Casino gambling problem under probability weighting

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Casino gambling

- Casino gambling is popular, but a typical casino bet has at most zero expected value
- The popularity of casino gambling cannot be explained by models in the expected utility framework with concave utility functions

A Model by Barberis (2012)

- Barberis (2012) was the first to employ the cumulative prospect theory (CPT) of Tversky and Kahneman (1992) to model and study casino gambling
- A gambler comes to a casino at time 0 and is offered a bet with an equal chance to win or lose $1
- If the gambler accepts, the bet is then played out and she either gains $1 or loses $1 at time 1
- Then, the gambler is offered the same bet, and she can choose to leave the casino or to continue gambling
- It is a five-period model: at time 5, the gambler must leave the casino

A Model by Barberis (2012) (Cont'd)

- The gambler decides the optimal exit time to maximize the CPT value of her cumulative gain and loss at the exit time
- Barberis (2012) consider path-independent strategies only
- Barberis (2012) finds the optimal exit time by enumeration
- In this model, time-inconsistency arises due to probability weighting in CPT
- Barberis (2012) compares the strategies of three types of gamblers:
  1. naive gamblers, who do not realize the inconsistency in the future and thus keeps changing their strategy
  2. sophisticated gamblers with pre-commitment, who realize the inconsistency and commit their future selves to the strategy planned today
  3. sophisticated gamblers without pre-commitment, who realize the inconsistency but fail to commit their future selves

A Model by Barberis (2012) (Cont'd)

- With reasonable parameter values, Barberis (2012) finds
  - sophisticated gamblers with pre-commitment tend to take loss-exit strategies
  - naive gamblers, by contrast, end up with gain-exit strategies
  - sophisticated agents without pre-commitment choose not to play at all
- CPT is a *descriptive* model for individuals' preferences
- A crucial contribution in Barberis (2012): show that the optimal strategy of a gambler with CPT preferences is consistent with several commonly observed gambling behaviors such as the popularity of casino gambling and the implementation of gain-exit and loss-exit strategies

Motivation

- Can CPT also explain other commonly observed gambling patterns?
  - Gamblers become more risk seeking in the presence of a prior gain, a phenomenon referred to as *house money effect* (Thaler and Johnson, 1990)
  - Individuals may use a random device, such as a coin flip, to aid their choices in various contexts (Dwenger, Kübler, and Weizsacker, 2013)
- Can we solve the casino gambling problem with CPT preferences analytically?

Contributions

- We consider the casino gambling problem without additional restrictions on time-horizon or set of available strategies
- We find that the gambler may strictly prefer path-dependent strategies over path-independent strategies, and may further strictly improve her preference value by tossing random coins
- We study theoretically the issue of why the gambler prefers randomized strategies
- We show that any path-dependent strategy is equivalent to a randomization of path-independent strategies
- We develop a systematic approach to solving the casino gambling problem

CPT

- Suppose an individual experiences a random gain/loss X. Then, in CPT, the preference value of X is

$$V(X) := \int_0^{\infty} u(x) d[-w_+(1 - F_X(x))] + \int_{-\infty}^{0} u(x) d[w_-(F_X(x))],$$

  - F_X is the CDF of X
  - u is an S-shaped utility function
  - $w_\pm$ are two inverse-S-shaped probability weighting functions

Casino Gambling Model

- At time 0, a gambler is offered a fair bet: win or lose one dollar with equal probability
- If she declines this bet, she does not enter the casino to gamble
- Otherwise, she starts the game and the outcome of the bet is played out at time 1, at which time she either wins or loses one dollar
- The gambler is then offered the same bet and she can again choose to play or not, and so forth
- At time 0, the gambler decides whether to enter the casino and, if yes, the optimal time to leave the casino
- The decision criterion is to maximize the CPT value of her cumulative gain and loss at the time when she leaves the casino

Binomial Tree Representation

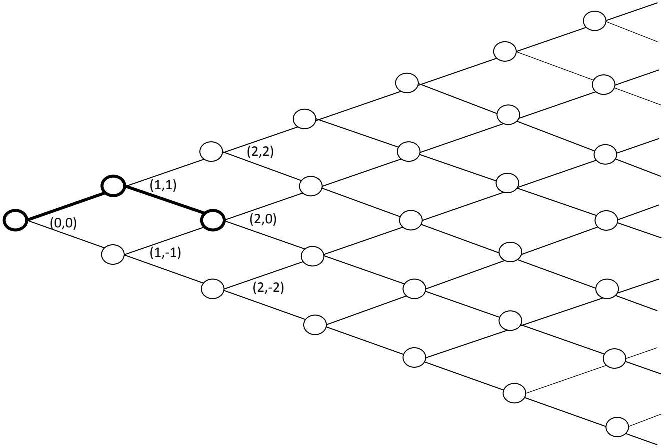

Figure: Gain/loss process represented as a binomial tree

Types of Feasible Stopping Strategies

- Denote $S_t, t \geq 0$ as the cumulative gain/loss of the gambler
- *Path-independent (Markovian) strategies*: for any $t \geq 0$, $\{\tau = t\}$ (conditioning on $\{\tau \geq t\}$) is determined by (t, S_t)
- *Path-dependent strategies*: for any $t \geq 0$, $\{\tau = t\}$ is determined by $(u, S_u), u \leq t$
- *Randomized, path-independent strategies*:
    - At each time the gambler tosses a coin and decides whether to leave the casino (continue with tails and stop with heads)
    - The coins are tossed independently
    - The probability that the coin tossed at t turns up tails is determined by (t, S_t), and is part of the gambler's strategy

Example

- Consider the following utility and probability weighting functions

$$u(x) = \begin{cases} x^{\alpha_+} & \text{for } x \geq 0 \\ -\lambda(-x)^{\alpha_-} & \text{for } x < 0, \end{cases} \qquad w_\pm(p) = \frac{p^{\delta_\pm}}{(p^{\delta_\pm} + (1-p)^{\delta_\pm})^{1/\delta_\pm}}.$$

  with $\alpha_+ = \alpha_- = 0.9$, $\delta_+ = \delta_- = 0.4$, and $\lambda = 2.25$
- Consider a 6-period horizon, and compare the CPT values for different strategies

Example (Cont'd)

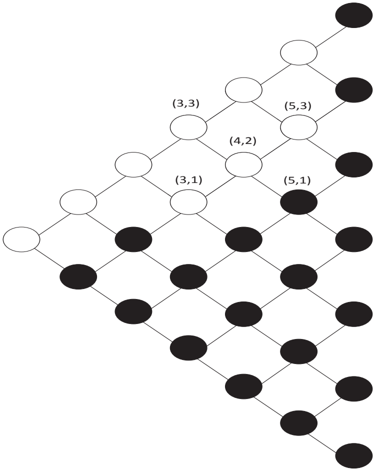

Figure: Optimal path-independent strategy. The CPT value is $V = 0.250440$

Example (Cont'd)

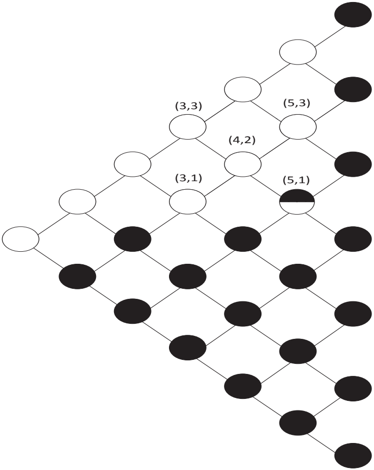

Figure: Optimal path-dependent strategy. The CPT value is $V = 0.250693$

Example (Cont'd)

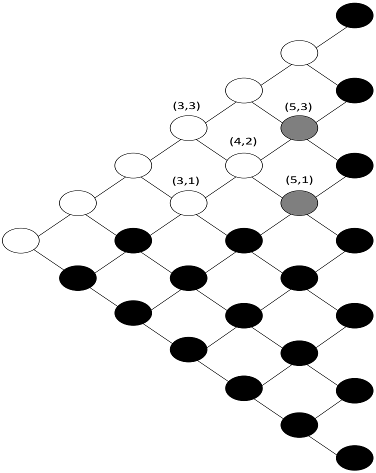

Figure: Randomized, path-independent strategy. The CPT value is $V = 0.250702$

Obsevations

- It is possible to strictly increase the gambler's CPT value by allowing for path-dependent strategies
- The CPT value can be further improved by switching to randomized, path-independent strategies
- Why?

A General Optimal Stopping Problem

- Consider a general discrete-time Markov chain $\{X_t\}_{t\geq 0}$ taking integer values
- Consider the optimal stopping problem

$$\sup_{\tau} V(X_\tau)$$

- Assume $V(\cdot)$ is law-invariant, i.e., $V(X) = \mathcal{V}(F_X)$, such as CPT and EUT

Randomized Strategies

Lemma

For any randomized, path-independent strategy τ, there exist Markovian strategies τ_i's such that the distribution of (X_τ, τ) is a convex combination of the distributions of (X_{τ_i}, τ_i)'s, i.e.,

$$F_{X_\tau,\tau} = \sum_i \alpha_i F_{X_{\tau_i},\tau_i}$$

for some $\alpha_i \geq 0$ such that $\sum_i \alpha_i = 1$.

Proposition

For any path-dependent strategy τ, there exists a randomized, path-independent strategy $\tilde{\tau}$ such that (X_τ, τ) has the same distribution as $(X_{\tilde{\tau}}, \tilde{\tau})$. More generally, for any randomized, path-dependent strategy τ', there exists a randomized, path-independent strategy $\tilde{\tau}$ such that $(X_{\tau'}, \tau')$ has the same distribution as $(X_{\tilde{\tau}}, \tilde{\tau})$.

Quasi-Convex Preferences

- The preference measure $V(X) = \mathcal{V}(F_{X_\tau})$ is *quasi-convex* if

$$\mathcal{V}(pF_1 + (1-p)F_2) \leq \max\{\mathcal{V}(F_1), \mathcal{V}(F_2)\}, \quad \forall F_1, F_2,\ \forall p \in [0,1].$$

- An agent with quasi-convex preferences does not prefer path-dependent or randomized strategies over Markovian strategies
- The gambler in the casino model strictly prefers path-dependent strategies over Markovian strategies, and strictly prefer randomized, path-independent strategies over path-dependent strategies because CPT is *not* quasi-convex
- EUT is quasi-convex

Casino Gambling in Infinite Horizon

- Consider the casino gambling problem on an infinite time horizon
- Consider randomized, path-independent strategies τ
- Consider uniformly integrable stopping times, i.e., $S_{\tau\wedge t}, t \geq 0$ is uniformly integrable
- Denote the set of feasible stopping times as $\mathcal{T}$
- Optimal stopping problem:

$$\sup_{\tau \in \mathcal{T}} V(S_\tau)$$

Change of Variable

- This problem is difficult to solve: Snell envelop and dynamic programming cannot apply due to probability weighting
- Idea: change the decision variable from τ to the distribution of S_τ
- The quantile formulation in Xu and Zhou (2012) cannot apply
- The key is the characterization of the set of feasible distributions
- Denote

$$\mathcal{M}_0(\mathbb{Z}) = \left\{ \mu : \sum_{n\in\mathbb{Z}} |n| \cdot \mu(\{n\}) < \infty, \sum_{n\in\mathbb{Z}} n \cdot \mu(\{n\}) = 0 \right\}.$$

Skorokhod Embedding

Theorem

For any $\mu \in \mathcal{M}_0(\mathbb{Z})$*, there exists* $\{r_i\}_{i\in\mathbb{Z}}$ *such that* S_τ *follows* μ *where*

$$\tau := \inf\{t \geq 0 | \xi_{t,S_t} = 0\}$$

and $\xi_{t,i}$, $t \in \mathbb{Z}^+$, $i \in \mathbb{Z}$ *are 0-1 random variables independent of each other and* S_t *with* $\mathbb{P}(\xi_{t,i} = 0) = r_i$, $i \in \mathbb{Z}$*. Furthermore,* $\{S_{\tau\wedge t}\}_{t\geq 0}$ *is uniformly integrable and does not visit states outside any interval that contains the support of* μ*. Conversely, for any randomized, path-independent strategy* τ *such that* $\{S_{\tau\wedge t}\}_{t\geq 0}$ *is uniformly integrable, the distribution of* S_τ *belongs to* $\mathcal{M}_0(\mathbb{Z})$*.*

Infinite-Dimensional Program

- The optimal stopping problem is equivalent to

$$
\begin{aligned}
\max_{\mathbf{x},\mathbf{y}} \quad & U(\mathbf{x},\mathbf{y}) \\
\text{subject to} \quad & 1 \geq x_1 \geq x_2 \geq \dots \geq x_n \geq \dots \geq 0, \\
& 1 \geq y_1 \geq y_2 \geq \dots \geq y_n \geq \dots \geq 0, \\
& x_1 + y_1 \leq 1, \\
& \textstyle\sum_{n=1}^{\infty} x_n = \sum_{n=1}^{\infty} y_n.
\end{aligned}
$$

where

- $\mathbf{x}$ and $\mathbf{y}$ stand for the cumulative gain distribution and decumulative loss distribution, respectively
- $$
\begin{aligned}
U(\mathbf{x},\mathbf{y}) := & \sum_{n=1}^{\infty} \left(u_+(n) - u_+(n-1)\right) w_+(x_n) \\
& - \sum_{n=1}^{\infty} \left(u_-(n) - u_-(n-1)\right) w_-(y_n).
\end{aligned}
$$

Infinite-Dimensional Program (Cont'd)

- After we find the optimal solution $(\mathbf{x}^*, \mathbf{y}^*)$ to the infinite-dimensional problem, we can recover the optimal stopping time τ^* using Skorokhod embedding
- Difficulties in finding the optimal solution $(\mathbf{x}^*, \mathbf{y}^*)$:
  - The objective function is neither concave nor convex in the decision variable
  - There are infinitely many constraints, so the standard Lagrange dual method is not directly applicable even if the objective function were concave

Infinite-Dimensional Program (Cont'd)

- Procedures to find the optimal $(\mathbf{x}^*, \mathbf{y}^*)$:
  1. Introduce $\mathbf{z}_+ = (z_{+,1}, z_{+,2}, \dots)$ with $z_{+,n} := x_{n+1}/x_1, n \geq 1$
     Introduce $\mathbf{z}_- = (z_{-,1}, z_{-,2}, \dots)$ with $z_{-,n} := y_{n+1}/y_1, n \geq 1$
     Introduce $s_+ := (\sum_{n=2}^{\infty} x_n)/x_1$, and $s_- := (\sum_{n=2}^{\infty} y_n)/y_1$
  2. Find the optimal $\mathbf{z}_+$ and $\mathbf{z}_-$ with constraint $\sum_{n=1}^{\infty} z_{\pm,n} = s_\pm$
  3. Find the optimal x_1, y_1, and $s_\pm$
  4. After we find optimal $\mathbf{z}_\pm$, x_1, y_1, and $s_\pm$, we can compute optimal $\mathbf{x}$ and $\mathbf{y}$ as $\mathbf{x} = (x_1, x_1\mathbf{z}_+)$ and $\mathbf{y} = (y_1, y_1\mathbf{z}_-)$, respectively
- $s_\pm$ imply the conditional expected gain and loss of S_τ, and the (asymptotically) optimal value $s_\pm$ will dictate whether the gambler takes a loss-exit, gain-exit, or non-exit strategy

Stop Loss, Disposition Effect, and Non-Exit

- Under some conditions the optimal strategy (asymptotically) is loss-exit, gain-exit, or non-exit strategy

Parameter Values	Exist. Opt. Solution	Strategy
$\alpha_+ > \delta_+$	No	loss-exit
$\alpha_- < \delta_-$	No	gain-exit
$\alpha_+ > \alpha_-$	No	non-exit
$\alpha_+ = \alpha_-, \lambda < \sup_{0<p<1} \frac{w_+(p)/p^{\alpha_+}}{w_-(1-p)/(1-p)^{\alpha_+}}$	No	non-exit

Table: Existence of optimal solutions. Utility function is given as $u_+(x) = x^{\alpha_+}, u_-(x) = \lambda x^{\alpha_-}$. Probability weighting function is given as $w_\pm(p) = \frac{p^{\delta_\pm}}{\left(p^{\delta_\pm}+(1-p)^{\delta_\pm}\right)^{1/\delta_\pm}}$, or $w_\pm(p) = a_\pm p^{\delta_\pm}/(a_\pm p^{\delta_\pm} + (1-p)^{\delta_\pm})$, or $w_\pm(p) = p^{\delta_\pm}$

Stop Loss, Disposition Effect, and Non-Exit (Cont'd)

- When $\alpha_+ > \delta_+$, the optimal strategy is essentially "loss-exit":
  - To exit once reaching a fixed loss level and not to stop in gain
  - Such a strategy produces a highly skewed distribution which is favored due to probability weighting
- When $\alpha_- < \delta_-$, the optimal strategy is essentially "gain-exit":
  - To exit when losing a large amount of dollars or when winning a small amount of dollars
  - This type of strategy exhibits disposition effect: sell the winners too soon and hold the losers too long

Stop Loss, Disposition Effect, and Non-Exit (Cont'd)

- When either (1) $\alpha_+ > \alpha_-$, or (2)

$$\alpha_+ = \alpha_-, \quad \lambda < \sup_{0<p<1} \frac{w_+(p)/p^{\alpha_+}}{w_-(1-p)/(1-p)^{\alpha_+}},$$

  the optimal strategy is essentially "non-exit":
  - To exit when the gain reaches a sufficiently high level or the loss reaches another sufficiently high level with the same magnitude
  - The second condition mean probability weighting on large gains dominates a combination of weighting on large losses and loss aversion

Optimal Solution

		Gamb.	Exist.	Strategies
$\alpha_+ = \alpha_- =$ $\delta_+ = \delta_- = 1$	$\lambda \geq 1$	No	Yes	not gamble
	$\lambda < 1$	Yes	No	non-exit
$\alpha_+ > \delta_+$	any λ	Yes	No	loss-exit
$\alpha_- < \delta_-$	any λ	Yes	No	gain-exit
$\alpha_- < \alpha_+$	any λ	Yes	No	non-exit
$\alpha_+ = \delta_+ < 1$	any λ	Yes	No	certain asymptotical strategy
$\delta_- \leq \delta_+$, $\alpha_+ < \delta_+$, and $\alpha_- > \max(\alpha_+, \delta_-)$	$\lambda \geq M_1$	No	Yes	not gamble
	$\lambda < M_1$	Yes	Yes	$(s_+^*, s_-^*) \in$ argmax$f\ (\hat{y}(s_+, s_-), s_+, s_-)$ $y_1^* = \hat{y}(s_+^*, s_-^*)$
$\delta_- \leq \delta_+$, $\alpha_+ < \delta_+$, and $\alpha_- = \delta_- \geq \alpha_+$	$\lambda \geq M_1$	No	Yes	not gamble
	$\lambda < M_1$	Yes	No	gain-exit
$\delta_- \leq \delta_+$, $\alpha_+ < \delta_+$, and $\alpha_- = \alpha_+ > \delta_-$	$\lambda \geq M_1$	No	Yes	not gamble
	$M_2 < \lambda < M_1$	Yes	Yes	$(s_+^*, s_-^*) \in$ argmax$f\ (\hat{y}(s_+, s_-), s_+, s_-)$ $y_1^* = \hat{y}(s_+^*, s_-^*)$
	$\lambda = M_2 < M_1$	Yes		
	$\lambda < M_2$	Yes	No	non-exit
$\alpha_- > \delta_- >$ $\delta_+ > \alpha_+$	$\lambda \geq M_3$	Yes	Yes	$s_+^* = s_1, s_-^* = 0$, and $y_1^* = \bar{y}(s_+^*, s_-^*)$
	$\lambda < M_3$	Yes	Yes	$(s_+^*, s_-^*) \in$ argmax$f(\bar{y}(s_+, s_-), s_+, s_-)$ and $y_1^* = \bar{y}(s_+^*, s_-^*)$
$\alpha_- = \delta_- >$ $\delta_+ > \alpha_+$	$\lambda > M_4$	Yes	Yes	$s_+^* = s_1$, $s_-^* = m$ for some integer m and $y_1^* = \bar{y}(s_+^*, s_-^*)$
	$\lambda \leq M_4$	Yes	No	gain-exit

Recover Optimal Stopping Time

- After finding the optimal solution $(\mathbf{x}^*, \mathbf{y}^*)$ to the infinite-dimensional problem, we establish a new Skorokhod embedding result to recover the optimal stopping time τ^*
- Randomized, path-independent strategy: For example, toss coins
- Randomized, path-dependent strategy: For example, randomized Azéma-Yor (AY) stopping time
  - Exit when relative loss (difference between cumulative gains and its running maximal) is large enough to exceed the bound
  - Possible to toss coins to determine exit or not when relative loss is very close to the bound

Randomized AY Stopping Time

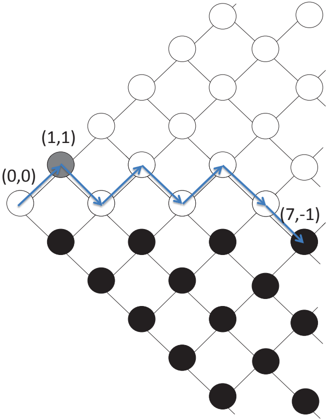

Figure: Two paths are drawn for illustrating randomized AY stopping rules. Black nodes mean stop. White nodes mean continue. Grey nodes mean randomization.

Numerical Examples

- Suppose $\alpha_+ = 0.6$, $\alpha_- = 0.8$, $\delta_\pm = 0.7$, $\lambda = 1.05$
  1. The probability distribution function of S_{τ^*} is

$$p_n^* = \begin{cases} 0.4465((n^{0.6} - (n-1)^{0.6})^{1/0.3} - ((n+1)^{0.6} - n^{0.6})^{1/0.3}), & n \geq 2, \\ 0.3297, & n = 1, \\ 0.6216, & n = -1, \\ 0, & \text{otherwise.} \end{cases}$$

     Note $p_n^* := \mathbb{P}(S_{\tau^*} = n)$.
  2. Recover the optimal stopping time τ^*

Numerical Examples (Cont'd)

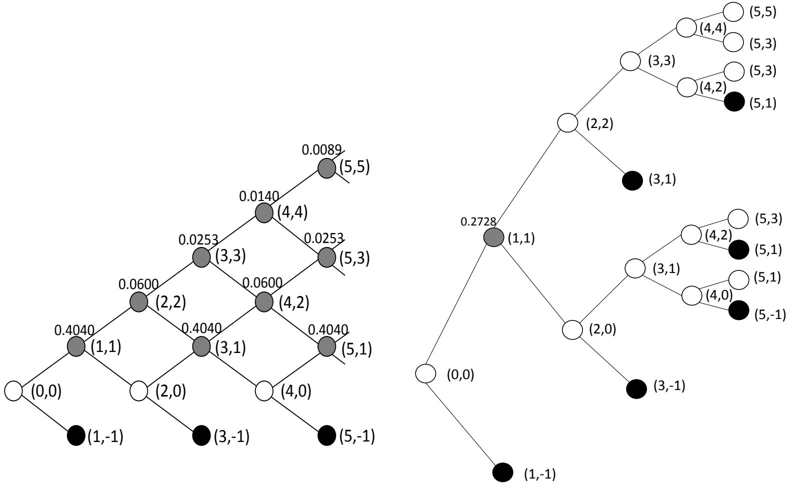

Figure: Randomized path-independent strategy (left-panel) and randomized AY strategy (right panel)

Naive Gamblers

- If a gambler revisits the gambling problem in the future, she may find the initial strategy is no longer optimal
  - If she cannot commit herself to following the initial strategy, she may change to a new strategy that can be totally different
  - Such type of gamblers are called *naive* gamblers

Naive Gamblers (Cont'd)

- Under some conditions, while a pre-committed gambler who follows the initial strategy after time 0 stops for sure before the loss hits a certain level, the naive gambler continues to play with a positive probability at any loss level
  - Either she simply continues, or else she might want to toss a coin to decide whether to continue to play or not
- Why?

Naive Gamblers (Cont'd)

- At time 0:
  - The probability of having losses strictly larger than certain level L is small from time 0's perspective
  - This small probability is exaggerated due to probability weighting
  - The gambler decides to stop when the loss reaches level L in the future
- At some time t when the naive gambler actually reaches loss level L:
  - The probability of having losses strictly larger than L is no longer small from time t's perspective
  - This large loss is not overweighted
  - The naive gambler chooses to take a chance and not to stop gambling

Naive Gamblers (Cont'd)

H	$n \le -7$	-6	-5	-4	-3	-2	-1	0	1	2	$n \ge 3$
-5	0	0.977	0	0	0	0	0	0	0	0.0096	$0.1477q_n$
-4	0	0	0.971	0	0	0	0	0	0.0063	$0.1453q_n$	$0.1453q_n$
-3	0	0	0	0.961	0	0	0	0	0.0161	$0.1424q_n$	$0.1424q_n$
-2	0	0	0	0	0.947	0	0	0	0.0308	$0.1393q_n$	$0.1393q_n$
-1	0	0	0	0	0	0.924	0	0	0.0543	$0.1364q_n$	$0.1364q_n$
0	0	0	0	0	0	0	0.885	0	0.0933	$0.1360q_n$	$0.1360q_n$
1	0	0	0	0	0	0	0	0.850	$0.1501q_n$	$0.1501q_n$	$0.1501q_n$
2	0	0	0	0	0	0	0	0.700	$0.3003q_n$	$0.3003q_n$	$0.3003q_n$
3	0	0	0	0	0	0	0	0.550	$0.4504q_n$	$0.4504q_n$	$0.4504q_n$
4	0	0	0	0	0	0	0	0.400	$0.6006q_n$	$0.6006q_n$	$0.6006q_n$
5	0	0	0	0	0	0	0	0.250	$0.7507q_n$	$0.7507q_n$	$0.7507q_n$

Table: Optimal distribution for $H = \pm 1, \pm 2, \ldots, \pm 5$. Suppose $\alpha_+ = 0.5$, $\alpha_- = 0.9$, $\delta_\pm = 0.52$, and $\lambda = 2.25$. The gambler does not change the reference point. Let $q_n = \left(n^{0.5} - (n-1)^{0.5}\right)^{1/0.48} - \left((n+1)^{0.5} - n^{0.5}\right)^{1/0.48}$, $n \ge 1$

Conclusion

- We considered a casino gambling problem with CPT preferences, and found that path-dependent stopping strategies and randomized stopping strategies strictly outperform path-independent strategies
- We showed that the improvement in performance brought by these strategies in the casino gambling problem is a consequence of lack of quasi-convexity of CPT preferences
- We developed a systematic approach to solving the casino gambling problem analytically
    - Change the decision variable
    - Solve an infinite-dimensional optimization problem
    - Establish a new Skorokhod embedding result
        1. Randomized, path-independent stopping time
        2. Randomized Azéma-Yor stopping time

Conclusion

- We have found the conditions under which the pre-committed gambler takes essentially loss-exit (stop-loss), gain-exit (disposition effect), and non-exit strategies
- We have also revealed that, under some conditions the initial optimal strategy and the actual strategy implemented by the naive gambler are totally different
    - While the pre-commitment strategy (initial one) is to stop if her cumulative loss reaches a certain level, the naive gambler continues to play with a positive probability at any loss level

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