Long-term Portfolio Optimization: Sense and Sensitivities

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Question

- How sensitive are optimal portfolios to changes in model parameters?
- We are focusing on the long term sensitivities for utility maximizing portfolios, i.e.,

$$\frac{\partial}{\partial \epsilon}\bigg|_{\epsilon=0} \sup_{\Pi \in \mathcal{X}^\epsilon} \mathbb{E}\big[U(\Pi_T)\big] \quad \xrightarrow{T\to\infty} \quad ?$$

- Here the set $\mathcal{X}^\epsilon$ of optimal portfolio depends on a perturbation parameter ϵ, perturbing some model parameter.

Literature

- **Sensitivity Analysis of Optimal Portfolios**: Kramkov and Sîrbu (AAP '06), Larsen and Žitković (SPA '07), Larsen, Mostovyi and Žitković (F&S '18), Mostovyi and Sîrbu (F&S '19), Mostovyi (Preprint); Backhoff and Silva (ESAIM '17), Backhoff and Silva (MFE '18); Monin and Zariphopoulou (JFE '14)
- **Long Term Portfolio Optimization / Risk-sensitive Control**: Fleming and McEneaney (SICON '95), Nagai (SICON '96), Fleming and Sheu (AAP '02), Kaise and Sheu (AMO '09), Knispel (AAP '12), Guasoni and Robertson (AAP '12), Robertson and Xing (SICON '15)
- **Hansen–Scheinkman Decomposition**: Hansen and Scheinkman (Econometrica '09), Qin and Linetsky (OR '16), Park (F&S '18)

Question

- Park (F&S '18) shows how the Hansen–Scheinkman decomposition can be used for an analysis of the sensitivity of **long term cash flows**.
- Can we apply this argument to optimal portfolios / stochastic control problems?
- This is relatively straightforward for complete markets (via the dual representation), but much more tricky for incomplete markets.
- While the setting is more restrictive than in other sensitivity analyses and is concerned only with the long-term limit, it allows to find closed-form solutions for sensitivities for CRRA preferences with negative power.

Results

- Consider the Kim–Omberg model for asset price S and the stochastic excess returns X:

$$\begin{aligned} dS_t &= X_t S_t \, dt + S_t \, dW_t, && S_0 = 1, \\ dX_t &= k(\overline{m} - X_t) \, dt + \sigma \, dZ_t, && X_0 = \chi \end{aligned}$$

- W and Z are correlated Brownian motions with correlation parameter $\rho \in (-1, 1)$.
- Assume an investor has constant relative risk aversion $1 - p > 1$ and denote the dual exponent by $q = -\frac{p}{1-p} > 0$.

Results

For

$$\alpha_1 = k + q\rho\sigma, \quad \alpha_2 = \frac{1-\rho^2 q}{1-q}\sigma^2, \quad \alpha_3 = k\overline{m}, \quad \alpha_4 = \sqrt{\alpha_1^2 + q(1-q)\alpha_2}$$

$$\alpha_5 = -\alpha_1\rho + q\rho^2\sigma - \sigma, \quad B = \frac{\alpha_4 - \alpha_1}{\alpha_2}, \qquad C = \frac{\alpha_3(\alpha_4 - \alpha_1)}{\alpha_2\alpha_4}$$

We have for $\sup_{\Pi\in\mathcal{X}} \mathbb{E}\big[U(\Pi_T)\big] = \frac{1}{p} v^{1-p}(\chi, T)$

$$\lim_{T\to\infty} \frac{\partial}{\partial\chi} \ln v(\chi, T) = -B\chi - C,$$

$$\lim_{T\to\infty} \frac{1}{T}\frac{\partial}{\partial k} \ln v(\chi, T) = \alpha_2\left(\frac{\overline{m}}{\alpha_3} - \frac{\alpha_4 + \alpha_1}{\alpha_4^2}\right)C^2 - \left(2\overline{m} - + \frac{\alpha_3(\alpha_4 + \alpha_1)}{\alpha_4^2}\right)C - \frac{\sigma^2 B}{2\alpha_4}$$

$$\lim_{T\to\infty} \frac{1}{T}\frac{\partial}{\partial\overline{m}} \ln v(\chi, T) = \frac{\alpha_2}{\alpha_3}kC^2 + 2kC$$

$$\lim_{T\to\infty} \frac{1}{T}\frac{\partial}{\partial\sigma} \ln v(\chi, T) = \left(\frac{kq\rho + q\sigma - \alpha_4 q\rho}{\alpha_4(\alpha_4 - \alpha_1)} - \frac{1}{\sigma} - \frac{kq\rho + q\sigma}{\alpha_4^2}\right)(\alpha_2 C - \alpha_3)C$$
$$- \frac{1}{2}\sigma^2\left(\frac{kq\rho + q\sigma - \alpha_4 q\rho}{\alpha_4(\alpha_4 - \alpha_1)}\right)B$$

Model

- We consider a factor model for the stock dynamics

$$\begin{aligned} dS_t &= b(X_t)S_t\,dt + \varsigma(X_t)S_t\,dW_{1,t}\,, & S_0 &= 1, \\ dX_t &= m(X_t)\,dt + \sigma_1(X_t)\,dW_{1,t} + \sigma_2(X_t)\,dW_{2,t}\,, & X_0 &= \chi, \end{aligned}$$

  for independent Brownian motions W_1, W_2.
- We assume that the investor has CRRA preferences with risk aversion $1-p>1$:

$$U(x) = \frac{1}{p}x^p.$$

Method - Step (1)

- From the dual formulation of utility maximization problem we know that

$$\sup_{\Pi \in \mathcal{X}} \mathbb{E}[U(\Pi_T)] = \frac{1}{p}\Big(\mathbb{E}^{\mathbb{P}}\big[\hat{Y}_T^q\big]\Big)^{1-p}$$

for some nonnegative supermartingale $\hat{Y}$ and $q = -\frac{p}{(1-p)}$. Define

$$v(\chi, T) = \mathbb{E}^{\mathbb{P}}\big[\hat{Y}_T^q \,\big|\, X_0 = \chi\big].$$

it suffices to evaluate the long-term behavior of

$$\frac{\partial}{\partial \chi} \ln v(\chi, T).$$

Method - Step (2)

- The function v satisfies a HJB equation

$$v_t = \frac{1}{2}\big(\sigma_1^2(x) + \sigma_2^2(x)\big) v_{xx} + \sup_{\xi \in \mathbb{R}} \big\{ l(\xi, x) v + h(\xi, x) v_x \big\},$$

$$v(x, 0) = 1$$

where

$$l(\xi, x) := -\frac{q(1-q)}{2}\big(\theta^2(x) + \xi^2\big)$$

$$h(\xi, x) := m(x) - q\theta(x)\sigma_1(x) - q\xi\sigma_2(x).$$

Method - Step (3)

- The function v can be approximated by a solution pair (λ, ϕ) of an ergodic HJB equation

$$\lambda\phi(x) = \frac{1}{2}\big(\sigma_1^2(x) + \sigma_2^2(x)\big)\phi_{xx} + \sup_{\xi\in\mathbb{R}}\big\{l(\xi, x)\phi + h(\xi, x)\phi_x\big\}$$

in the sense that $e^{-\lambda T}\phi(\chi)$ is asymptotically equal to $v(\chi, T)$ up to a constant factor, that is,

$$v(\chi, T) \simeq e^{-\lambda T}\phi(\chi)$$

We call (λ, ϕ) the eigenpair of the problem

Method - Step (4)

- By taking the partial derivative to the above asymptotics, one can anticipate that

$$\frac{\partial}{\partial \chi} \ln v(\chi, T) \simeq \frac{\phi'(\chi)}{\phi(\chi)}$$

- Similarly, for parameter perturbations

$$\frac{1}{T} \frac{\partial}{\partial \epsilon}\Big|_{\epsilon=0} \ln v_\epsilon(\chi, T) \simeq -\frac{\partial}{\partial \epsilon}\Big|_{\epsilon=0} \lambda_\epsilon.$$

Method - Step (5)

- Making the above method rigorous we need a precise error analysis.
- This relies on the following result, which can be seen as generalization of the Hansen-Scheinkman factorization for some time-inhomogeneous Markov processes.
- Under some measure $\mathbb{Q}$ we have

$$v(\chi, T) = e^{-\lambda T}\phi(\chi)\,\mathbb{E}^{\mathbb{Q}}\Big[\frac{1}{\phi(X_T)}\,e^{\int_0^T f(X_s,t;T)\,ds}\Big]$$

Method - Step (5)

A way to understand the above theorem is to consider the commutative diagram

$$
\begin{array}{ccc}
(\Omega, \mathcal{F}, \hat{\mathbb{P}}) & \rightsquigarrow & (\Omega, \mathcal{F}, \mathbb{Q}) \\
\Big\downarrow \frac{d\tilde{\mathbb{P}}}{d\hat{\mathbb{P}}} & & \frac{d\mathbb{Q}}{d\overline{\mathbb{P}}} \Big\uparrow \\
(\Omega, \mathcal{F}, \tilde{\mathbb{P}}) & \xrightarrow{M_T = \frac{d\overline{\mathbb{P}}}{d\tilde{\mathbb{P}}}} & (\Omega, \mathcal{F}, \overline{\mathbb{P}})
\end{array}
$$

On the top level we are in a non-ergodic regime (control $\hat{\xi}(x, t; T)$), in the lower level in the ergodic regime (control $\xi^*(x)$).

Time-inhomogeneous Hansen–Scheinkman Decomposition

Theorem

The optimal expected utility has the representation

$$v(\chi, T) = e^{-\lambda T}\phi(\chi)\,\mathbb{E}^{\mathbb{Q}}\left[\frac{1}{\phi(X_T)}\,e^{-\frac{q}{2}(1-q)\int_0^T |\xi^*(X_s)-\hat{\xi}(X_s,t;T)|^2\,ds}\right]$$

with

$$\begin{aligned} dX_t = \Big(& m(X_t) - q\theta(X_t)\sigma_1(X_t) - q\hat{\xi}(X_t, t; T)\sigma_2(X_t) \\ &+ \frac{\phi'(X_t)}{\phi(X_t)}\left(\sigma_1^2(X_t) + \sigma_2^2(X_t)\right)\Big)\,dt \\ &+ \sigma_1(X_t)\,dW_{1,t}^{\mathbb{Q}} + \sigma_2(X_t)\,dW_{2,t}^{\mathbb{Q}} \end{aligned}$$

Time-inhomogeneous Hansen–Scheinkman Decomposition

- Here $\hat{\xi}(\,\cdot\,,\,\cdot\,;T)$ is the optimal control for the dual HJB equation for time horizon T and ξ^* is the optimal control for the ergodic HJB equation
- Thus, if the controls converge fast enough and X is ergodic under $\mathbb{Q}$, the expectation term converges to a constant
- The expectation term can be seen as multiplicative error term

Sensitivity wrt. the initial factor

Theorem

Under suitable (integrability and smoothness) assumptions and if additionally

$$\chi \mapsto \mathbb{E}^{\mathbb{Q}}\Big[\frac{1}{\phi(X_T)}\, e^{\int_0^T f(X_s, s; T)\, ds} \,\Big|\, X_0 = \chi\Big]$$

is continuously differentiable with derivative converging to zero as $T \to \infty$*. Then*

$$\lim_{T\to\infty} \frac{\partial}{\partial\chi} \ln v(\chi, T) = \frac{\phi'(\chi)}{\phi(\chi)}.$$

Sensitivity wrt. the initial factor

- This follows directly from the chain rule

$$\frac{\partial}{\partial \chi} \ln v(\chi, T) = \frac{\phi'(\chi)}{\phi(\chi)} + \frac{\partial}{\partial \chi} \ln \mathbb{E}^{\mathbb{Q}}\Big[\frac{1}{\phi(X_T)}\, e^{\int_0^T f(X_s, s; T)\, ds}\Big].$$

- To get the needed continuous differentiability, one can use (uniform) integrability conditions on X and its first variation process Y.

Sensitivity wrt. factor process parameters

- To discuss the factor process parameter sensitivities, we consider

$$dX_t^\epsilon = m_\epsilon(X_t^\epsilon)\,dt + \sigma_{1,\epsilon}(X_t^\epsilon)\,dW_{1,t} + \sigma_{2,\epsilon}(X_t^\epsilon)\,dW_{2,t}, \qquad X_0^\epsilon = \chi$$

  for some ϵ perturbations

Sensitivity wrt. factor process parameters

Theorem

Under suitable (integrability and smoothness) assumptions and if additionally

$$\epsilon \mapsto \mathbb{E}^{\mathbb{Q}_\epsilon}\left[\frac{1}{\phi(X_T^\epsilon)}\, e^{\int_0^T f(X_s^\epsilon, s; T)\, ds}\right]$$

is continuously differentiable at $\epsilon = 0$ with

$$\frac{1}{T}\frac{\partial}{\partial \epsilon}\Big|_{\epsilon=0} \ln \mathbb{E}^{\mathbb{Q}_\epsilon}\left[\frac{1}{\phi(X_T^\epsilon)}\, e^{\int_0^T f(X_s^\epsilon, s; T)\, ds}\right]$$

converging to zero as $T \to \infty$. Then

$$\lim_{T\to\infty} \frac{1}{T}\frac{\partial}{\partial \epsilon}\Big|_{\epsilon=0} \ln v_\epsilon(\chi, T) = -\frac{\partial \lambda_\epsilon}{\partial \epsilon}\Big|_{\epsilon=0}.$$

Sensitivity wrt. factor process parameters

- This follows again directly from the chain rule

$$\frac{1}{T}\frac{\partial}{\partial \epsilon}\Big|_{\epsilon=0} \ln v_\epsilon(\chi, T) = -\frac{\partial \lambda_\epsilon}{\partial \epsilon}\Big|_{\epsilon=0} + \frac{1}{T}\frac{\partial}{\partial \epsilon}\Big|_{\epsilon=0} \ln \phi_\epsilon(\chi) + \frac{1}{T}\frac{\partial}{\partial \epsilon}\Big|_{\epsilon=0} \ln \mathbb{E}^{\mathbb{Q}_\epsilon}\left[\frac{1}{\phi_\epsilon(X_T^\epsilon)}\, e^{\int_0^T f_\epsilon(X_s^\epsilon, s; T)\, ds}\right]$$

- To get the needed continuous differentiability, one can use (uniform) integrability conditions on X^ε and one splits the functional and process perturbations by considering

$$w_{\eta,\epsilon}(\chi, T) := \mathbb{E}^{\mathbb{Q}_\epsilon}\left[\frac{1}{\phi_\eta(X_T^\epsilon)}\, e^{\int_0^T f_\eta(X_s^\epsilon, s; T)\, ds}\right]$$

- For considering the sensitivities wrt. the factor utilities, one can use a Lameprti transform.