Optimal incentives for a system of interacting Agents

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work in progress

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Motivation

- Large economic literature on contract design in order to revisit equilibrium theory by incorporating incentives and asymmetry of information ;

Customers know more about their tastes than firms,
firms know more about their costs than the government
and all agents take actions that are at least partly unobservable.
B. Salanié, The economics of contracts

- Optimal contracting between a Principal and an Agent ;
- Mainly static or discrete time problems : *Spear & Srivastava*, *Salanié*, *Tirole*, *Laffont*, *Martimort*, *Radner* $\Longrightarrow$ Limited computations ;
- Extension to continuous time models : *Holmstrom & Milgrom*, *Sannikov*, *Cvitanic*,... ;
- More explicit solutions and strong connexion with the theory of BSDEs. ;
- Optimal contracting between a Principal and a system of Agents in interactions...

The classical Principal Agent problem

The Principal wishes to design a contract that :

- the Agent will accept, i.e. which provides him his reservation utility ;
- maximizes the expected utility of the Principal.

Three main cases depending on how much information is available to both the Principal and the Agent

1. Risk sharing or first best
   Information is symmetric and the Principal and the Agent agree on how they share the risk
   $\Longrightarrow$ Principal chooses both the contract and the actions of the Agent

2. Moral hasard or second best
   The actions of the Agent are unobservable
   $\Longrightarrow$ First solve the Agent problem for any contract, and then solve the Principal problem

3. Adverse selection of third best
   The type/characteristics of the Agent is unknown for the Principal
   $\Longrightarrow$ The Principal offers a menu of contracts, one for each type of Agent, forcing the Agent to reveal himself.

Revisiting the Holmstrom and Milgrom moral hazard problem

- Controlled output process (such as a production good, or cash flows ...)

$$dX_t^a = a_t dt + \sigma dB_t$$

- The control a is s.t. $\mathcal{E}(\int_0^T a_t/\sigma dB_t)$ is unif. integrable ;

- Weak formulation : choice of a probability $\mathbb{P}^a$ with Brownian Motion B^a

$$dX_t = a_t dt + \sigma dB_t^a$$

- The Agent observes B^a whereas the Principal only observes X ;

- The Agent chooses a control $(a_t)_t$ and receives a payment ξ at time T. He solves

$$\sup_a \mathbb{E}^{\mathbb{P}^a}\left[U_A\left(\xi - \int_0^T k(a_t)dt\right)\right]$$

- The Principal chooses and pays the terminal payment ξ and solves

$$\sup_\xi \mathbb{E}^{\mathbb{P}^a}\left[U_P(X_T - \xi)\right]$$

- U_A and U_P are exponential utility functions with risk aversions R_A and R_P.

Solving the Agent's problem

- Consider a given $\mathcal{F}_T^X$-measurable payment contract ξ
- The problem of the Agent rewrites as follows

$$U_0^A = \sup_a \mathbb{E}^{\mathbb{P}^a}\left[U_A\left(\xi - \int_0^T k(a_s)ds\right)\right]$$

- The dynamic version at time t is

$$U_t^A(a) = \operatorname*{ess\,sup}_{a' \, on[t,T]} J_t^{a,a'} \qquad \text{where} \qquad J_t^{a,a'} = \mathbb{E}^{\mathbb{P}^{a'}}\left[U_A\left(\xi - \int_t^T k(a'_s)ds\right) \mid \mathcal{F}_t^B\right]$$

- $e^{R_A \int_0^t k(a_s)ds} J_t^{a,a'}$ is a $\mathbb{P}^{a'}$- martingale
- Introduce the log target process $Y^{a,a'} := \frac{-ln(J_\cdot^{a,a'})}{R_A}$
- Ito's formula and Girsanov Theorem imply that $Y^{a,a'}$ solves the BSDE

$$Y_t^{a,a'} = \xi + \int_t^T f(a', Z_s^{a,a'})ds - \int_t^T Z_s^{a,a'} \sigma dB_s$$

with the family of drivers $f : (a', z) \mapsto -\frac{R_A}{2}\sigma^2 z^2 + a'z - k(a')$

Solving the Agent's problem

$$U_t^A(a) = \underset{a' \, on [t,T]}{\text{ess sup}} \, J_t^{a,a'} \qquad \text{where} \qquad J_t^{a,a'} = \mathbb{E}^{\mathbb{P}^{a'}} \left[U_A \left(\xi - \int_t^T k(a'_s) ds \right) \mid \mathcal{F}_t^B \right]$$

- $Y^{a,a'} := \frac{-ln(J^{a,a'}_{.})}{R_A}$ solves the BSDE

$$Y_t^{a,a'} = \xi + \int_t^T f(a', Z_s^{a,a'}) ds - \int_t^T Z_s^{a,a'} \sigma dB_s$$

  with the family of drivers $f : (a', z) \mapsto -\frac{R_A}{2} \sigma^2 z^2 + a' z - k(a')$

- We want to maximize $J^{a,a'}$ or similarly $Y^{a,a'}$ over a'
- Comparison for BSDEs $\Longrightarrow$ take $f^* := \sup_{a'} f(a', .)$
- Optimal control a^* which maximizes the driver : $a^* = \bar{k}(z)$
- The class of admissible contracts relies on integrability conditions on ξ for the BSDE to be well posed

Back to the Principal's problem

- For a given terminal payment ξ, the log expected utility Y of the Agent is

$$Y_t = \xi + \int_t^T f^*(Z_s^*)ds - \int_t^T Z_s^* \sigma dB_s$$

- The participation constraint of the Agent requires $Y_0 \geq y_0$

  $\Longrightarrow$ Consider terminal payment ξ of the form

$$\xi = y_0 + \int_t^T f^*(Z_s^*)ds - \int_t^T Z_s^* \sigma dB_s$$

- The problem of the Principal is

$$\sup_{\xi} \mathbb{E}^{P^{a^*(Z^*)}}[U_P(X_T - \xi)]$$

- Maximizing over Z^*, it rewrites

$$\sup_{Z^*} \mathbb{E}^{P^{a^*(z^*)}}[\mathcal{E}(-R_p \int_0^T \sigma(1 - Z_s^*)dB_s^{a^*(Z^*)})U_p(\int_0^T \beta(Z_s^*)ds)]$$

  with $\beta : z \mapsto a^*(z) + f^*(z) - za^*(z) - \frac{R_p}{2}\sigma^2(1-z)^2$

- Optimal $Z_.$ is deterministic and given by a constant pointwise sup z^* of β

Solution to the Holstrom and Milgrom moral hazard problem

- The optimal contract is linear in the output process X

$$\xi_T^* = y_0 - Tf^*(z^*) + z^*(X_T - Ta^*(z^*))$$

- It provides exactly his utility of reservation to the Agent

- Example : quadratic cost $k : a \mapsto \frac{ka^2}{2}$

  Optimal proportion of X in the contract

$$z^* = \frac{R_P + \frac{1}{k\sigma^2}}{R_A + R_P + \frac{1}{k\sigma^2}}$$

  Higher than in the first best case :

$$\frac{R_P}{R_A + R_P}$$

Some extensions and related literature

Sannikov : infinite horizon and intertemporal payments

- For the Agent : similar approach via BSDEs with infinite horizons
- For the Principal : mixed control and optimal stopping problem
  She looks for the optimal retirement stopping time, when she offers to the Agent its utility of continuation.
- Key idea : identify the continuation value of the Agent as a new state variable
  $\Longrightarrow$ Dynamically consistent problem, HJB equation...
- The Agent can get paid more than its reservation utility

Cvitanic et al : adverse selection via offering a menu of contracts
$\Longrightarrow$ multidimensional HJB equation

Cvitanic, Possamai, Touzi : control of the volatility process
$\Longrightarrow$ Path dependent PDEs

In general : optimal choice of the Agent via a stochastic maximum principle,
$\Longrightarrow$ fully coupled BSDEs, for which well-posedness is not clear

A Principal wishes to hire N Agents

- A Principal requires to handle an N-dimensional output process X

$$dX_t = \Sigma_t dB_t \quad \text{with} \quad \Sigma \text{ bounded and invertible}$$

- She wishes to hire N Agents

- Each Agent will be assigned to one project but he can choose to impact (positively or negatively) any project.

- The control process for Agent j is a vector a^j.
  a^{ij} is the control used by Agent j in order to impact the project i.

- The controled process X is given by

$$dX_t = b(t, a_t)dt + \Sigma_t dB_t^a$$

where B^a is a BM under $\mathbb{P}^a$ defined by $\frac{d\mathbb{P}^a}{d\mathbb{P}} = \mathcal{E}\left(-\int_0^T b(s, a_s) \cdot \Sigma_s^{-1} dB_s\right)$

Assumption

The drift $b(t,.)$ has linear growth and is C_b^1.

Agents with relative performance concerns

Men do not desire to be rich, but to be richer than other men.
J.S. Mill

The satisfaction of each Agent comes from

- His terminal payment ξ^i ;
- How well his project performed in comparison to the other Agents

The Agent optimization problem is

$$\sup_{a^{:,i}} \mathbb{E}^{\mathbb{P}^{a}} \left[U_i^A \left(\xi^i + \gamma_i (X_T^i - \bar{X}_T^{-i}) - \int_0^T k^i(s, a_s^{:,i}) ds \right) \right]$$

where $\bar{X}_T^{-i}$ represents the average output of all Agents except Agent i

- γ_i is a competition index for Agent i.
- Similar relative performance concerns as in e.g. *Espinosa-Touzi*.

- The Agent optimization problem is

$$\sup_{a^{:,i}} \mathbb{E}^{\mathbb{P}^a}\left[U_i^A\left(\xi^i + \gamma_i(X_T^i - \bar{X}_T^{-i}) - \int_0^T k^i(s, a_s^{:,i})ds\right)\right]$$

Assumption

The cost function $k(t,.)$ is C^1, has each coordinates increasing and strictly convex. There exists $C > 0$ and $\ell \geq 2$ s.t.

$$\varliminf_{\|x\|\to+\infty} \frac{\|k(t,x)\|}{\|x\|} = +\infty, \ \|k(t,x)\| \leq C\left(1 + \|x\|^\ell\right),$$

$$\text{and} \quad \|\nabla k(t,x)\| \leq C\left(1 + \|x\|^{\ell-1}\right).$$

- First best problem
  $\Longrightarrow$ The Principal chooses the payments and the actions of the Agents.

- Admissible contracts with exponential moments :

$$\mathcal{C}^{FB} := \left\{\xi,\ \mathcal{F}_T\text{-measurable with } \xi \in M^\phi(\mathbb{R}^N)\right\},$$

where $M^\phi(\mathbb{R}^N)$ is the Morse-Transue space

- We denote

$$\bar{\gamma}^{-i} := \frac{1}{N-1}\sum_{j\neq i}\gamma_j \qquad \text{and} \qquad \frac{1}{\overline{R}_A} = \frac{1}{N}\sum_{i=1}^{N}\frac{1}{R_A^i}$$

Theorem

Given the reservation utilities (U_0^i) of the Agents, the optimal first best payment is as follows :

$$\xi_{FB}^i := \frac{R_P\overline{R}_A}{R_A^i(\overline{R}_A+NR_P)}(\mathbf{1}_N+\gamma-\overline{\gamma}^-)\cdot X_T-\gamma_i\left(X_T^i-\overline{X}_T^{-i}\right)+\int_0^T k^i(s,(a^*)_s^{:,i})ds$$
$$+\frac{R_P\overline{R}_A}{R_A^i(\overline{R}_A+NR_P)}\int_0^T b(s,a_s^*)\cdot(\overline{\gamma}^- -\gamma-\mathbf{1}_N)ds-\frac{1}{R_A^i}\log(-U_0^i)$$
$$+\frac{1}{2R_A^i}\left(\frac{R_P\overline{R}_A}{\overline{R}_A+NR_P}\right)^2\int_0^T\|\Sigma_s(\overline{\gamma}^- -\gamma-\mathbf{1}_N)\|^2ds.$$

where the optimal action a_t^* is any minimizer of the map

$$a\longmapsto(\overline{\gamma}^- -\gamma-\mathbf{1}_N)\cdot b(t,a)+\mathbf{1}_N\cdot k(t,a)\ .$$

Economic interpretation

- The optimal contract is of the form :

$$(\xi^*)^i(a^*) = C_i + \frac{R_P \overline{R}_A}{R_A^i(\overline{R}_A + NR_P)}\left(\mathbf{1}_N + \gamma - \overline{\gamma}^-\right) \cdot X_T - \gamma_i \left(X_T^i - \overline{X}_T^{-i}\right),$$

  for some constant C_i.

- Each Agent is penalized with the amount $-\gamma^i(X_T^i - \bar{X}_T^{-i})$, so as to suppress the appetence for competition of the Agents.

- Moreover, each Agent is paid a positive part of each projects, the percentage depending on the risk aversion of the Agent, and of the universal vector

$$\frac{R_P \overline{R}_A}{\overline{R}_A + NR_P}\left(\mathbf{1}_N + \gamma - \overline{\gamma}^-\right).$$

- If an Agent is particularly competitive, then any Agent will receive a large part of his project

- If an Agent is not very competitive, other Agents have incitation to reduce the value of his project as much as possible.

Linear quadratic case with 2 Agents

Linear drift and quadratic cost functions :

$$b(t,a) := \begin{pmatrix} a^{11} - a^{12} \\ a^{22} - a^{21} \end{pmatrix} \quad \text{and} \quad k(t,a) := \begin{pmatrix} \frac{k^{11}}{2}\left|a^{11}\right|^2 + \frac{k^{21}}{2}\left|a^{21}\right|^2 \\ \frac{k^{22}}{2}\left|a^{22}\right|^2 + \frac{k^{12}}{2}\left|a^{12}\right|^2 \end{pmatrix}$$

$\Longrightarrow$ Optimal actions of the two Agents are

$$\text{Agent 1 :} \qquad a^{11} = \frac{1+\gamma_1-\gamma_2}{k^{11}} \qquad a^{21} = -\frac{1+\gamma_2-\gamma_1}{k^{21}}$$

$$\text{Agent 2 :} \qquad a^{12} = \frac{1+\gamma_2-\gamma_1}{k^{12}} \qquad a^{22} = -\frac{1+\gamma_1-\gamma_2}{k^{22}}$$

If Agent 1 is much more competitive than Agent 2 :

- Agent 1 will work towards his project and will also work to decrease the value of the project of Agent 2 ;
- Agent 2 will work to decrease the value of his own project and to increase the value of the project of Agent 1.

Optimal recruitment scheme

- Let consider Agents with similar reservation utilities.

  What is the optimal competition scheme between Agents from the Principal viewpoint ?

- Maximizing the value function of the Principal boils down to minimizing

$$g : (\gamma_1, \gamma_2) \mapsto (1+\gamma_1-\gamma_2)^2 \alpha_1 + (1+\gamma_2-\gamma_1)^2 \alpha_2,$$

where

$$\alpha_1 := \frac{R_P \overline{R}_A}{\overline{R}_A + 2R_P} \sigma_1^2 - \left(\frac{1}{k^{11}} + \frac{1}{k^{22}}\right), \ \alpha_2 := \frac{R_P \overline{R}_A}{\overline{R}_A + 2R_P} \sigma_2^2 - \left(\frac{1}{k^{12}} + \frac{1}{k^{21}}\right).$$

- For low working costs, $\alpha_1 + \alpha_2 \leq 0$

  The Principal would like to hire Agents with $|\gamma_1 - \gamma_2| \longrightarrow +\infty$

- For high working costs, $\alpha_1 + \alpha_2 > 0$,

  The Principal wants to hire Agents with $\gamma_1 - \gamma_2 = \dfrac{\alpha_2 - \alpha_1}{\alpha_1 + \alpha_2}$.

- More competitive Agents must work on less volatile projects.

Moral hazard with multiple Agents

- We now turn to the second best/moral hazard problem ;
- The Principal can not observe the actions of the Agents and only controls the salary ξ that he offers ;
- Similar ideas as in the BSDE scheme of proof derived for the Principal - unique Agent case ;
- Stackelberg equilibrium between the Principal and the system of Agents ;
- Nash equilibrium between all the interacting Agents.

Nash equilibrium between Agents

Definition

Given a contract ξ, a Nash equilibrium for the N Agents is an action $a^*(\xi) \in \mathcal{A}$ such that for any $i = 1, \dots, N$, we have

$$\sup_{a \in \mathcal{A}^i((a^*)^{:,-i}(\xi))} U_0^i(a, (a^*)^{:,-i}(\xi), \xi^i) = U_0^i((a^*)^{:,i}(\xi), (a^*)^{:,-i}(\xi), \xi^i).$$

- Need a rule chosen by the system of Agents for selecting collectively one Nash equilibrium between possibly several.
- For instance, choice of the equilibrium with the highest global utility of all Agents :

$$x \succeq y, \qquad \text{iff} \qquad \sum_{i=1}^{N} \mathcal{U}^i(x^i) \geq \sum_{i=1}^{N} \mathcal{U}^i(y^i),$$

- If the system of Agents is indifferent between several equilibria, the Principal chooses the best one for her

Identifying the best reaction functions

- Let the terminal payment ξ be given
- Consider Agent i given the actions a^{-i} of the others

$$U_0^i(a^{-i},\xi^i) := \sup_{a\in\mathcal{A}^i(a^{-i})} \mathbb{E}^{\mathbb{P}^{a\otimes_i a^{-i}}}\left[U_A^i\left(\xi^i + \gamma_i\left(X_T^i - \overline{X}_T^{-i}\right) - \int_0^T k^i(s,a_s)ds\right)\right].$$

- As in the unique Agent case, this leads to the consideration of BSDEs for the log target process.

$$Y_t^{i,a^{-i},\xi^i} = \xi^i + \Gamma_i(X_T) + \int_t^T \tilde{f}^{i,a^{-i}}\left(s, Z_s^{i,a^{-i},\xi^i}, a\right) ds - \int_t^T Z_s^{i,a^{-i},\xi^i}\cdot\Sigma_s dW_s$$

with $\tilde{f}^{i,a^{-i}}(t,\omega,z,a) := -\frac{R_A^i}{2}\|\Sigma(t)z\|^2 + b(t, a\otimes_i a_t^{-i}(\omega))\cdot z - k^i(t,a_t).$

- Consider the maximal solution of the BSDE, if it exists.
- Hoping for comparison results for the BSDE, let introduce

$$f^{i,a^{-i}}(t,\omega,z) := \sup_{a\in\mathcal{A}^i(a^{-i})} \tilde{f}^{i,a^{-i}}(t,\omega,z,a)$$

Theorem

There is a one-to-one correspondence between

(i) *a Nash equilibrium* $a^*(\xi) \in \mathcal{A}$ *such that for any* $i = 1, \dots, N$, *there exists some* $p > 1$ *such that*

$$\left(\mathbb{E}^{\mathbb{P}^{a^*(\xi)}} \left[-e^{-R_A^i \left(\xi^i + \Gamma_i(X_T) - \int_0^T k^i(s,(a_s^*(\xi))^{:,i})ds\right)} \middle| \mathcal{F}_t \right] \right)_{t \in [0,T]}$$

the reverse Holder inequality of order p *for* $\mathbb{P}^{a^*(\xi)}$.

(ii) *a solution* (Y, Z) *to the BSDE*

$$Y_t^\xi = \xi + \Gamma(X_T) + \int_t^T f(s, Z_s^\xi, X_s)ds - \int_t^T Z_s^\xi \cdot \Sigma_s dW_s,$$

such that in addition $Z \in \mathbb{H}^2_{\mathrm{BMO}}(\mathbb{P}, \mathcal{M}_N(\mathbb{R}))$.

The correspondence is given by, for any $i = 1, \dots, N$

$$(a^*{}_s(\xi))^{:,i} \in \underset{a \in \mathcal{A}^i((a^*)^{:,-i})}{\operatorname{argmax}} \left\{ \sum_{j=1}^N b^j(s, (a \otimes_i (a^*{}_s)^{:,-i}(s, Z_s))^{:,j}) Z_s^{ij} - k^i(s, a_s) \right\},$$

Identifying the admissible contracts

Finding a Nash equilibria reduces to solving the multidimensional quadratic BSDE.

- For small bounded terminal condition ξ, results of *Tevzadze*.
- counter examples in general, e.g. *Frei and Dos Reis*
- particular structure not valid in our setting : *Cheredito & Nam*, *Kramkov & Pulido*, *Hu & Tang*, *Jamneshan et al*, *Luo & Tangpi*

The Principal requires to offer a terminal payment ξ which produces a Nash equilibrium for the system of Agents.

$$\mathrm{NA}(\xi) := \{\text{Nash equilibria associated to } \xi \text{ satisfying the reverse Hölder cond.}\}$$

$$\mathrm{NAI}(\xi) := \{a \in \mathrm{NA}(\xi),\ a \succeq b,\ \text{for any } b \in \mathrm{NA}(\xi)\}\,.$$

Admissible contracts for the second best problem :

$$\mathcal{C}^{SB} := \Big\{\xi \in \mathcal{C}^{FB},\ \mathrm{NAI}(\xi) \text{ is non-empty}\Big\}\,.$$

- The Principal problem is

$$\sup_{\xi \in \mathcal{C}^{SB}} \sup_{a \in \mathrm{NAI}(\xi)} \mathbb{E}^{\mathbb{P}^a}\left[-e^{-R_P(X_T - \xi)\cdot \mathbf{1}_N} - \sum_{i=1}^{N} \rho_i e^{-R_A^i\left(\xi^i + \gamma_i\left(X_T^i - \overline{X}_T^{-i}\right) - \int_0^T k^i(s, a_s^{:,i})ds\right)} \right]$$

- The $(\rho_i)_i$ are the Lagrange multipliers of the participation constraints.

- For any $\xi \in \mathcal{C}^{SB}$, there exists a pair $(Y_0^\xi, Z^\xi) \in \mathbb{R}^N \times \mathbb{H}^2_{\mathrm{BMO}}(\mathbb{P}, \mathcal{M}_N(\mathbb{R}))$ s. t.

$$\xi = Y_0^\xi - \Gamma(X_T) - \int_0^T f(s, Z_s^\xi, X_s)ds + \int_0^T Z_s^\xi \cdot \Sigma_s dW_s, \ a.s. \quad (1)$$

- Optimization over the process Z^ξ.

- We know the actions a^* of the system of Agents in response to a payment $(Z_t^\xi)_t$.

- Optimal deterministic process Z^*.

Theorem

An optimal contract $\xi_{SB} \in \mathcal{C}^{SB}$ with reservation utilities $(U_0^i)_{1\leq i\leq N}$, is given by

$$\xi^i_{SB} := -\frac{1}{R_A^i}\log(-U_0^i) - \gamma_i(X_T^i - \overline{X}_T^{-i}) + \left(\int_0^T z_s^* dX_s\right)^i + \int_0^T k^i(s, (a_s^*((z_s^*)^{i,:}))^{:,i})ds + \frac{R_A^i}{2}\int_0^T \|\Sigma(z_s^*)^{:,i}\|^2 ds,$$

where the matrix $a_s^*(z) \in \mathcal{M}_N(\mathbb{R})$ is defined by

$$(a_s^*(z))^{:,i} \in \underset{a\in\mathcal{A}^i((a^*)^{:,-i})}{\operatorname{argmax}} \left\{ b(t, a\otimes_i (a_s^*(z))^{:,-i})\cdot z^{:,i} - k^i(t,a)\right\},$$

and where z_t^* is any deterministic maximizer of the map

$$z \longmapsto \left(\mathbf{1}_N + \gamma - \overline{\gamma}^-\right)\cdot b(t, a_t^*(z)) - k(t, a_t^*(z))\cdot \mathbf{1}_N - \sum_{i=1}^N \frac{R_A^i}{2}\|\Sigma(t)z^{:,i}\|^2 - \frac{R_P}{2}\left\|\Sigma\left(z^\top \mathbf{1}_N + \mathbf{1}_N + \gamma - \overline{\gamma}^-\right)\right\|^2.$$

Properties of the optimal contract

- z^* deterministic implies existence and uniqueness of solution for the optimal BSDE.
- For non-time dependent cost and drift function, the optimal contract is linear of the form

$$(\xi^*)^i(a^*) = C_i + z^* \cdot X_T - \gamma_i \left(X_T^i - \overline{X}_T^{-i} \right),$$

  for some constant C_i.
- Each Agent gets his utility of reservation.
- Each Agent is paid a fixed part of each project.
- Each Agent is penalized with the amount $-\gamma^i(X_T^i - \bar{X}_T^{-i})$, so as to suppress the appetence for competition of the Agents.
- For the linear drift/quadratic cost case, we obtain more explicit formulae where in particular

$$a^*(z) = \begin{pmatrix} \frac{z^{11}}{k^{11}} & \frac{z^{12}}{k^{12}} \\ \frac{z^{21}}{k^{21}} & \frac{z^{22}}{k^{22}} \end{pmatrix}.$$

  and z^* maximizes a 4-dimensional linear-quadratic function.

More general framework

- More general dynamics/drift for the controlled output

$$dX_t = b(t, a_t, X_t)dt + \Sigma_t dB_t^a$$

- Cost function $k(t, a_t, X_t)$;

- More general performance concerns :

$$\gamma^i(X_T^i - \bar{X}_T^{-i}) \quad \text{replaced by} \quad \Gamma^i(X_T)$$

with any linear growth function Γ^i.

- Same resolution for the Agent problem ;

- HJB characterization for the Principal problem ;

- Recovers in particular the framework of *Goukasian & Wan* with relative payments concerns.

Conclusion

- Contracting between a Principal and a system of interacting Agents ;
- Explicit treatment of a *toy example* : linear contracts
- Clarification of the BSDE methodology for these problems ;
- Mean field limit : what happens when N goes to ∞ ?
- Consideration of MacKean-Vlasov output dynamics

$$dX_t = b(t, a_t, X_t, \mathcal{L}_t^X)dt + \Sigma_t dB_t^a$$