Sensitivity analysis of the expected utility maximization problem with respect to model perturbations

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based on joint work with

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Outline

The starting point

- consider the perturbation analysis in Larsen, Mostovyi and Žitković
- do a similar analysis with a general rather than power utility

The mathematics

- present a method to approximate
  1. value functions to second order
  2. optimizers to the first order
- stochastic control problems which are convex, but not convex with respect to a parameter
- abstract version (over random variables)
- back to the original model, write approximation of strategies as Kunita-Watanabe decomposition under risk tolerance wealth process as numeraire

Utility Maximization Problem

Agent

initial wealth

utility

Market

stock | no arbitrage

bank account | frictionless

Initial wealth x — Controlling investment H $\longrightarrow$ Wealth at time t: $x + \int_0^t H_u dS_u$

Value function: $$u(x) \triangleq \max_{X \in \mathcal{X}(x)} \mathbb{E}\left[U(X_T)\right].$$

Utility Maximization Problem

Utility function U:

- $(0,\infty) \to \mathbb{R}$: strictly increasing, strictly concave, C^1,
- Satisfies the Inada conditions:

$$\lim_{x\to 0} U'(x) = \infty \quad \textit{and} \quad \lim_{x\to\infty} U'(x) = 0.$$

Standard results in the literature:

- existence and uniqueness of solutions,
- properties of the value function,
- properties of the solutions.

Merton, Cox, Huang, Karatzas, Lehoczky, Shreve, Xu, Kramkov, Schachermayer...
Under certain (quite weak) conditions, the optimal $\widehat{H}$ and $\widehat{X}$

exist and are unique.

Stability and asymptotics

- stability: continuous dependence on parameters (goes back to Hadamard)
- asymptotics: higher order dependence (needs differentiability structure on parameters)

Stability and asymptotics: literature

Existing results (small fraction):

- dependence on x: Kramkov and Schachermayer (1999, 2003)
- dependence on U (and/or $\mathbb{P}$): Jouini and Napp (2004), Carasus and Rasonyi (2005), Larsen (2006), Kardaras and Žitković (2011),
- dependence on the parametrization of the stock price: Prigent (2003), Larsen and Žitković (2007), Larsen, Mostovyi, and Žitković (2014).
- on random endowment: Henderson (2002), Kramkov and S. (2006, 2007), Kallsen, Muhle-Karbe, and Vierthauer (2014).

Our model: the family of markets

(from Larsen, Mostovyi, Žitković)

A family of markets is parametrized by δ. Every market consist of a stock and a bond.
(Return of) the stock prince process evolves as

$$dS_t^{\delta} \triangleq (\lambda_s + \delta\nu_s)\, d\langle M\rangle_s + dM_t$$

(see Hulley and Schweizer (2010), Delbaen and Schachermayer (1995));

The price process of the bond equals to 1 at all times.

Goal: study dependence on δ.

Primal problem

Define

$$\mathcal{X}(x,\delta) \triangleq \left\{X : \; X_t = x + \int_0^t H_u dS_u^{\delta}, t \in [0,T] \text{ and } X \geq 0\right\}, \quad x > 0.$$

A utility function $U : (0,\infty) \to \mathbb{R}$ is strictly increasing, strictly concave, two times continuously differentiable on $(0,\infty)$ and there exist positive constants c_1 and c_2, such that

$$c_1 \leq A(x) \triangleq -\frac{U''(x)x}{U'(x)} \leq c_2,$$

and define the value function as:

$$u(x,\delta) \triangleq \sup_{X \in \mathcal{X}(x,\delta)} \mathbb{E}\left[U(X_T)\right], \quad (x,\delta) \in (0,\infty) \times \mathbb{R}.$$

Mathematical goal

How to establish an expansion with respect to δ of

- the value function $u(x, \delta)$ (second order),
- the corresponding trading strategy (first order)?

Remark

Dual problem can be helpful.

Dual problem

$$V(y) \triangleq \sup_{x>0} \left(U(x) - xy \right), \quad y > 0,$$
$$-\frac{V''(y)y}{V'(y)} = \frac{1}{A(x)}, \quad \textit{if} \quad y = U'(x).$$

Let $\mathcal{Y}(y, \delta)$ be a set of nonnegative supermartingales such that:

1. $Y_0 = y$,
2. $(X_t Y_t)_{t \in [0,T]}$ is a supermartingale for every $X \in \mathcal{X}(1, \delta)$.

The dual value function is

$$v(y, \delta) \triangleq \inf_{Y \in \mathcal{Y}(y,\delta)} \mathbb{E}\left[V(Y_T)\right], \quad (y, \delta) \in (0, \infty) \times \mathbb{R}.$$

Structural lemma

Lemma

For every $\delta \in \mathbb{R}$, *we have*

$$\begin{aligned} \mathcal{Y}(1,\delta) &= \mathcal{Y}(1,0)\mathcal{E}\left(-\delta\nu \cdot S^0\right), \\ \mathcal{X}(1,\delta) &= \mathcal{X}(1,0)\frac{1}{\mathcal{E}\left(-\delta\nu \cdot S^0\right)}. \end{aligned}$$

Remark

Looks like a multiplicative (and non-linear) random endowment.

Abstract theorems

In the spirit of Kramkov-Schachermayer (99), consider the sets $\mathcal{C}$ and $\mathcal{D}$ polar in $\mathbf{L}^0_+$:

Assumption

Both $\mathcal{C}$ and $\mathcal{D}$ contain a stricly positive element and

$$\xi \in \mathcal{C} \quad \textit{iff} \quad \mathbb{E}[\xi\eta] \leq 1 \quad \textit{for every } \eta \in \mathcal{D},$$

as well as

$$\eta \in \mathcal{D} \quad \textit{iff} \quad \mathbb{E}[\xi\eta] \leq 1 \quad \textit{for every } \xi \in \mathcal{C}.$$

Primal and dual problems for 0-model

We set

$$\mathcal{C}(x,0) \triangleq x\mathcal{C} \quad \textit{and} \quad \mathcal{D}(x,0) \triangleq x\mathcal{D}, \quad x > 0.$$

Now we can state the abstract primal and dual problems as

$$u(x,0) \triangleq \sup_{\xi \in \mathcal{C}(x,0)} \mathbb{E}\left[U(\xi)\right], \quad x > 0,$$

$$v(y,0) \triangleq \inf_{\eta \in \mathcal{D}(y,0)} \mathbb{E}\left[V(\eta)\right], \quad y > 0.$$

Abstract version for δ-models

For some random variables F and $G \geq 0$, we set

$$L^{\delta} \triangleq \exp\left(-(\delta F + \tfrac{1}{2}\delta^2 G)\right),$$

$$\mathcal{C}(x,\delta) \triangleq \mathcal{C}(x,0)\frac{1}{L^{\delta}} \quad \textit{and} \quad \mathcal{D}(y,\delta) \triangleq \mathcal{D}(y,0)L^{\delta}, \quad \delta \in \mathbb{R}.$$

The abstract versions of the perturbed optimization problems:

$$u(x,\delta) \triangleq \sup_{\xi \in \mathcal{C}(x,\delta)} \mathbb{E}\left[U(\xi)\right] = \sup_{\xi \in \mathcal{C}(x,0)} \mathbb{E}\left[U\left(\xi \frac{1}{L^{\delta}}\right)\right], \quad x > 0, \delta \in \mathbb{R},$$

$$v(y,\delta) \triangleq \inf_{\eta \in \mathcal{D}(y,\delta)} \mathbb{E}\left[V(\eta)\right] = \inf_{\eta \in \mathcal{D}(y,0)} \mathbb{E}\left[V\left(\eta L^{\delta}\right)\right], \quad y > 0, \delta \in \mathbb{R}.$$

The approach

Follows Henderson (2002).

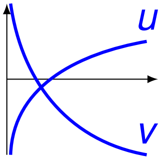
- find lower bound up to second order for u
- upper bound up to second order for v
- "match" them

Matching one-sided bounds can be found using quadratic optimization problems:

- (Kramkov, S. (2006))

Lack of convexity in δ

The value functions $-u$ and v are convex in x, y.



But for the parametrized family of markets, we do not have convexity in δ.

No-convexity in δ, cont'd

Can use convexity only in direction of x, y. For

$$y = u_x(x, \delta), \quad u(x, \delta) - xy = v(y, \delta)$$

Even if we fix x and vary δ alone, y depends on δ: need to approximate at least v in both directions (y, δ).
Summary: better provide joint expansion for both

- (x, δ) for u
- (y, δ) for v

The 0-model:

If u is finite at some point

(i) $u(x,0) < \infty$, for every $x > 0$, and $v(y,0) > -\infty$, for every $y > 0$. The functions u and v are Legendre conjugate

$$
\begin{aligned}
v(y,0) &= \sup_{x>0}\left(u(x,0) - xy\right), \quad y > 0, \\
u(x,0) &= \inf_{y>0}\left(v(y,0) + xy\right), \quad x > 0.
\end{aligned}
$$

(ii) The functions u and $-v$ are continuously differentiable on $(0,\infty)$, strictly concave, strictly increasing and satisfy the Inada conditions

$$
\begin{aligned}
\lim_{x\downarrow 0} u_x(x,0) &= \infty, & \lim_{y\downarrow 0}(-v_x(y,0)) &= \infty, \\
\lim_{x\uparrow \infty} u_x(x,0) &= 0, & \lim_{y\uparrow \infty}(-v_y(y,0)) &= 0.
\end{aligned}
$$

(iii) For every $x > 0$ and $y > 0$, the solutions $\widehat{X}(x,0)$ and $\widehat{Y}(y,0)$ exist and are unique and, if $y = u'(x)$, we have

$$
\widehat{Y}_T(y) = U'\left(\widehat{X}_T(x)\right), \quad \mathbb{P}\text{-a.s.}
$$

Assumption on perturbations

- First, we set:

$$\frac{d\mathbb{R}(x,0)}{d\mathbb{P}} \triangleq \frac{\widehat{X}_T(x,0)\widehat{Y}_T(y,0)}{xy}.$$

- Let $x > 0$ be fixed. There exists $c > 0$, such that

$$\mathbb{E}^{\mathbb{R}(x,0)}\left[\exp\left(c(|\nu \cdot S^0_T| + \langle \nu \cdot S^0 \rangle_T)\right)\right] < \infty.$$

Assumption under $\mathbb{P}$ and original numéraire

- Let us assume that $c_1 > 1$, i.e. that relative-risk aversion of U is strictly greater than 1 (relative risk aversion uniformly exceeds 1)
- A sufficient condition for the previous slide Assumption to hold is the existence of some positive exponential moments under $\mathbb{P}$

First-order analysis

Theorem (Envelope)

Let $x > 0$ be fixed and assumptions above hold. Then we have

- *There exists $\delta_0 > 0$ such that for every $\delta \in (-\delta_0, \delta_0)$, we have*

$$u(z, \delta) \in \mathbb{R} \quad \textit{and} \quad v(z, \delta) \in \mathbb{R}, \quad z > 0.$$

- *The first-order derivatives are*

$$u_\delta(x, 0) = v_\delta(y, 0) = xy\mathbb{E}^{\mathbb{R}(x,0)}\left[\nu \cdot S_T^0\right], \quad y = u_x(x, 0).$$

- *The value functions u and v are continuous at $(x, 0)$ and $(y, 0)$, respectively.*

Remark

$u_\delta(x, 0)$ and $v_\delta(y, 0)$ are linear in ν.

Second-order analysis

- Let $S^{X(x,0)}$ be the price process of the traded securities under the numéraire $\frac{\widehat{X}(x,0)}{x}$, i.e.

$$S^{X(x,0)} = \left(\frac{x}{\widehat{X}(x,0)}, \frac{xS^0}{\widehat{X}(x,0)} \right).$$

- For every $x > 0$, let $\mathbf{H}_0^2(\mathbb{R}(x,0))$ denote the space of square integrable martingales under $\mathbb{R}(x,0)$, such that

$$\begin{aligned} \mathcal{M}^2(x,0) &\triangleq \left\{ M \in \mathbf{H}_0^2(\mathbb{R}(x,0)) : M = H \cdot S^{X(x,0)} \right\}, \\ \mathcal{N}^2(y,0) &\triangleq \left\{ N \in \mathbf{H}_0^2(\mathbb{R}(x,0)) : MN \textit{ is } \mathbb{R}(x,0)\text{-martingale} \right. \\ &\qquad \left. \text{for every } M \in \mathcal{M}^2(x,0) \right\}, \quad y = u_x(x,0). \end{aligned}$$

Auxiliary minimization problems (for u_{xx} and v_{yy})

Let us set

$$a(x,x) \triangleq \inf_{M \in \mathcal{M}^2(x,0)} \mathbb{E}^{\mathbb{R}(x,0)} \left[A(\widehat{X}_T(x,0))(1+M_T)^2 \right],$$

$$b(y,y) \triangleq \inf_{N \in \mathcal{N}^2(y,0)} \mathbb{E}^{\mathbb{R}(x,0)} \left[B(\widehat{Y}_T(y,0))(1+N_T)^2 \right],$$

where $y = u_x(x,0)$,

$$A(x) = -\frac{U''(x)x}{U'(x)} \quad \textit{and} \quad B(y) = -\frac{V''(y)y}{V'(y)} = \frac{1}{A(x)}.$$

Second-order derivatives with respect to x and y

Proved in Kramkov and S. (2006):

- auxiliary minimization problems admit unique solutions $M^0(x,0)$ and $N^0(y,0)$;
- the value functions are two-times differentiable and

$$\begin{aligned} u_{xx}(x,0) &= -\tfrac{y}{x} a(x,x), \\ v_{yy}(y,0) &= \tfrac{x}{y} b(y,y); \end{aligned}$$

- u_{xx} and v_{yy} are linked via

$$\begin{aligned} u_{xx}(x,0)v_{yy}(y,0) &= -1, \\ a(x,x)b(y,y) &= 1; \end{aligned}$$

- the optimizers to auxiliary problems satisfy

$$A(\widehat{X}_T(x,0))(1+M^0_T(x,0)) = a(x,x)(1+N^0_T(y,0)).$$

Auxiliary minimization problem (for $u_{\delta\delta}$ and $v_{\delta\delta}$)

With

$$F \triangleq \nu \cdot S^0_T \quad and \quad G \triangleq \nu^2 \cdot \langle M \rangle_T,$$

we consider the following minimization problems.

$$a(d,d) \triangleq \inf_{M \in \mathcal{M}^2(x,0)} \mathbb{E}^{\mathbb{R}(x,0)} \Big[A(\widehat{X}_T(x,0))(M_T + xF)^2 - 2xFM_T - x^2(F^2 + G) \Big],$$

$$b(d,d) \triangleq \inf_{N \in \mathcal{N}^2(y,0)} \mathbb{E}^{\mathbb{R}(x,0)} \Big[B(\widehat{Y}_T(y,0))(N_T - yF)^2 + 2yFN_T - y^2(F^2 - G) \Big].$$

Structure of $u_{x\delta}$ and $v_{y\delta}$

Denoting by $M^1(x,0)$ and $N^1(y,0)$ the unique solutions the auxiliary problems above, we set

$$a(x,d) \triangleq \mathbb{E}^{\mathbb{R}(x,0)}\Big[A(\widehat{X}_T(x,0))(1+M^0_T(x,0))(xF+M^1_T(x,0)) \\ -xF(1+M^0_T(x,0))\Big],$$

$$b(y,d) \triangleq \mathbb{E}^{\mathbb{R}(x,0)}\Big[B(\widehat{Y}_T(y,0))(1+N^0_T(y,0))(N^1_T(y,0)-yF) \\ +yF(1+N^0_T(y,0))\Big].$$

Theorem (Mostovyi., S.)

Let $x > 0$ be fixed. Let the assumptions above hold and $y = u_x(x, 0)$. Define

$$H_u(x,0) \triangleq -\frac{y}{x}\begin{pmatrix} a(x,x) & a(x,d) \\ a(x,d) & a(d,d) \end{pmatrix},$$

$$H_v(y,0) \triangleq \frac{x}{y}\begin{pmatrix} b(y,y) & b(y,d) \\ b(y,d) & b(d,d) \end{pmatrix}.$$

Then, the value functions u and v admit the second-order expansions around $(x, 0)$ and $(y, 0)$, respectively,

$$\begin{aligned} u(x+\Delta x,\delta) &= u(x,0) + (\Delta x \quad \delta)\nabla u(x,0) \\ &\quad + \tfrac{1}{2}(\Delta x \quad \delta) H_u(x,0)\begin{pmatrix}\Delta x \\ \delta\end{pmatrix} + o(\Delta x^2 + \delta^2), \\ v(y+\Delta y,\delta) &= v(y,0) + (\Delta y \quad \delta)\nabla v(y,0) \\ &\quad + \tfrac{1}{2}(\Delta y \quad \delta) H_v(y,0)\begin{pmatrix}\Delta y \\ \delta\end{pmatrix} + o(\Delta y^2 + \delta^2). \end{aligned}$$

Theorem (Mostovyi, S.)

(*i*) *The values of quadratic optimizations*

$$\begin{pmatrix} a(x,x) & 0 \\ a(x,d) & -\frac{x}{y} \end{pmatrix} \begin{pmatrix} b(y,y) & 0 \\ b(y,d) & -\frac{y}{x} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$\frac{y}{x} a(d,d) + \frac{x}{y} b(d,d) = a(x,d) b(y,d).$$

(*ii*) *The optimizers of the quadratic problems are related*

$$U''(\widehat{X}_T(x,0))\widehat{X}_T^0(x,0) \begin{pmatrix} M_T^0(x,0)+1 \\ M_T^1(x,0)+xF \end{pmatrix} = -\begin{pmatrix} a(x,x) & 0 \\ a(x,d) & -\frac{x}{y} \end{pmatrix} \widehat{Y}_T^0(y,0) \begin{pmatrix} N_T^0(y,0)+1 \\ N_T^1(y,0)-yF \end{pmatrix},$$

$$V''(\widehat{Y}_T(y,0))\widehat{Y}_T(y,0) \begin{pmatrix} 1+N_T^0(y,0) \\ -yF+N_T^1(y,0) \end{pmatrix} = \begin{pmatrix} b(y,y) & 0 \\ b(y,d) & -\frac{y}{x} \end{pmatrix} \widehat{X}_T(x,0) \begin{pmatrix} 1+M_T^0(x,0) \\ xF+M_T^1(x,0) \end{pmatrix}.$$

(*iii*) *The product of any of* $\widehat{X}(x,0)$, $\widehat{X}(x,0)M^0(x,0)$, $\widehat{X}(x,0)M^1(x,0)$ *and any of* $\widehat{Y}(y,0)$, $\widehat{Y}(y,0)N^0(y,0)$, $\widehat{Y}(y,0)N^1(y,0)$ *is a* $\mathbb{P}$*-martingale.*

Derivatives of the optimizers

Theorem (Mostovyi, S.)

Let us set

$$X'_T(x,0) \triangleq \frac{\widehat{X}_T(x,0)}{x}(1+M^0_T(x,0)), \quad Y'_T(x,0) \triangleq \frac{\widehat{Y}_T(y,0)}{y}(1+N^0_T(y,0)),$$

and

$$X^d_T(x,0) \triangleq \frac{\widehat{X}_T(x,0)}{x}(M^1_T(x,0)+xF), \quad Y^d_T(y,0) \triangleq \frac{\widehat{Y}_T(y,0)}{y}(N^1_T(y,0)-yF).$$

Then, we have

$$\lim_{|\Delta x|+|\delta|\to 0} \tfrac{1}{|\Delta x|+|\delta|} \left|\widehat{X}_T(x+\Delta x,\delta) - \widehat{X}_T(x,0) - \Delta x X'_T(x,0) - \delta X^d_T(x,0)\right| = 0,$$

$$\lim_{|\Delta y|+|\delta|\to 0} \tfrac{1}{|\Delta y|+|\delta|} \left|\widehat{Y}_T(y+\Delta y,\delta) - \widehat{Y}_T(y,0) - \Delta y Y'_T(y,0) - \delta Y^d_T(y,0)\right| = 0,$$

where the convergence takes place in $\mathbb{P}$-probability.

Approximation of the optimal trading strategies

Observation: because the "random endowment" is multiplicative, proportions work better.
With

$$M^R = S^0 - \widehat{\pi}(x,0)\cdot\langle M\rangle,$$

let

$$\gamma^0 \cdot M^R = \frac{M^0(x,0)}{x} \quad and \quad \gamma^1 \cdot M^R = \frac{M^1(x,0)}{x},$$

and

$$\begin{aligned}\sigma_\varepsilon &\triangleq \inf\left\{t\in[0,T]:\ |M^0_t(x,0)|\geq \tfrac{x}{\varepsilon}\ \ or\ \ \langle M^0(x,0)\rangle_t \geq \tfrac{x}{\varepsilon}\right\},\\ \tau_\varepsilon &\triangleq \inf\left\{t\in[0,T]:\ |M^1_t(x,0)|\geq \tfrac{x}{\varepsilon}\ \ or\ \ \langle M^1(x,0)\rangle_t \geq \tfrac{x}{\varepsilon}\right\},\\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \varepsilon>0,\end{aligned}$$

as well as

$$\gamma^{0,\varepsilon} = \gamma^0 1_{\{[0,\sigma_\varepsilon]\}} \quad and \quad \gamma^{1,\varepsilon} = \gamma^1 1_{\{[0,\tau_\varepsilon]\}}, \quad \varepsilon>0.$$

Approximation of the optimal trading strategies

Let us set

$$\begin{aligned} dX_t^{\Delta x,\delta,\varepsilon} &= X_t^{\Delta x,\delta,\varepsilon}(\widehat{\pi}_t(x,0) + \Delta x \gamma_t^{0,\varepsilon} + \delta(\nu_t + \gamma_t^{1,\varepsilon}))dS_t^\delta, \\ X_0^{\Delta x,\delta,\varepsilon} &= x + \Delta x. \end{aligned}$$

Note that

$$X^{\Delta x,\delta,\varepsilon} = (x+\Delta x)\frac{\widehat{X}(x,0)}{x}\frac{\mathcal{E}\left((\Delta x \gamma^{0,\varepsilon} + \delta\gamma^{1,\varepsilon})\cdot M^R\right)}{\mathcal{E}(-\delta\nu\cdot S^0)}.$$

Theorem (Mostovyi, S.)

There exists a function $\varepsilon = \varepsilon(\Delta x, \delta)$, *such that*

$$\mathbb{E}\left[U\left(X_T^{\Delta x,\delta,\varepsilon(\Delta x,\delta)}\right)\right] = u(x+\Delta x,\delta) - o(\Delta x^2 + \delta^2).$$

Risk-tolerance wealth process

Definition

For $x > 0$ and $\delta \in \mathbb{R}$, the risk-tolerance wealth process is a maximal wealth process $R(x, \delta)$, such that

$$R_T(x, \delta) = -\frac{U'(\widehat{X}_T(x, \delta))}{U''(\widehat{X}_T(x, \delta))}.$$

Remark

This process was introduced in Kramkov and S. (2006) in the context of asymptotic analysis of utility-based prices.

Theorem (Kramkov and S. (2006))

The following assertions are equivalent:

(1) *The risk-tolerance wealth process* $R(x,0)$ *exists.*

(2) *The value function* u *admits the expansion quadratic expansion at* $(x,0)$ *and* $u_{xx}(x,0) = -\frac{y}{x}a(x,x)$ *satisfies*

$$\frac{(u_x(x,0))^2}{u_{xx}(x,0)} = \mathbb{E}\left[\frac{\left(U'(\widehat{X}_T(x,0)\right)^2}{U''(\widehat{X}_T(x,0))}\right],$$

$$u_{xx}(x,0) = \mathbb{E}\left[U''(\widehat{X}_T(x,0)\left(\frac{R_T(x,0)}{R_0(x,0)}\right)^2\right].$$

(3) *The value function* v *admits the quadratic expansion at* $(y,0)$ *and* $v_{yy}(y,0) = \frac{x}{y}b(y,y)$ *satisfies*

$$y^2 v_{yy}(y,0) = \mathbb{E}\left[\left(\widehat{Y}_T(y,0)\right)^2 V''(\widehat{Y}_T(y,0))\right] = xy\mathbb{E}^{\mathbb{R}(x,0)}\left[B(\widehat{Y}_T(y,0))\right].$$

Theorem (..Continued)

In addition, if these assertions are valid, then the initial value of $R(x)$ *is given by*

$$R_0(x,0) = -\frac{u_x(x,0)}{u_{xx}(x,0)} = \frac{x}{a(x,x)},$$

the product $R(x,0)Y(y,0) = (R_t(x,0)Y_t(y,0))_{t\in[0,T]}$ *is a uniformly integrable martingale and*

$$\lim_{\Delta x\to 0} \frac{\widehat{X}_T(x+\Delta x,0) - \widehat{X}_T(x,0)}{\Delta x} = \frac{R_T(x,0)}{R_0(x,0)},$$

$$\lim_{\Delta y\to 0} \frac{\widehat{Y}_T(y+\Delta y,0) - \widehat{Y}_T(y,0)}{\Delta y} = \frac{\widehat{Y}_T(y,0)}{y},$$

where the limits take place in $\mathbb{P}$*-probability.*

For $x > 0$ and with $y = u_x(x, 0)$, let us define

$$\frac{d\widetilde{\mathbb{R}}(x,0)}{d\mathbb{P}} \triangleq \frac{R_T(x,0)\widehat{Y}_T(y,0)}{R_0(x,0)y},$$

and choose $\frac{R(x,0)}{R_0(x,0)}$ as a numéraire, i.e., let us set

$$S^{R(x,0)} \triangleq \left(\frac{R_0(x,0)}{R(x,0)}, \frac{R_0(x,0)S}{R(x,0)}\right).$$

We define the spaces of martingales

$$\widetilde{\mathcal{M}}^2(x,0) \triangleq \left\{M \in \mathbf{H}_0^2(\widetilde{\mathbb{R}}(x,0)) : \; M = H \cdot S^{R(x,0)}\right\},$$

and $\widetilde{\mathcal{N}}^2(y,0)$ it the orthogonal complement in $\mathbf{H}_0^2(\widetilde{\mathbb{R}}(x,0))$.

Risk-tolerance wealth process and a Kunita-Watanabe decomposition

Theorem (Mostovyi, S.)

Let us assume that the risk-tolerance process $R(x,0)$ *exists. Consider the Kunita-Watanabe decomposition of the square integrable martingale*

$$P_t \triangleq \mathbb{E}^{\widetilde{\mathbb{R}}(x,0)}\left[\left(A(\widehat{X}_T(x,0))-1\right)xF|\mathcal{F}_t\right], \quad t\in[0,T].$$

given by

$$P = P_0 - \widetilde{M}^1 - \widetilde{N}^1, \quad \textit{where} \quad \widetilde{M}^1 \in \widetilde{\mathcal{M}}^2(x,0), \quad \widetilde{N}^1 \in \widetilde{\mathcal{N}}^2(y,0), \quad P_0 \in \mathbb{R}.$$

Theorem (..Continued)

Then, the optimal solutions $M^1(x,0)$ and $N^1(y,0)$ of the auxiliary quadratic optimization problems for $u_{\delta\delta}$ and $v_{\delta\delta}$ can be obtained from the Kunita-Watanabe decomposition (above) by reverting to the original numéraire, through the identities:

$$\widetilde{M}_t^1 = \frac{\widehat{X}_t(x,0)}{R_t(x,0)} M_t^1(x,0), \quad \widetilde{N}_t^1 = \frac{x}{y} N_t^1(y,0), \quad t \in [0,T].$$

In addition, the Hessian terms in the quadratic expansion of u and v can be identified as

$$\begin{aligned} a(d,d) &= \frac{R_0(x,0)}{x} \inf_{\widetilde{M} \in \widetilde{\mathcal{M}}^2(x,0)} \mathbb{E}^{\widetilde{\mathbb{R}}(x,0)} \left[\left(\widetilde{M}_T + xF \left(A \left(\widehat{X}_T(x,0) \right) - 1 \right) \right)^2 \right] + C_a. \\ &= \frac{R_0(x,0)}{x} \mathbb{E}^{\widetilde{\mathbb{R}}(x,0)} \left[\left(\widetilde{N}_T^1 \right)^2 \right] + \frac{R_0(x,0)}{x} P_0^2 + C_a, \end{aligned}$$

where $C_a \triangleq x^2 \mathbb{E}^{\mathbb{R}(x,0)} \left[F^2 \frac{A(\widehat{X}_T(x,0)) - 1}{A(\widehat{X}_T(x,0))} - G \right].$

Theorem (..Continued)

$$
\begin{aligned}
b(d,d) &= \frac{R_0(x,0)}{x} \inf_{\widetilde{N}\in\mathcal{N}^2(y,0)} \mathbb{E}^{\widetilde{\mathbb{R}}(y,0)}\left[\left(\widetilde{N}_T + yF\left(A\left(\widehat{X}_T(x,0)\right)-1\right)\right)^2\right] + C_b. \\
&= \frac{R_0(x,0)}{x}\left(\frac{y}{x}\right)^2 \mathbb{E}^{\widetilde{\mathbb{R}}(y,0)}\left[\left(\widetilde{M}^1_T\right)^2\right] + \frac{R_0(x,0)}{x}\left(\frac{y}{x}\right)^2 P_0^2 + C_b,
\end{aligned}
$$

where $C_b \triangleq y^2\mathbb{E}^{\mathbb{R}(x,0)}\left[G + F^2\left(1 - A\left(\widehat{X}_T(x,0)\right)\right)\right]$. *The cross terms in the Hessians of* u *and* v *are identified as*

$$a(x,d) = P_0$$

and $b(y,d)$ *is given by*

$$b(y,d) = \frac{y}{x}\frac{P_0}{a(x,x)}.$$

With these identifications, all the expansions of the value functions above hold.

Summary

- look at the simultaneous perturbations of the market price of risk and the initial wealth
- formulate quadratic optimization problems and relate the second-order approximations of both primal and dual value functions to these problems.
- in case when the risk-tolerance wealth process exists, we used it as a numéraire, and changed the measure accordingly, to identify solutions to the quadratic optimization problems above in terms of a Kunita-Watanabe decomposition.