nu}) \in D_t(Y)} & (1 + Z_t^P) f(e_t, \hat{\mu}_t, \hat{\nu}_t, t, Y_t) \\ & - \frac{1}{2}\left(\gamma_A \beta_t^2 + \gamma_P (1 - \beta_t + Z_t^P)^2\right)\sigma^2(\hat{\nu}, t, Y_t).\end{aligned} \tag{23}$$

- That is, the second-best contract is

$$S = \mathcal{W}_0 + \int_0^1 ...dt + \int_0^1 \theta_t d\langle Y_t \rangle + \int_0^1 \beta_t dY_t.$$

  - If $\gamma_A = 0$, then $\theta_t = 0$ and $\beta_t = 1 + Z_t^P$.
  - If $\gamma_A > 0$, then $\theta_t > 0$ and $0 < \beta_t < 1 + Z_t^P$.

- Two striking implications from Theorem 4.

  1. $\theta_t > 0 \Rightarrow$ the agent gets paid for realized volatility.

Realized volatility, $\langle Y_t \rangle \uparrow \Longrightarrow$ Realized compensation, $S \uparrow$.

Consistent with compensation practices of stock option grants!

2. $(\mu_t^A, \nu_t^A) = (\mu_t^P, \nu_t^P) = (\mu_t^c, \nu_t^c)$.

The worst priors of the two parties are symmetrized.

- Suppose the two worst priors are not equal.
  $\Rightarrow$ asymmetric perceptions
  $\rightsquigarrow$ asymmetric information without learning
  $\Rightarrow$ inefficiency in contracting.
- The principal uses the volatility-sharing contract to eliminate the asymmetry of the perceptions.

Symmetrization through volatility sharing

- The principal can, in effect, influence the agent's risk aversion.

  $\theta_t \uparrow \Rightarrow$ agent's risk burden $\downarrow \Rightarrow$ his effective risk aversion $\downarrow$
  $\Rightarrow$ risk premium on compensation uncertainty $\downarrow$.

- If $\nu_A > \nu_P$, then the agent demands **an excessively high risk premium**.

  $\Rightarrow$ The principal shifts the agent's uncertainty exposure to herself by increasing θ_t.

- If $\nu_A < \nu_P$, then the agent demands lower risk premium than the principal perceives.

  $\Rightarrow$ The principal shifts her own uncertainty exposure to the agent by decreasing θ_t.

- At optimum, $\nu_A = \nu_P$.

Then, Why No Volatility-Sharing in the First Best?

- The 1st-best contract: uncertainty sharing. No incentive issues.
  - Under the 1st-best outcome-sharing contract,
    both parties share uncertainties symmetrically.
    Thus, their worst priors are symmetrized.
- The 2nd-best contract: uncertainty sharing and incentives.
  - For incentives, the 2nd-best outcome-sharing sensitivity has to be greater than that of the 1st best.

    $\Rightarrow$ the two worst priors become asymmetrical.

    $\Rightarrow$ contract inefficiency.

    $\Rightarrow$ the principal uses a volatility sharing contract in order to achieve the symmetrization.

5 A Linear-Quadratic (Markovian) Case

The outcome process:

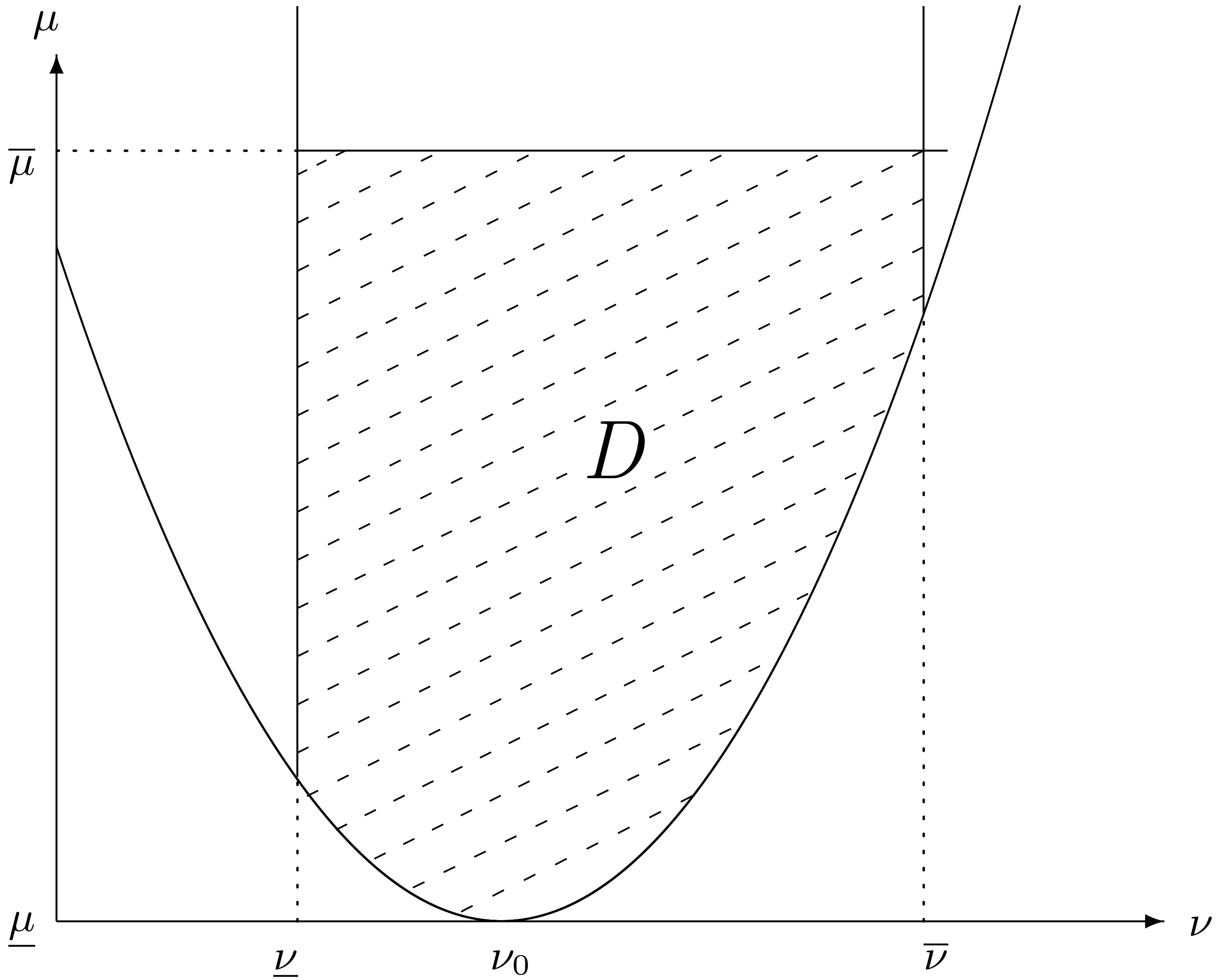
$$dY_t = (\eta Y_t + e_t + \mu_t)dt + \nu_t dB_t^{u,v}.$$

- **If** $\eta > (=, <)0$**, then the outcome exhibits 'increasing (constant, decreasing) returns to scale.'**

Assume:

$$c(e) = \frac{\kappa}{2}e^2,$$
$$D = \left\{(\mu, \nu) \in [\underline{\mu}, \overline{\mu}] \times [\underline{\nu}, \overline{\nu}] \;\middle|\; \pi = \mu - \frac{\alpha}{2}(\nu - \nu^0)^2 \geq 0\right\}.$$

- The set D becomes Epstein-Schneider's (2010) quadratic ambiguity set if $\nu^0 = 0$.


μ
$\overline{\mu}$
D
$\underline{\mu}$
ν
$\underline{\nu}$
ν_0
$\overline{\nu}$

Proposition 2. *Assume an interior optimum in* (μ, ν). *The common worst prior* (μ_t^c, ν_t^c) *at optimum is a unique solution to the following equations,*

$$\mu_t^c = \frac{\alpha}{2}(\nu_t^c - \nu^0)^2; \tag{24}$$

$$\left\{1 + (1 + R_t)\, \gamma_P \kappa (\nu_t^c)^2\right\} \frac{\gamma_A \exp(\eta(1-t))}{\alpha R_t^2} + \frac{\nu^0}{\nu_t^c} - 1 = 0, \tag{25}$$

where $R_t := 1 + (\gamma_A + \gamma_P)\kappa(\nu_t^c)^2$. *Moreover, the value function* $\mathcal{V}_t$ *is given by*

$$\mathcal{V}(t, Y_t) = -e^{-\gamma_P(\zeta(t)Y_t + \rho(t))}, \quad \mathcal{P}\text{-}q.s.,$$

where

$$\zeta(t) = e^{\eta(1-t)} - 1 = Z_t^P, \tag{26}$$

$$\rho(t) = Y_0 - \mathcal{W}_0 + \int_t^1 [.....]\, ds. \tag{27}$$

The sharing sensitivities (β_t, θ_t) *of the optimal contract are:*

$$\beta_t = \frac{1 + \gamma_P \kappa(\nu_t^c)^2}{1 + (\gamma_A + \gamma_P)\kappa(\nu_t^c)^2} e^{\eta(1-t)}, \tag{28}$$

$$\theta_t = \frac{1}{2e^{\eta(1-t)}} \beta_t \left(e^{\eta(1-t)} - \beta_t\right) \left[\gamma_A \beta_t - \left(e^{\eta(1-t)} - \beta_t\right) \gamma_P\right]. \tag{29}$$

Remark 1: In 'the linear-quadratic case,'
$Z_t^P > (=, <)\, 0$**, iff** $\eta > (=, <)\, 0$**,**
iff the outcome exhibits increasing (constant, decreasing) returns to scale.

Remark 2: If $\eta = 0$, then the optimal contract is linear in Y_1 and $\langle Y_1 \rangle$: an ambiguity version of the Holmstrom-Milgrom stationary case.

Corollary 2. *(Comparative statics.)*

i. *Both the commonly perceived mean and volatility increase with the returns-to-scale parameter* η*, the degree of ambiguity* $1/\alpha$*, and the agent's ability* $1/\kappa$*. That is,* $\frac{\partial \mu_t^c}{\partial \eta}, \frac{\partial \nu_t^c}{\partial \eta} > 0$*, and* $\frac{\partial \mu_t^c}{\partial \alpha}, \frac{\partial \nu_t^c}{\partial \alpha}, \frac{\partial \mu_t^c}{\partial \kappa}, \frac{\partial \nu_t^c}{\partial \kappa} < 0$.

ii. *The outcome-sharing sensitivity increases with the agent's ability, but decreases with the degree of ambiguity: i.e.,* $\frac{\partial \beta_t}{\partial \kappa} < 0$*, and* $\frac{\partial \beta_t}{\partial \alpha} > 0$*. Moreover, if the principal is risk neutral, i.e.,* $\gamma_P = 0$*, then the sensitivity increases with the returns-to-scale parameter: i.e.,* $\frac{\partial \beta_t}{\partial \eta} > 0$.

Remark 1: $\frac{\partial \mu_t^c}{\partial \alpha}, \frac{\partial \nu_t^c}{\partial \alpha} \Rightarrow$ ambiguity increases both mean and volatility perceptions.

Remark 2: $\frac{\partial \beta_t}{\partial \alpha} > 0 \Rightarrow$ **ambiguity decreases the pay-for-performance sensitivity.**

6 Conclusion

- We have developed a martingale method for principal-agent problems under the joint ambiguity.
- We have distinguished between ex-post realized and ex-ante perceived volatilities.
- The second-best contract in general consists of two sharing rules: one for realized outcome and the other for realized volatility.
- We have shown that the compensation level is positively associated with the realized volatility. **Consistent with stock option granting practices in managerial compensation.**
- Their worst priors are equalized across the principal and agent.

Thank you!