

Market Models of Competing Brownian Particles with Splits and Mergers

Andrey Sarantsev (joint with Ioannis Karatzas)

University of California, Santa Barbara

September 28, 2015

Informal Introduction and Main Results

Two Competing Brownian Particles

- $X_1(t), X_2(t)$
- The particle which is currently maximal moves at this moment as a Brownian motion with drift g_1 and variance σ_1^2
- The particle which is currently minimal moves at this moment as a Brownian motion with drift g_2 and variance σ_2^2

System of Competing Brownian Particles (CBP)

Fix $N \geq 2$, the number of particles $X_1(t), \dots, X_N(t)$. At each moment t , rank them: $X_{(1)}(t) \geq \dots \geq X_{(N)}(t)$

- The particle $X_{(1)}$ which is currently at the top moves at this moment as a Brownian motion with drift g_1 and variance σ_1^2
- The particle $X_{(2)}$ which is currently the second top one moves at this moment as a Brownian motion with drift g_2 and variance σ_2^2
- etc.
- The particle $X_{(N)}$ which is currently the bottom one moves at this moment as a Brownian motion with drift g_N and variance σ_N^2

Capitalizations of N stocks

$$Y_1(t) = e^{X_1(t)}, \dots, Y_N(t) = e^{X_N(t)}.$$

A feature of real-world markets is that stocks with larger capitalizations have smaller drift and smaller volatility. This model can capture this: just let

$$g_1 \leq g_2 \leq \dots \leq g_N, \quad \sigma_1^2 \leq \sigma_2^2 \leq \dots \leq \sigma_N^2.$$

Studied by BANNER, FERNHOLZ, ICHIBA, KARATZAS, PAL, SHKOLNIKOV, JOURDAIN and others

Market weight of the i th company

$$\mu_i(t) = \frac{Y_i(t)}{Y_1(t) + \dots + Y_N(t)}$$

A market model is **diverse** if $\mu_i(t) \leq 1 - \delta$ for all i and t , where $\delta \in (0, 1)$ is fixed

Such models allow an arbitrage: a portfolio which beats the market: FERNHOLZ, KARATZAS (2002, 2005, 2009)

Fact: The CBP-based market model is not diverse.

Try to amend the CBP model to make it diverse. Allow for a changing number of stocks: STRONG (2011)

Splits: When a company's market weight reaches $1 - \delta$, forcefully split it into two companies of random size.

Mergers: Set an exponential clock with rate λ_N , where N is the current number of companies. This is a positive random variable with Lebesgue density

$$\lambda_N e^{-\lambda_N s} ds$$

and mean λ_N^{-1} . When it rings, choose randomly two companies and merge them.

Mechanism of Splits and Mergers

Immediately after the quantity of companies became equal to N , set the exponential clock with rate λ_N .

Two events:

- Some company has market weight reached $1 - \delta$ (a split)
- Exponential clock rings (a merger)

If the first event happens sooner than the second one, then we split this company into two random parts.

If the second event happens sooner than the first one, then we randomly choose two companies and merge them. If the resulting company has market weight greater than or equal to $1 - \delta$, this merger is suppressed. Otherwise, it is allowed to proceed.

N stocks at time t : $(\pi_0(t), \dots, \pi_N(t))$

- $\pi_0(t) + \dots + \pi_N(t) = 1$
- $|\pi_i(t)| \leq K$ for K independent of i, N, t
- $\pi_0(t)$ is the share of the current capital invested in money
- $\pi_i(t)$ is the share invested in the i th stock

Portfolio Behavior at Jumps

How does a portfolio behave when companies split or merge?

When two companies **merge**, then the portfolio weight corresponding to the new company's stock is the sum of the portfolio weights corresponding to the two old stocks.

When a company gets **split** into two new ones, then its weight in the portfolio is partitioned in proportion to the weights of the newly created companies.

An investor starts with initial capital 1 and invests in the stock market according to a portfolio π .

The **wealth process** V^π is given by

$$\frac{dV^\pi(t)}{V^\pi(t)} = \sum_{i=1}^N \pi_i(t) \frac{dX_i(t)}{X_i(t)},$$

while there are N stocks.

The wealth process does not change at the moment of jumps.

A portfolio π represents an **arbitrage opportunity** relative to another portfolio ρ if

$$V^\pi(T) \geq V^\rho(T) \text{ a.s.}$$

$$V^\pi(T) > V^\rho(T) \text{ with positive probability}$$

Under certain conditions on parameters,

- This model is **non-explosive**: the quantity of stocks does not go to infinity in finite time
- An equivalent martingale measure exists, so there is **no arbitrage**: we cannot beat the market

For market models with constant number of stocks, diversity leads to arbitrage. For varying number of stocks, it is no longer true. On this, see also FOUQUE, STRONG (2011)

Formal Description

While the quantity of stocks is constant

If the quantity of stocks is N , then the capitalizations of stocks are

$$Y_1(t) = e^{X_1(t)}, \dots, Y_N(t) = e^{X_N(t)},$$

where X_1, \dots, X_N behave as competing Brownian particles with drifts g_{N1}, \dots, g_{NN} and variances $\sigma_{N1}^2, \dots, \sigma_{NN}^2$.

The k th top particle among X_1, \dots, X_N behaves as a Brownian motion with drift g_{Nk} and variance σ_{Nk}^2 .

What happens at the split

The model has a certain fixed distribution F on $[1/2, 1 - \varepsilon]$.

If $\mu_i(t)$, the market weight of the i th company, reaches $1 - \delta$, we generate a random variable $\xi \sim F$, independent of the past, and

split $X_i(t)$ into $\xi X_i(t)$ and $(1 - \xi)X_i(t)$.

What happens at the merger

For every N , we have a certain family $\mathcal{P}_N(x), x \in (0, \infty)^N$ of probability distributions on the set of subsets of $\{1, \dots, N\}$ which contain exactly two elements.

If the exponential clock rings at time t , we pick a subset $\{i, j\}$, $i \neq j$ according to $\mathcal{P}_N(X(t))$, and try to merge the companies X_i and X_j .

If $\mu_i(t) + \mu_j(t) \geq 1 - \delta$, the the merger is suppressed. Otherwise, it is allowed to proceed.

In this article, we choose two companies to be merged as follows.

Exclude the company with the **largest capitalization**. From the remaining $N - 1$ companies, choose a subset of two companies uniformly. Each subset is chosen with probability $\binom{N-1}{2}^{-1}$.

How should the rate of mergers λ_N depend on N , the quantity of companies?

Since there are $\binom{N-1}{2} \sim N^2/2$ ways to choose two companies to merge them, the **natural assumption** is $\lambda_N \sim cN^2$ as $N \rightarrow \infty$.

Conditions for Main Results

Top stock has minimal drift (this is consistent with the feature that stocks with larger capitalizations have smaller drifts):

$$g_{N1} \leq \min_{2 \leq k \leq N} g_{Nk}, \quad N = 2, 3, \dots$$

Also,

$$\sup_{N,k} g_{Nk} < \infty, \quad 0 < C_1 \leq \sigma_{Nk} \leq C_2 < \infty.$$

Additional conditions:

$$0 < \delta < \frac{1}{6}.$$

For some parameters $c, \alpha > 0$,

$$\lambda_N \sim cN^\alpha \quad \text{as } N \rightarrow \infty.$$

Techniques of Proof

Proof of Non-Explosion: Fast Mergers, Slow Splits

Let N_k be the quantity of stocks after the k th jump. Consider the process $(N_k)_{k \geq 0}$. Suppose that its current value is N .

When there are N stocks and N is large, they merge quickly. The rate λ_N is going to infinity as $N \rightarrow \infty$. But they do not split with high rate.

What will happen earlier, a split ($N \rightarrow N + 1$) or a merger ($N \rightarrow N - 1$)? With high probability, the merger will be earlier.

Proof of Non-Explosion: Why Slow Splits?

Why don't splits become faster as the quantity of stocks grows?

Let $\mu_{(k)}$ be the k th largest market weight. A split occurs when the top market weight $\mu_{(1)}$ hits $1 - \delta$. We have the condition

$$g_{N1} \leq \min_{2 \leq k \leq N} g_{Nk}, \quad N \geq 2.$$

Write the SDE for the logarithm of the top market weight $\mu_{(1)}$. It will have a non-positive drift and reflection at $\log \mu_{(2)} \leq \log(1/2)$. One can show that it is dominated by a reflected Brownian motion (RBM) on the half-line $[\log(1/2), \infty)$. So $\log \mu_{(1)}$ hits $\log(1 - \delta)$ no earlier than the RBM hits $\log(1 - \delta)$. And this RBM is independent of N .

Proof of No Arbitrage

We construct an equivalent martingale measure using the GIBSANOV theorem.

We need some bound on the quantity of stocks $N(t)$ for $t \in [0, T]$. The key step is to prove that the maximal number of stocks

$$\max_{0 \leq t \leq T} N(t)$$

has sub-exponential tail. That is, for every $C > 0$ we have:

$$\mathbf{E} \exp\left(C \max_{0 \leq t \leq T} N(t)\right) < \infty.$$

This allows us to use the NOVIKOV condition to show that the new measure is indeed a probability measure.

Thanks!