Wiener-Hopf Factorization for Time-Inhomogeneous Markov Chains

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Motivation

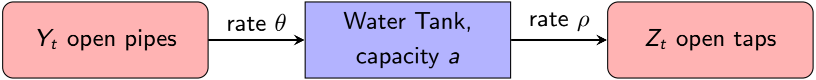
In many applications one needs to compute conditional expectations of the form

$$\mathbb{E}\left(e^{-c\tau} f(\widehat{X}_\tau) \,\middle|\, X_s\right), \quad s \geq 0, \quad c \in (0, \infty),$$

where X is a Markov process, defined on a stochastic basis $(\Omega, \mathscr{F}, \mathbb{F}, \mathbb{P})$, $\widehat{X}$ is a process related to X (usually $\widehat{X} = X$), and τ is an $\mathbb{F}$-stopping time.

- Monte Carlo method: slow and inaccurate, curse of dimensionality.
- Feynman-Kac representation (for $\mathbb{R}^d$-valued X): not easy in practice, especially if d is large or the associated integro-PDE is highly nonlinear.
- Wiener-Hopf Factorization: pure analytic method.

Example: A Simple Fluid Model (Rogers' 94)



- The water volume in the tank at time t, ξ_t, satisfies

$$\frac{d\xi_t}{dt} = \theta Y_t - \rho Z_t, \quad \text{if } 0 < \xi_t < a.$$

- We model $X_t = (Y_t, Z_t)$ by a finite-state time-homogeneous Markov chain.
- Let $v(y,z) := \theta y - \rho z \neq 0$. Then $\xi(t) = \xi_0 + \int_0^t v(X_s)ds$, $t \geq 0$. Let

$$\tau_\ell^+ := \inf\left\{u \geq 0 : \xi(u) > \ell\right\}, \quad \tau_\ell^- := \inf\left\{u \geq 0 : \xi(u) < \ell\right\}, \quad \ell \geq 0.$$

- τ_ℓ^+ ($\ell \in (\xi_0, a]$) is the first time the tank has ℓ amount of water.
- If $a = \infty$, τ_ℓ^- ($\ell \in [0, \xi_0)$) is the first time the tank has ℓ water amount left.
- The problem of interest is to find the joint distribution of $(\tau_\ell^\pm, X_{\tau_\ell^\pm})$, or

$$\mathbb{E}\left(e^{-c\tau_\ell^\pm}\mathbf{1}_{\{X_{\tau_\ell^\pm}=x'\}} \,\middle|\, X_0 = x\right), \quad c > 0.$$

Example: Two Barrier Ruin Problem (Avram, Pistorius & Usabel' 03)

- The reserve process U of an insurance company is modeled by

$$U_t = u - \sum_{k=1}^{N_t} Z_k + pt, \quad t \geq 0,$$

where $u > 0$ is the initial capital, $p > 0$ is the premium rate.

- The claims $(Z_k)_{k \in \mathbb{N}}$ are i.i.d. random variables with common phase-type (n, β, B) distribution.

- N is an independent renewal process with inter-arrival distribution of phase-type (m, α, A).

- Let $\tau_K := \inf\{t > 0 : \ U_t \notin [0, K)\}$, $K > 0$. The ruin problem is to find expressions for

$$\mathbb{E}\left(e^{-\delta \tau_K} \mathbf{1}_{\{U_{\tau_k} = K\}} \,\middle|\, U_0 = u\right) \quad \text{and} \quad \mathbb{E}\left(e^{-\delta \tau_K} \mathbf{1}_{\{U_{\tau_k} \leq 0\}} \,\middle|\, U_0 = x\right),$$

for any $\delta > 0$.

Example: CDS on Defaultable Stock in Markov-Modulated Models

- Let Z be a continuous-time Markov chain with finite state space $E \cup \partial$, where ∂ is an absorbing cemetery state. Let $\zeta := \inf\{t \geq 0 : Z_t = \partial\}$.
- Consider a defautable stock with pre-default dynamics $S_t = e^{X_t}$, where

$$X_t = X_0 + \int_0^t b(Z_s)\,ds + \int_0^t \sigma(Z_s)\,dW_s, \quad t \geq 0.$$

- W is a standard Brownian motion, and is independent of Z.
- The default occurs when X_t reaches the barrier $\ell < X_0$.
- Consider a credit default swap written on S. The expected payment when default occurs is then given by

$$\mathbb{E}\Big(h(Z_{\tau_\ell^-}) \mathbf{1}_{\{\tau_\ell^- < \zeta\}} \,\Big|\, Z_0 = i,\, X_0 = x \Big).$$

Literature on Applications

- Fluid Flow Models: Rogers '94, Rogers & Shi '94, Asmussen '95
- Ruin Problem: Avram, Pistorius, & Usabel '03, Avram & Usabel' 04
- Option Pricing: Guo & Zhang' 04, Jobert & Rogers '06, Jiang & Pistorius '08, Levendorskiĭ '08, Mijatović & Pistorius '11
- Optimal Control: Jiang & Pistorius '12
- Battery Charges, Network Loading, etc.

Theoretical Significance

- A significant aspect of the Wiener-Hopf factorization theory for Markov processes is the probabilistic interpretation of purely algebraic factorizations of linear operators that are related to these processes.
- The results of the factorization are interpreted in terms of functionals of appropriately time-changed processes that are related to the underlying Markov process.
- The Wiener-Hopf factorization is a vital component in the theory of Markov processes path decomposition or splitting time theorems.

Time-Inhomogeneous Wiener-Hopf Factorization

- The various existing forms of the Wiener-Hopf factorization for Markov chains, strong Markov processes, Lévy processes, and Markov additive process, have been obtained and applied only in the time-homogeneous case.
- However, there are abundant real life dynamical systems that are modeled in terms of time-inhomogeneous processes, and yet the corresponding Wiener-Hopf factorization theory is not available for those models.
- Our research is aimed at developing and applying the Wiener-Hopf factorization framework for a large class of time-inhomogeneous processes.

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Spaces of Matrices

- For any $n \in \mathbb{N}$, let $\mathcal{Q}(n)$ be the set of $n \times n$ generator matrices, i.e.,

$$\mathrm{Q}(i,j) \geq 0, \;\; i \neq j; \quad \sum_{j \in \mathbf{E}} \mathrm{Q}(i,j) \leq 0, \quad \text{for } \mathrm{Q} \in \mathcal{Q}(n).$$

- Let $\mathcal{Q}_0(n) \subset \mathcal{Q}(n)$ be the set of irreducible $n \times n$ generator matrices.
- $\mathrm{Q} \in \mathcal{Q}_0(n)$ is called recurrent if all its rows sum up to zero; otherwise, Q is called transient.
- For any $n, n' \in \mathbb{N}$, let $\mathcal{P}(n, n')$ be the set of $n \times n'$ matrices whose rows are sub-probability vectors.

The Partition of State Space

- Let $\mathbf{E}$ be a finite state space with cardinality m, and let ∂ be a coffin state.
- Let $v : \mathbf{E} \cup \{\partial\} \to \mathbb{R}$ and $\sigma : \mathbf{E} \cup \{\partial\} \to \mathbb{R}$ with $v(\partial) = \sigma(\partial) = 0$. Denote by

$$\mathrm{V} := \mathrm{diag}\big\{v(i),\, i \in \mathbf{E}\big\}, \quad \Sigma := \mathrm{diag}\big\{\sigma(i),\, i \in \mathbf{E}\big\}.$$

  Assume that v and σ are NOT equal to zero simultaneously on $\mathbf{E}$.
- The state space $\mathbf{E}$ is partitioned into $\mathbf{E} = \mathbf{E}_0 \cup \mathbf{E}_+ \cup \mathbf{E}_-$, where

$$\mathbf{E}_0 := \big\{i \in \mathbf{E} : \sigma(i) \neq 0\big\}, \quad \mathbf{E}_\pm := \big\{i \in \mathbf{E} : \sigma(i) = 0,\, \pm v(i) > 0\big\}.$$

  The cardinalities of $\mathbf{E}_0$ and $\mathbf{E}_\pm$ are denoted respectively by m_0 and $m_\pm$.

The Probabilistic Model

- Let $(\Omega, \mathscr{F}, \mathbb{F}, \mathbb{P})$ be a stochastic basis, where $\mathbb{F}$ is the augmented filtration generated by X and W.
- Let $X = (X_t)_{t \geq 0}$ be a continuous-time Markov chain taking values in $\mathbf{E} \cup \{\partial\}$, with generator $\mathrm{Q} \in \mathcal{Q}(m)$.
- Let $W = (W_t)_{t \geq 0}$ be a standard Brownian motion, independent of X.
- Define the additive functional with noise

$$\varphi_t := \int_0^t v(X_s)\, ds + \int_0^t \sigma(X_s)\, dW_s, \quad t \geq 0.$$

- For any $\ell \geq 0$, define the passage times

$$\tau_\ell^{\pm} := \inf \left\{ t \geq 0 : \pm\varphi_t > \ell \right\}.$$

- The time-changed processes $(X_{\tau_\ell^+})_{\ell \geq 0}$ and $(X_{\tau_\ell^-})_{\ell \geq 0}$ are continuous-time Markov chains with respective state spaces $\mathbf{E}_+ \cup \mathbf{E}_0 \cup \{\partial\}$ and $\mathbf{E}_- \cup \mathbf{E}_0 \cup \{\partial\}$.

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An Algebraic Factorization for Generator Matrices

Consider first the case when $\sigma \equiv 0$. Then $v \neq 0$ on $\mathbf{E}$, and

$$\mathbf{E}_0 = \emptyset, \quad \mathbf{E}_\pm = \left\{ i \in \mathbf{E} : \pm v(i) > 0 \right\}.$$

Theorem 2.1 (Barlow, Rogers, & Williams '80)

Let $\mathsf{Q} \in \mathcal{Q}(m)$ and $c > 0$. Then, there exists a unique quadruple of matrices $(\Pi_c^+, \widetilde{\mathsf{Q}}_c^+, \Pi_c^-, \widetilde{\mathsf{Q}}_c^-)$, where $\Pi_c^\pm \in \mathcal{P}(m_\mp, m_\pm)$ and $\widetilde{\mathsf{Q}}_c^\pm \in \mathcal{Q}(m_\pm)$, such that

$$\mathsf{V}^{-1}(\mathsf{Q} - c\mathsf{I}) = \begin{pmatrix} \mathsf{I}^+ & \Pi_c^- \\ \Pi_c^+ & \mathsf{I}^- \end{pmatrix} \begin{pmatrix} \widetilde{\mathsf{Q}}_c^+ & 0 \\ 0 & -\widetilde{\mathsf{Q}}_c^- \end{pmatrix} \begin{pmatrix} \mathsf{I}^+ & \Pi_c^- \\ \Pi_c^+ & \mathsf{I}^- \end{pmatrix}^{-1}, \quad (2.1)$$

where I (respectively, $\mathsf{I}^\pm$) is the $m \times m$ (respectively, $m_\pm \times m_\pm$) identity matrix.

Probabilistic Interpretation of Matrix Wiener-Hopf

Theorem 2.2 (Barlow, Rogers, & Williams '80)

$$\Pi_c^{\pm}(i,j) = \mathbb{E}\left(e^{-c\tau_0^{\pm}}\mathbf{1}_{\{X_{\tau_0^{\pm}}=j\}}\,\middle|\,X_0=i\right),\quad i\in\mathbf{E}_{\mp},\ j\in\mathbf{E}_{\pm}, \tag{2.2}$$

$$e^{\ell\,\widetilde{\mathsf{Q}}_c^{\pm}}(i,j) = \mathbb{E}\left(e^{-c\tau_\ell^{\pm}}\mathbf{1}_{\{X_{\tau_\ell^{\pm}}=j\}}\,\middle|\,X_0=i\right),\quad i,j\in\mathbf{E}_{\pm},\ \ell>0. \tag{2.3}$$

- $\widetilde{\mathsf{Q}}_c^{\pm}$ is the generator matrix of the Markov chain obtained by first killing X at rate c, and then applying the time-change $(\tau_\ell^{\pm})_{\ell\geq 0}$.
- If $c=0$, the factorization (2.1) still holds true, but not necessarily unique when row sums of Q are all zero. In this case, the quadruple $(\Pi_0^+,\widetilde{\mathsf{Q}}_0^+,\Pi_0^-,\widetilde{\mathsf{Q}}_0^-)$ given in Theorem 2.2 is not the only solution to (2.1).
- By Theorems 2.1 and 2.2, in order to compute expectation of the form (2.2) or (2.3), one only needs to solve the algebraic equation (2.1).

Computing $\Pi_c^{\pm}$ and $\widetilde{Q}_c^{\pm}$

- The "plus" and "minus" parts in (2.1) can be separated as

$$V^{-1}(Q - cI)\begin{pmatrix} I^+ \\ \Pi_c^+ \end{pmatrix} = \begin{pmatrix} I^+ \\ \Pi_c^+ \end{pmatrix}\widetilde{Q}_c^+, \tag{2.4}$$

$$V^{-1}(Q - cI)\begin{pmatrix} I^- \\ \Pi_c^- \end{pmatrix} = -\begin{pmatrix} I^- \\ \Pi_c^- \end{pmatrix}\widetilde{Q}_c^-. \tag{2.5}$$

  Hence, $(\Pi_c^+, \widetilde{Q}_c^+)$ and $(\Pi_c^-, \widetilde{Q}_c^-)$ can be computed independently.

- Write Q and V in their respective block forms:

$$Q = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \quad \text{and} \quad V = \begin{pmatrix} V^+ & \mathbf{0} \\ \mathbf{0} & V^- \end{pmatrix}.$$

  Then, $\widetilde{Q}_c^+ = (V^+)^{-1}((A - cI^+) + B\Pi_c^+)$, and Π_c^+ satisfies following algebraic Riccati equation

$$-(V^-)^{-1}(D - cI^-)\Pi_c^+ + \Pi_c^+(V^+)^{-1}(A - cI^+) + \Pi_c^+(V^+)^{-1}B\Pi_c^+ - (V^-)^{-1}C = \mathbf{0}.$$

  Similar computation can be done for $(\Pi_c^-, \widetilde{Q}_c^-)$.

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Existence and Uniqueness

Consider the general case when σ is any function on $\mathbf{E}$. Let $\mathrm{Q} \in \mathcal{Q}_0(m)$, namely, Q is irreducible.

Theorem 2.3 (Jiang & Pistorius '08)

Let $\mathrm{Q} \in \mathcal{Q}_0(m)$ be transient. Then there exists a unique quadruple of matrices $(\Pi^+, \widetilde{\mathrm{Q}}^+, \Pi^-, \widetilde{\mathrm{Q}}^-)$, where $\Pi^\pm \in \mathcal{P}(m_\mp, m_\pm + m_0)$ and $\widetilde{\mathrm{Q}}^\pm \in \mathcal{Q}(m_\pm + m_0)$, such that

$$\frac{1}{2}\Sigma^2 \begin{pmatrix} \mathrm{I}^+ & \mathbf{0} \\ \mathbf{0} & \mathrm{I}_0 \\ \multicolumn{2}{c}{\Pi^+} \end{pmatrix} \left(\widetilde{\mathrm{Q}}^+\right)^2 - \mathrm{V} \begin{pmatrix} \mathrm{I}^+ & \mathbf{0} \\ \mathbf{0} & \mathrm{I}_0 \\ \multicolumn{2}{c}{\Pi^+} \end{pmatrix} \widetilde{\mathrm{Q}}^+ + \mathrm{Q} \begin{pmatrix} \mathrm{I}^+ & \mathbf{0} \\ \mathbf{0} & \mathrm{I}_0 \\ \multicolumn{2}{c}{\Pi^+} \end{pmatrix} = \mathbf{0}, \qquad (2.6)$$

$$\frac{1}{2}\Sigma^2 \begin{pmatrix} \multicolumn{2}{c}{\Pi^-} \\ \mathrm{I}_0 & \mathbf{0} \\ \mathbf{0} & \mathrm{I}^- \end{pmatrix} \left(\widetilde{\mathrm{Q}}^-\right)^2 + \mathrm{V} \begin{pmatrix} \multicolumn{2}{c}{\Pi^-} \\ \mathrm{I}_0 & \mathbf{0} \\ \mathbf{0} & \mathrm{I}^- \end{pmatrix} \widetilde{\mathrm{Q}}^- + \mathrm{Q} \begin{pmatrix} \multicolumn{2}{c}{\Pi^-} \\ \mathrm{I}_0 & \mathbf{0} \\ \mathbf{0} & \mathrm{I}^- \end{pmatrix} = \mathbf{0}. \qquad (2.7)$$

Probabilistic Interpretation

Theorem 2.4 (Jiang & Pistorius '08)

$$\Pi^{\pm}(i,j) = \mathbb{P}\Big(X_{\tau_0^{\pm}} = j,\ \tau_0^{\pm} < \infty \,\Big|\, X_0 = i\Big), \quad i \in \mathbf{E}_{\mp},\ j \in \mathbf{E}_{\pm} \cup \mathbf{E}_0,$$

$$e^{\ell \widetilde{Q}^{\pm}}(i,j) = \mathbb{P}\Big(X_{\tau_\ell^{\pm}} = j,\ \tau_\ell^{\pm} < \infty \,\Big|\, X_0 = i\Big), \quad i,j \in \mathbf{E}_{\pm},\ \ell > 0.$$

- When $\sigma \equiv 0$ (so that $\mathbf{E}_0 = \emptyset$ and $\mathbf{E} = \mathbf{E}_+ \cup \mathbf{E}_-$), (2.6)−(2.7) reduce to the "plus" and "minus" part of (2.1), given as in (2.4)−(2.5), respectively.
- When σ is constant (so that $\mathbf{E}_+ = \mathbf{E}_- = \emptyset$ and $\mathbf{E} = \mathbf{E}_0$), (2.6)−(2.7) reduce to Kennedy & Williams '90.

Application: Two-Sided Exit Problem

- The two-sided exit problem of φ from the interval $[k, \ell]$, $-\infty < k < \ell < \infty$, is to find the joint distribution of the position (X_τ, φ_τ) at the first-exit time

$$\tau = \tau_{k,\ell} := \inf\left\{t \geq 0 : \varphi_t \notin [k, \ell]\right\}.$$

- Such problem can be solved explicitly in terms of the output of the Wiener-Hopf factorization $(\Pi^+, \widetilde{Q}^+, \Pi^-, \widetilde{Q}^-)$.

- To this end, define

$$W^+ := \begin{pmatrix} I^+ & \mathbf{0} \\ \mathbf{0} & I_0 \\ \multicolumn{2}{c}{\Pi^+} \end{pmatrix}, \quad W^- := \begin{pmatrix} \multicolumn{2}{c}{\Pi^-} \\ I_0 & \mathbf{0} \\ \mathbf{0} & I^- \end{pmatrix}, \quad J^+ := \begin{pmatrix} I^+ & \mathbf{0} \\ \mathbf{0} & I_0 \\ \multicolumn{2}{c}{\mathbf{0}} \end{pmatrix}, \quad J^- := \begin{pmatrix} \multicolumn{2}{c}{\mathbf{0}} \\ I_0 & \mathbf{0} \\ \mathbf{0} & I^- \end{pmatrix}$$

$$Z^+ := \begin{pmatrix} \mathbf{0} & I_0 \\ \multicolumn{2}{c}{\Pi^+} \end{pmatrix} e^{\widetilde{Q}^+(\ell-k)}, \quad Z^- := \begin{pmatrix} \multicolumn{2}{c}{\Pi^-} \\ \mathbf{0} & I_0 \end{pmatrix} e^{\widetilde{Q}^-(\ell-k)}.$$

Application: Two-Sided Exit Problem (Cont'd)

$$\Psi^+(a) := \left(\mathrm{W}^+ e^{\widetilde{\mathrm{Q}}^+(\ell-a)} - \mathrm{W}^- e^{\widetilde{\mathrm{Q}}^-(a-k)} \mathrm{Z}^+\right)\left(\mathrm{I}_0^+ - \mathrm{Z}^-\mathrm{Z}^+\right)^{-1},$$

$$\Psi^-(a) := \left(\mathrm{W}^- e^{\widetilde{\mathrm{Q}}^-(\ell-a)} - \mathrm{W}^+ e^{\widetilde{\mathrm{Q}}^+(a-k)} \mathrm{Z}^-\right)\left(\mathrm{I}_0^- - \mathrm{Z}^+\mathrm{Z}^-\right)^{-1},$$

$$\Psi_0(s,a) := \left(e^{sa}\mathrm{I} - e^{s\ell}\Psi^+(a)\mathrm{J}^+ - e^{sk}\Psi^-(a)\mathrm{J}^-\right)\left(-\frac{1}{2}\Sigma^2 s^2 - \mathrm{V}s - \mathrm{Q}\right)^{-1}.$$

Proposition 2.5 (Jiang & Pistorius '08)

Let $h^{\pm}$ and h be real-valued functions on $\mathbf{E}_0 \cup \mathbf{E}_{\pm}$ and $\mathbf{E}$, respectively. For any $a \in [k, \ell]$ and $i \in \mathbf{E}$,

$$\mathbb{E}\left(h^+(X_\tau)\mathbf{1}_{\{\varphi_\tau=\ell\}}\mathbf{1}_{\{\tau<\zeta\}} \,\middle|\, X_0 = i,\, \varphi_0 = a\right) = \mathbf{e}_i^{\mathsf{T}}\Psi^+(a)\,h^+,$$

$$\mathbb{E}\left(h^-(X_\tau)\mathbf{1}_{\{\varphi_\tau=k\}}\mathbf{1}_{\{\tau<\zeta\}} \,\middle|\, X_0 = i,\, \varphi_0 = a\right) = \mathbf{e}_i^{\mathsf{T}}\Psi^-(a)\,h^-,$$

$$\mathbb{E}\left(e^{s\varphi_{\zeta-}} h(X_{\zeta-})\mathbf{1}_{\{\zeta<\tau\}} \,\middle|\, X_0 = i,\, \varphi_0 = a\right) = \mathbf{e}_i^{\mathsf{T}}\Psi_0(s,a)\,\mathrm{H}(-\mathrm{Q})\mathbf{1}.$$

where $\mathrm{H} := \mathrm{diag}\{h(i),\, i \in \mathbf{E}\}$.

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Highly Non-trivial Extension to Time-Inhomogeneous Framework

- Let X be a time-inhomogeneous Markov chain taking values in $\mathbf{E} \cup \partial$, with generator function $\mathsf{Q}_t \in \mathcal{Q}(m)$, $t \geq 0$.
- The factorization of $\mathsf{V}^{-1}(\mathsf{Q}_t - c\mathsf{I})$ can be done for each $t \geq 0$ separately, exactly as described in Theorem 2.1.
- However, the resulting matrices $\Pi_c^{\pm}(t)$ and $\widetilde{\mathsf{Q}}_c^{\pm}(t)$, $t \geq 0$, do not have similar probabilistic interpretations as shown in Theorem 2.2.
- In particular, they are not useful for computing expectations of the form

$$\mathbb{E}\left(e^{-c\tau_\ell^{\pm}(s)}\mathbf{1}_{\{X_{\tau_\ell^{\pm}(s)}=j\}} \,\Big|\, X_s = i\right), \tag{3.1}$$

where

$$\tau_\ell^{\pm}(s) := \inf\left\{t \geq s : \int_s^t v(X_u)\,du > \ell\right\}.$$

First Attempt: Piecewise Constant Generator

- Bielecki, Cialenco, Gong, & Huang '18: Wiener-Hopf Factorization for time-inhomogeneous Markov chain X with piecewise-constant generator $\boldsymbol{\Lambda}$:

$$\Lambda_t = \Lambda_k, \quad s_{k-1} \leq t < s_k, \quad k = 1, \ldots, n; \qquad \Lambda_t = \Lambda_{n+1}, \quad t \geq s_n,$$

where $\Lambda_1, \ldots, \Lambda_{n+1} \in \mathcal{Q}(m)$, $n \in \mathbb{N}$, $0 = s_0 < s_1 < \ldots < s_n$.

- Time-inhomogeneous Markov chains with piecewise-constant generators are adequate models for, among others, seasonal phenomena, Erlang loss systems with moving boundaries, or structural breaks in credit migrations.
- The main goal is to apply the Wiener-Hopf factorization technique to compute expectations of the form (3.1).
- This special setup allows to use some appropriately tailored and original randomization techniques.

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The Time-Inhomogeneous Generator

In Bielecki, Cheng, Cialenco, & Gong '19, we study Wiener-Hopf Factorization for time-inhomogeneous Markov chain with continuous bounded generators.

- Let $\mathbf{E}$ be a finite set with $|\mathbf{E}| = m > 1$.
- Let $(\Lambda_s)_{s\in\mathbb{R}_+}$, $\mathbb{R}_+ := [0,\infty)$, where $\Lambda_s \in \mathcal{Q}(m)$.
- Main Assumptions:
  - $\exists K \in (0,\infty)$, such that $|\Lambda_s(i,j)| \leq K$, for all $i, j \in \mathbf{E}$ and $s \in \mathbb{R}_+$;
  - $(\Lambda_s)_{s\in\mathbb{R}_+}$, considered as a mapping from $\mathbb{R}_+$ to the set of $m \times m$ generator matrices, is continuous with respect to s on $\mathbb{R}_+$.

The Partition of State Space

- Recall $v : \mathbf{E} \to \mathbb{R} \setminus \{0\}$, $\mathrm{V} := \mathrm{diag}\{v(i) : i \in \mathbf{E}\}$.
- Assume that both $\mathbf{E}_+$ and $\mathbf{E}_-$ are non-empty, where

$$\mathbf{E}_+ := \{i \in \mathbf{E} :\ v(i) > 0\} \quad \text{and} \quad \mathbf{E}_- := \{i \in \mathbf{E} :\ v(i) < 0\}.$$

- Accordingly, we write Λ_s and V in the block form :

$$\Lambda_s = \begin{array}{c} \\ \mathbf{E}_+ \\ \mathbf{E}_- \end{array} \begin{array}{c} \begin{array}{cc} \mathbf{E}_+ & \mathbf{E}_- \end{array} \\ \begin{pmatrix} \mathrm{A}_s & \mathrm{B}_s \\ \mathrm{C}_s & \mathrm{D}_s \end{pmatrix} \end{array}, \quad \mathrm{V} = \begin{array}{c} \\ \mathbf{E}_+ \\ \mathbf{E}_- \end{array} \begin{array}{c} \begin{array}{cc} \mathbf{E}_+ & \mathbf{E}_- \end{array} \\ \begin{pmatrix} \mathrm{V}_+ & 0 \\ 0 & \mathrm{V}_- \end{pmatrix} \end{array}.$$

- Denote by $\mathscr{X} := \mathbb{R}_+ \times \mathbf{E}$ and $\mathscr{X}_\pm := \mathbb{R}_+ \times \mathbf{E}_\pm$.
- Define $\widetilde{\Lambda} : L^\infty(\mathscr{X}) \to L^\infty(\mathscr{X})$, associated with $(\Lambda_s)_{s \in \mathbb{R}_+}$, by

$$(\widetilde{\Lambda}\, g)(s,i) := (\Lambda_s\, g(s,\cdot))(i), \quad (s,i) \in \mathscr{X},$$

and similarly for $\widetilde{\mathrm{A}}$, $\widetilde{\mathrm{B}}$, $\widetilde{\mathrm{C}}$, and $\widetilde{\mathrm{D}}$.

The Time-Inhomogeneous Markov Family

- Let Ω be the collection of $\mathbf{E}$-valued càdlàg functions on $\mathbb{R}_+$.
- There exists a standard Markov family

$$\mathcal{M} := \left\{ \left(\Omega, \mathscr{F}, \mathbb{F}_s, (X_t)_{t\in[s,\infty)}, \mathbb{P}_{s,i}\right), \, (s,i) \in \mathscr{X} \right\},$$

such the the associated evolution system $\mathrm{U} := (\mathrm{U}_{s,t})_{0\leq s\leq t<\infty}$, defined by

$$(\mathrm{U}_{s,t} f)(i) := \mathbb{E}_{s,i}\big(f(X_t)\big), \quad 0 \leq s \leq t < \infty, \quad i \in \mathbf{E},$$

for any $f : \mathbf{E} \to \mathbb{R}$, admits the generator $\boldsymbol{\Lambda}$, namely

$$\lim_{h\downarrow 0} \frac{1}{h}\big((\mathrm{U}_{s,s+h} f)(i) - f(i)\big) = (\Lambda_s f)(i), \quad \text{for any } (s,i) \in \mathscr{X}.$$

- Define the passage times

$$\tau_\ell^{\pm}(s) := \inf\left\{ t \in [s,\infty) : \pm \int_s^t v(X_u)\, du > \ell \right\}.$$

Main Goal

- Goal: To derive a Wiener-Hopf type method for computing expectations of the form

$$\mathbb{E}_{s,i}\left(g\left(\tau_\ell^{\pm}, X_{\tau_\ell^{\pm}}\right)\right),$$

for $g \in L^\infty(\mathscr{X})$, $\ell \in \mathbb{R}_+$, and $(s,i) \in \mathscr{X}$.

- Note that $X_{\tau_\ell^{\pm}(s)} \in \mathbf{E}_\pm \cup \{\partial\}$. Hence, it is sufficient to compute expectations of the following form

$$\mathbb{E}_{s,i}\left(g^{\pm}\left(\tau_\ell^{\pm}, X_{\tau_\ell^{\pm}}\right)\right), \tag{3.2}$$

for $g^\pm \in L^\infty(\mathscr{X}_\pm)$, $\ell \in \mathbb{R}_+$, and $(s,i) \in \mathscr{X}$.

Some Important Operators

- $J^{\pm} : L^{\infty}(\mathscr{X}_{\pm}) \to L^{\infty}(\mathscr{X}_{\mp})$ is defined as

$$\left(J^{\pm}g^{\pm}\right)(s,i) := \mathbb{E}_{s,i}\left(g^{\pm}\left(\tau_0^{\pm}, X_{\tau_0^{\pm}}\right)\right), \quad (s,i) \in \mathscr{X}_{\mp}.$$

- For any $\ell \in \mathbb{R}_+$, $\mathcal{P}_{\ell}^{\pm} : L^{\infty}(\mathscr{X}_{\pm}) \to L^{\infty}(\mathscr{X}_{\pm})$ is defined as

$$\left(\mathcal{P}_{\ell}^{\pm}g^{\pm}\right)(s,i) := \mathbb{E}_{s,i}\left(g^{\pm}\left(\tau_{\ell}^{\pm}, X_{\tau_{\ell}^{\pm}}\right)\right), \quad (s,i) \in \mathscr{X}_{\pm}.$$

- For any $(s,i) \in \mathscr{X}_{\pm}$, we define

$$\left(G^{\pm}g^{\pm}\right)(s,i) := \lim_{\ell \to 0+} \frac{1}{\ell}\left(\mathcal{P}_{\ell}^{\pm}g^{\pm}(s,i) - g^{\pm}(s,i)\right),$$

  for any $g^{\pm} \in C_0(\mathscr{X}_{\pm})$ such that the above limit exists and is finite.

- It can be shown that any expectation of the form (3.2) can be represented in terms of the operators $J^{\pm}$ and $\mathcal{P}_{\ell}^{\pm}$.

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Existence and Uniqueness

Theorem 3.1 (Bielecki, Cialenco, Cheng, & Gong '19)

There exists a unique quadruple of operators (S^+, H^+, S^-, H^-) which solves the following operator equation: for any $g^\pm \in C_0^1(\mathscr{X}_\pm)$,

$$\mathsf{V}^{-1}\left(\frac{\partial}{\partial s}+\widetilde{\Lambda}\right)\begin{pmatrix} I^+ & S^- \\ S^+ & I^- \end{pmatrix}\begin{pmatrix} g^+ \\ g^- \end{pmatrix} = \begin{pmatrix} I^+ & S^- \\ S^+ & I^- \end{pmatrix}\begin{pmatrix} H^+ & 0 \\ 0 & -H^- \end{pmatrix}\begin{pmatrix} g^+ \\ g^- \end{pmatrix},$$

subject to the conditions below:

(a) $S^\pm : C_0(\mathscr{X}_\pm) \to C_0(\mathscr{X}_\mp)$ is a bounded operator such that

  (i) for any $g^\pm \in C_c(\mathscr{X}_\pm)$ with $\operatorname{supp} g^\pm \subset [0, \eta_{g^\pm}] \times \mathbf{E}_\pm$ for some constant $\eta_{g^\pm} \in (0,\infty)$, we have $\operatorname{supp} S^\pm g^\pm \subset [0, \eta_{g^\pm}] \times \mathbf{E}_\mp$;

  (ii) for any $g^\pm \in C_0^1(\mathscr{X}_\pm)$, we have $S^\pm g^\pm \in C_0^1(\mathscr{X}_\mp)$.

(b) $H^\pm$ is the strong generator of a strongly continuous positive contraction semigroup $(\mathcal{Q}_\ell^\pm)_{\ell\in\mathbb{R}_+}$ on $C_0(\mathscr{X}_\pm)$ with domain $\mathscr{D}(H^\pm) = C_0^1(\mathscr{X}_\pm)$.

Probabilistic Interpretation

Theorem 3.2 (Bielecki, Cialenco, Cheng, & Gong '19)

For any $g^{\pm} \in C_0(\mathscr{X}_{\pm})$, we have

$$S^{\pm} g^{\pm} = J^{\pm} g^{\pm} \quad \text{and} \quad \mathcal{Q}_{\ell}^{\pm} g^{\pm} = \mathcal{P}_{\ell}^{\pm} g^{\pm}, \quad \text{for any } \ell \in \mathbb{R}_+,$$

Moreover, $G^{\pm}$ is the strong generator of $(\mathcal{P}_{\ell}^{\pm})_{\ell \in \mathbb{R}_+}$ on $C_0(\mathscr{X}_{\pm})$ with domain $\mathscr{D}(G^{\pm}) = C_0^1(\mathscr{X}_{\pm})$.

From Theorems 3.1 and 3.2, we see that (J^+, G^+, J^-, G^-) is the unique quadruple of operators, subject to conditions (a) and (b), that solves the operator equation: for any $g^{\pm} \in C_0^1(\mathscr{X}_{\pm})$,

$$\mathsf{V}^{-1}\left(\frac{\partial}{\partial s} + \widetilde{\Lambda}\right)\begin{pmatrix} I^+ & J^- \\ J^+ & I^- \end{pmatrix}\begin{pmatrix} g^+ \\ g^- \end{pmatrix} = \begin{pmatrix} I^+ & J^- \\ J^+ & I^- \end{pmatrix}\begin{pmatrix} G^+ & 0 \\ 0 & -G^- \end{pmatrix}\begin{pmatrix} g^+ \\ g^- \end{pmatrix}. \quad (3.3)$$

Remarks

- $\partial/\partial s + \widetilde{\mathrm{Q}}$ is the generator of the following time-homogenized process $\widetilde{X}$ on the enlarged probability space $\widetilde{\Omega} := \mathbb{R}_+ \times \Omega$:
$$\widetilde{X}_t(\widetilde{\omega}) = \widetilde{X}_t((s,\omega)) = \big(s+t, X_{t+s}(\omega)\big).$$
The equations (3.3) can be regarded as the Wiener-Hopf factorization corresponding to the time-homogeneous Markov process $\widetilde{X}$. Note that this is not a direct application of Barlow, Rogers, & Williams '80 since the state space of $\widetilde{X}$ is no longer a finite space.
- The operators $J^{\pm}$ and $G^{\pm}$ are counterparts of the matrices $\Pi_c^{\pm}$ and $\widetilde{\mathrm{Q}}_c^{\pm}$ in Theorem 2.1. When X is a time-homogeneous Markov chain with generator matrix Q, by taking $g^{\pm}(s,i) = e^{-cs}\mathbf{1}_{\{i=k\}}$, for $c \geq 0$ and $k \in \mathbf{E}_{\pm}$, we reduce (3.3) to the time-homogeneous factorization (2.1).

Computing the Operators $J^{\pm}$ and $G^{\pm}$

- By Theorems 3.1 and 3.2, for any $g^{\pm} \in C_0^1(\mathscr{X}_{\pm})$, $J^{\pm}g^{\pm}$ and $G^{\pm}g^{\pm}$ (and thus $\mathcal{P}_{\ell}^{\pm}g^{\pm}$, for any $\ell \in \mathbb{R}_+$) can be computed from the Wiener-Hopf equation (3.3) subject to conditions (a) and (b).
- Note that (3.3) can be decomposed into:

$$\mathsf{V}^{-1}\left(\frac{\partial}{\partial s} + \widetilde{\Lambda}\right)\begin{pmatrix} I^{+} \\ J^{+} \end{pmatrix} g^{+} = \begin{pmatrix} I^{+} \\ J^{+} \end{pmatrix} G^{+}g^{+}, \quad g^{+} \in C_0^1(\mathscr{X}_{+}), \tag{3.4}$$

$$\mathsf{V}^{-1}\left(\frac{\partial}{\partial s} + \widetilde{\Lambda}\right)\begin{pmatrix} I^{-} \\ J^{-} \end{pmatrix} g^{+} = -\begin{pmatrix} I^{-} \\ J^{-} \end{pmatrix} G^{-}g^{-}, \quad g^{-} \in C_0^1(\mathscr{X}_{-}).$$

Hence, one can compute $J^{+}g^{+}$ and $G^{+}g^{+}$ (and thus $\mathcal{P}_{\ell}^{+}g^{+}$) separately from $J^{-}g^{-}$ and $G^{-}g^{-}$ (and thus $\mathcal{P}_{\ell}^{-}g^{-}$).

Computing the Operators $J^{\pm}$ and $G^{\pm}$ (Cont'd)

- Using the block form of $\widetilde{\Lambda}$ and V, (3.4) can be written as

$$(\mathrm{V}^{+})^{-1}\left(\frac{\partial}{\partial s} + \widetilde{\mathrm{A}} + \widetilde{\mathrm{B}}J^{+}\right)g^{+} = G^{+}g^{+}, \tag{3.5}$$

$$(\mathrm{V}^{-})^{-1}\left(\frac{\partial}{\partial s}J^{+} + \widetilde{\mathrm{C}} + \widetilde{\mathrm{D}}J^{+}\right)g^{+} = J^{+}G^{+}g^{+}. \tag{3.6}$$

- From (3.5), we see that G^{+} is determined by J^{+}.
- Furthermore, from (3.5) and (3.6) we deduce the following operator Riccati equation for J^{+}:

$$\left(J^{+}(\mathrm{V}^{+})^{-1}\widetilde{\mathrm{B}}J^{+} + J^{+}(\mathrm{V}^{+})^{-1}\left(\frac{\partial}{\partial s} + \widetilde{\mathrm{A}}\right) - (\mathrm{V}^{-})^{-1}\left(\frac{\partial}{\partial s} + \widetilde{\mathrm{D}}\right)J^{+} - (\mathrm{V}^{-})^{-1}\widetilde{\mathrm{C}}\right)g^{+} = 0.$$

- In conclusion, solving (3.4) for G^{+} and J^{+} boils down to solving the above Ricatti equation for J^{+} .

Computing the Operators $J^{\pm}$ and $G^{\pm}$ (Cont'd)

- In practice, take $g_j^{\pm} \in C_0^1(\mathscr{X}_{\pm})$ with

$$g_j^{\pm}(s,i) := e^{-cs}\mathbf{1}_{\{j\}}(i), \quad (s,i) \in \mathscr{X}_{\pm},$$

  for any $c > 0$ and $j \in \mathbf{E}_{\pm}$.

- By solving $J^{\pm}g^{\pm}$ and $G^{\pm}g^{\pm}$ for such $g^{\pm}$, we obtain the Laplace transform for the pair $(\tau_\ell^{\pm}, X_{\tau_\ell^{\pm}})$.

- We then perform the inverse Laplace transform with respect to c to obtain the join distribution of $(\tau_\ell^{\pm}, X_{\tau_\ell^{\pm}})$ under $\mathbb{P}_{s,i}$, which enables us to compute the expectations (3.2) for any $g^{\pm} \in L^{\infty}(\mathscr{X}_{\pm})$.

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Future Research

- Noisy Wiener-Hopf factorization for time-inhomogeneous Markov chains
- Wiener-Hopf factorization for time-homogeneous/inhomogeneous strong Markov processes with general state space

  The Wiener-Hopf factorization for time-homogeneous strong Markov processes, with general state spaces, was studied by Williams '08, but only for the associated resolvent operators.
- Connection with the classical Wiener-Hopf factorization for Lévy processes and Markov additive processes
- Wiener-Hopf factorization for time-homogeneous Lévy processes
- Wiener-Hopf factorization for time-homogeneous Markov additive processes
- Ultimate Goal: a unifying framework in terms of functional of time-changed processes and in terms of algebraic factorizations of linear operators, which encompasses all the processes above.

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THANK YOU FOR YOUR ATTENTION!