A Stochastic Game and Free Boundary Problem

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Joint work with Xin Guo (UC Berkeley) and Wenpin Tang (UCLA)

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- **Dynamics of each player i:**

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- **Admissibility:**
  - $(\xi^{i,+}, \xi^{i,-}) \in \mathcal{U}$: measurability, adaptiveness, non-decreasing càdlàg, $\int_0^\infty e^{-\alpha_i t} d\check{\xi}_t^i < \infty$, where $\check{\xi}_t^i = \xi_t^{i,+} + \xi_t^{i,-}$

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$$\boldsymbol{F}(\check{\xi}^1, \cdots, \check{\xi}^N) \leq \boldsymbol{C}$$

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- **Objective (cost) :**

$$J^i(\boldsymbol{x}, \boldsymbol{\xi}) = \mathbb{E} \int_0^\infty e^{-\alpha^i t} \left[h^i(X_t^1, \cdots .X_t^N)dt + \lambda^i d\check{\xi}_t^i \right]$$

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- **Admissibility:**
  - $(\xi^+, \xi^-) \in \mathcal{U}_N^i$: non-decreasing càdlàg, measurability, adaptiveness, $\int_0^\infty e^{-\alpha_i t} d\check{\xi}_t^i < \infty$, where $\check{\xi}^i = \xi^{i,+} + \xi^{i,-}$
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Stochastic Game: Choice of RAC

- **Pooling game:** $\sum_{i=1}^{N} \int_0^\infty d\check{\xi}_t^i \leq y$

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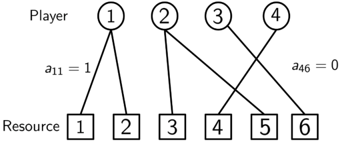
$A =$

1	1				
		1		1	
					1
			1		



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$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}$$

(a) Pooling

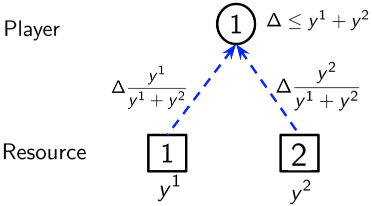
$$A = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

(b) Dividing

Stochastic Game: Choice of RAC

- **Pooling game:** $\sum_{i=1}^{N} \int_0^\infty d\check{\xi}_t^i \leq y$, $\Delta \leq y^1 + y^2$
- **Dividing game:** $\int_0^\infty d\check{\xi}_t^i \leq y^i$ $(i = 1, 2, \cdots, N)$
- **Sharing game:** N players M resources:
  - **Adjacent matrix:** $A = (a_{ij})_{1 \leq i \leq N, 1 \leq j \leq M}$, $a_{ij} = 0$ or 1
  - **Resource allocation constraint:** $(Y_t^1, \ldots, Y_t^M;\ t \geq 0)$

$$Y_t^j = y^j - \sum_{i=1}^{N} \int_0^t \frac{a_{ij} Y_{s-}^j}{\sum_{j=1}^{M} a_{ij} Y_{s-}^j} d\check{\xi}_s^i \geq 0, \quad \text{and} \quad Y_{0-}^j = y^j$$



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  - **Well-definedness:** $\sum_{j=1}^{M} a_{ij} \geq 1$

Stochastic Game: Set-up

Sharing game

$$
\begin{aligned}
J^i(\boldsymbol{x},\boldsymbol{y};\boldsymbol{\xi}) &:= \mathbb{E}\int_0^\infty e^{-\alpha^i t}(h^i(X_t^1,\cdots,X_t^N)dt+\lambda_i d\check{\xi}_t^i) \\
dX_t^i &= b^i(\boldsymbol{X}_t)dt+\boldsymbol{\sigma}^i(\boldsymbol{X}_t)d\boldsymbol{B}_t+d\xi_t^{i,+}-d\xi_t^{i-}, \qquad X_{0-}^i = x^i \\
dY_t^j &= -\sum_{i=1}^N \frac{a_{ij}Y_{t-}^j}{\sum_{k=1}^M a_{ik}Y_{t-}^k}d\check{\xi}_t^i, \qquad Y_{0-}^j = y^j
\end{aligned}
$$

- $\boldsymbol{\xi}\in\mathcal{S}(\boldsymbol{y}) := \left\{\boldsymbol{\xi}\ :\ \xi^i\in\mathcal{U},\ Y_t^j\geq 0,\ \ \forall i,j\right\}$
- $$\mathcal{U} := \{(\xi^+,\xi^-) : \xi^+ \text{ and } \xi^- \text{ are } \mathcal{F}^{\boldsymbol{X},\boldsymbol{Y}}\text{-progressively measurable, càdlàg, and non-decreasing, with } \xi_{0-}^+ = \xi_{0-}^- = 0\},$$
- h^i: convex, symmetric, $0<k\leq h''<K$, $\alpha^i>0$: discount factor

Stochastic Game: Special Case ($N = 1$)

- **Two-dimensional control problem:**

$$\begin{aligned} v(x,y) &= \inf_{\xi \in \mathcal{S}(y)} \int_0^\infty e^{-\alpha t} \left[h(X_t) dt + \lambda d\check{\xi}_t \right] \\ dX_t &= \mu(X_t) dt + \sigma(X_t) dB_t + d\xi_t^+ - d\xi_t^-, \quad X_{0-} = x \\ dY_t &= -d\xi_t^+ - d\xi_t^-, \quad Y_{0-} = y \end{aligned}$$

- **Partial references:**
  - **Finite fuel problem:** Beneš, Shepp & Witsenhausen (1980), Karatzas (1983), Ma (1993)
  - **Transaction cost analysis:** Davis & Norman (1990), Soner & Shreve (1994), Dai & Yi (2009), Kallsen & Muhle-Karbe (2010)
  - **Optimal execution/price impact:** Guo & Zervos (2015), Motairi & Zervos (2017)

Stochastic Game: Measure of Performance

- **Nash Equilibrium:** Stability
- **Pareto Optimality:** Efficiency

Definition (Nash equilibrium)

A tuple of admissible controls $\boldsymbol{\xi}^* := (\xi^{1*}, \ldots \xi^{N*})$ is a Markovian Nash equillibrium strategy (NES) of the N-player game with the cost functions $(J^1(\boldsymbol{x}, \boldsymbol{y}; \boldsymbol{\xi}), \ldots, J^N(\boldsymbol{x}, \boldsymbol{y}; \boldsymbol{\xi}))$ if for each $i = 1, \ldots, N$, and each ξ^i such that $(\boldsymbol{\xi}^{-i*}, \xi^i)$ is admissible,

$$J^i(\boldsymbol{x}, \boldsymbol{y}; \boldsymbol{\xi}^*) \leq J^i\left(\boldsymbol{x}, \boldsymbol{y}; \left(\boldsymbol{\xi}^{-i*}, \xi^i\right)\right).$$

Here the strategies ξ^{i*} and ξ^i are deterministic functions of time t and $\boldsymbol{X}_t = (X_t^1, \ldots, X_t^N)$ with $\boldsymbol{X}_{0-} = \boldsymbol{x}$.

Stochastic Game: Different Regions

Definition (Action and waiting regions)

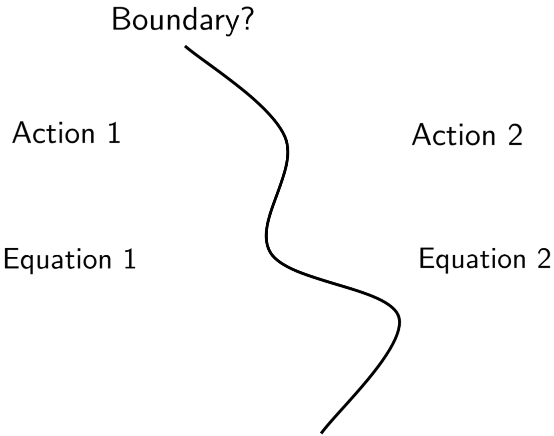
The i^{th} player's action region is

$$\mathcal{A}_i := \{(\boldsymbol{x}, \boldsymbol{y}) \in \mathbb{R}^N \times \mathbb{R}_+^M : d\xi^i(\boldsymbol{x}, \boldsymbol{y}) \neq 0\},$$

and the waiting region is $\mathcal{W}_i := (\mathbb{R}^N \times \mathbb{R}_+^M) \setminus \mathcal{A}_i$.

- $\mathcal{W}_{-i} := \cap_{j \neq i} \mathcal{W}_j$: common waiting region other than player i
- $\mathcal{W}_{NE}$: common waiting region of all players
- $\mathcal{W}_{NE}(\boldsymbol{y}) := \{\boldsymbol{x} \in \mathbb{R}^N : (\boldsymbol{x}, \boldsymbol{y}) \in \mathcal{W}_{NE}\}$

Stochastic Game: Free Boundary Problem



Theorem (Verification Theorem (Guo, Tang & X. (2018)))

For each $i = 1, \ldots, N$, *if* $\xi^{i*} \in \mathcal{U}$ *satisfies the following conditions, then* $\boldsymbol{\xi}^*$ *is an NES with game value* $v^i(\cdot) = J^i(\cdot; \boldsymbol{\xi}^*)$.

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(3) $v^i(\cdot)$ *satisfies the HJB equation*

$$\min_{(x^i, \boldsymbol{y}) \in \mathbb{R} \times \mathbb{R}_+^M} \Big\{ -\alpha v^i + h^i + \mathcal{L} v^i, -\sum_{j=1}^M \frac{a_{ij} y^j}{\sum_{k=1}^M a_{ik} y^k} v^i_{y^j} + v^i_{x^i}, \\ -\sum_{j=1}^M \frac{a_{ij} y^j}{\sum_{k=1}^M a_{ik} y^k} v^i_{y^j} - v^i_{x^i} \Big\} = 0, \ \text{for } (\boldsymbol{x}, \boldsymbol{y}) \in \mathcal{W}_{-i},$$

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Theorem (Verification Theorem (cont'))

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(6) $v^i(\boldsymbol{x}, \boldsymbol{y}) \in \mathcal{C}^2(\overline{\mathcal{W}_{-i}})$, *and there exists* $u^i(\boldsymbol{x}, \boldsymbol{y}) \in \mathcal{C}^2(\mathbb{R}^N \times \mathbb{R}^M_+)$ *convex such that* $u^i(\boldsymbol{x}, \boldsymbol{y}) = v^i(\boldsymbol{x}, \boldsymbol{y})$ *for all* $(\boldsymbol{x}, \boldsymbol{y}) \in \overline{\mathcal{W}_{-i}}$,

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(7) $v^i_{x^j}$ *is bounded in* $\overline{\mathcal{W}_{-i}}$ *for each* $j = 1, 2, \cdots, N$,

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(7) $v^i_{x^j}$ *is bounded in* $\overline{\mathcal{W}_{-i}}$ *for each* $j = 1, 2, \cdots, N$,

(8) $\xi^i \in \mathcal{U}$ *such that* $(\boldsymbol{\xi}^{-i*}, \xi^i) \in \mathcal{S}(\boldsymbol{y})$,

$$\mathbb{P}((\boldsymbol{X}_t^{-i*}, X_t^i, \boldsymbol{Y}_t) \in \overline{\mathcal{W}_{-i}},\ \forall t \geq 0) = 1.$$

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Stochastic Game: Different Scenarios

We focus on:

1		1	1
	1		1
		1	1
1		1	1

(a) Sharing game $N = M$

1	1	1	1
1	1	1	1
1	1	1	1
1	1	1	1

(b) Special case: pooling game

1			
	1		
		1	
			1

(c) Special case: dividing game

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Pooling: Set-up

Pooling game (on $\mathbb{R}^N \times \mathbb{R}_+$)

$$
\begin{aligned}
J^i(\mathbf{x}, y; \boldsymbol{\xi}) &:= \mathbb{E}\int_0^\infty e^{-\alpha t} h(X_t^i - \overline{X}_t)dt \\
dX_t^i &= dB_t^i + d\xi_t^{i,+} - d\xi_t^{i-}, \quad X_{0-}^i = x^i \\
dY_t &= -\sum_{i=1}^N d\check{\xi}_t^i, \qquad Y_{0-} = y
\end{aligned}
$$

- $\overline{X}_t = \frac{\sum_{i=1}^N X_t^i}{N}$: mean position
- h: convex, symmetric, $0 < k \leq h'' < K$
- $\alpha > 0$: discount factor
- **Constraints:**
  - **Zero-borrowing:** $Y_t \geq 0$ for all t a.s.
  - **No simultaneous jump:** $\mathbb{P}(d\xi_t^i d\xi_t^j \neq 0) = 1,\ i \neq j$

Pooling: Solution Derivation

Step 1: HJB system for N players

Step 2: Candidate solution of game value

Step 3: NE strategies via

- Skorokhod problem
- Sequential jumps at time 0

Step 1: HJB System

HJB system (on $\mathbb{R}^N \times \mathbb{R}_+$)

$$\min_{(x^i,y)\in\mathbb{R}\times\mathbb{R}_+}\left\{-\alpha v^i + h + \frac{1}{2}\sum_{j=1}^{N} v^i_{x^j x^j}, -v^i_y + v^i_{x^i}, -v^i_y - v^i_{x^i}\right\} = 0$$

$$\text{in } \mathcal{W}_{-i}$$

- First equation. Player i solves a usual control problem with three choices

Step 1: HJB System

HJB system (on $\mathbb{R}^N \times \mathbb{R}_+$)

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$$\text{in } \mathcal{A}_j, j \neq i$$

- Second equation. If player j intervenes, by the definition of Nash equilibrium, we expect that player i has no incentive to move

Step 1: HJB System

HJB system (on $\mathbb{R}^N \times \mathbb{R}_+$)

$$\mathcal{A}_i \cap \mathcal{A}_j = \emptyset$$

- Third equation. No simultanuous jump

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HJB system (on $\mathbb{R}^N \times \mathbb{R}_+$)

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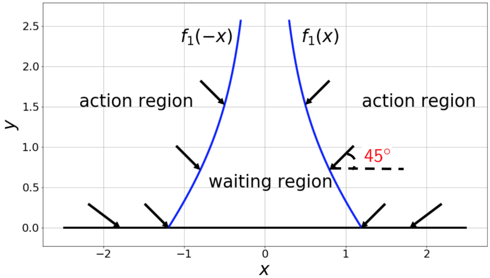
$$\mathcal{A}_i \cap \mathcal{A}_j = \emptyset$$

- First equation. Player i solves a usual control problem with three choices
- Second equation. If player j intervenes, by the definition of Nash equilibrium, we expect that player i has no incentive to move
- Third equation. No simultanuous jump

Step 2: Game Value (Special Case $N = 1$)[1]

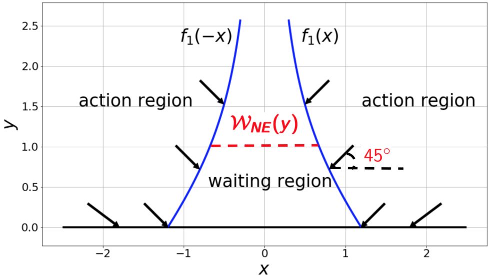
$$v(x,y) = \begin{cases} \frac{x^2}{\alpha} + \frac{1}{\alpha^2} + A_1(y)\cosh(x\sqrt{2\alpha}) & \text{if} \quad |x| \le f_1^{-1}(y) \\ v(x_+, f_1(x_+)) & \text{if} \quad x > f_1^{-1}(y) \\ v(x_-, f_1(x_-)) & \text{if} \quad x < -f_1^{-1}(y) \end{cases}$$



[1] Beneš, Shepp and Witsenhausen (1980)

Step 2: Game Value (Special Case $N = 1$)[1]

$$v(x,y) = \begin{cases} \frac{x^2}{\alpha} + \frac{1}{\alpha^2} + A_1(y)\cosh(x\sqrt{2\alpha}) & \text{if} \quad |x| \leq f_1^{-1}(y) \\ v(x_+, f_1(x_+)) & \text{if} \quad x > f_1^{-1}(y) \\ v(x_-, f_1(x_-)) & \text{if} \quad x < -f_1^{-1}(y) \end{cases}$$



[1] Beneš, Shepp and Witsenhausen (1980)

Step 2: Boundary of Free Boundary Problem

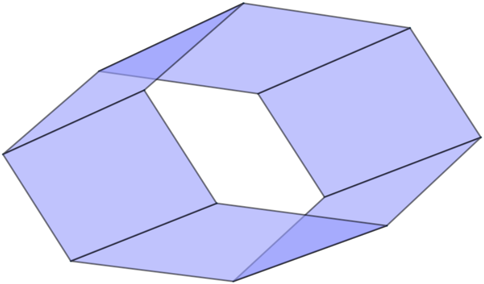


Figure: $\mathcal{W}_{NE}(y)$ when $N = 3$

Step 2: Candidate Game Value

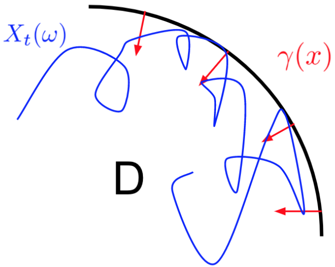
Candidate game value (Guo, Tang & X. (2018))

$$v^i(\boldsymbol{x}, y) = \begin{cases} p_N(\widetilde{x}^i) + A_N(y)\cosh(\widetilde{x}^i\sqrt{\frac{2(N-1)\alpha}{N}}) & \text{in } \mathcal{W}_i \\ v^i\left(\boldsymbol{x}^{-i}, x^i_+ + \frac{\sum_{k\neq i} x^k}{N-1}, f_N(x^i_+)\right) & \text{in } \mathcal{A}_i^+ \\ v^i\left(\boldsymbol{x}^{-i}, \frac{\sum_{k\neq i} x^k}{N-1} - x^i_-, f_N(x^i_-)\right) & \text{in } \mathcal{A}_i^- \\ v^i\left(\boldsymbol{x}^{-j}, x^j_+ + \frac{\sum_{k\neq j} x^k}{N-1}, f_N(x^j_+)\right) & \text{in } \mathcal{A}_j^+, j \neq i \\ v^i\left(\boldsymbol{x}^{-j}, \frac{\sum_{k\neq j} x^k}{N-1} - x^j_-, f_N(x^j_-)\right) & \text{in } \mathcal{A}_j^-, j \neq i \end{cases}$$

- $\widetilde{x}^i = x^i - \frac{\sum_{j\neq i} x^j}{N-1}$, $x^i_\pm$: unique positive root of $z \mp f_N(z) = \widetilde{x}^i \mp y$
- $f_N(\cdot)$: threshold function

Step 3: NE via Skorokhod

A heuristic description of Skorokhod: Given a domain D with a vector field $\gamma(.)$ on the boundary ∂D, obliquely reflecting Brownian motion behaves infinitesimally like Brownian motion in the interior. Every time it hits ∂D , there will be a "minimum push" to keep it within the closure $\bar{D}$ of the domain and spends zero Lebesgue time on the boundary.



Step 3: NE via Skorokhod

Partial references on Skorokhod problem

- **Region**:
  - **Smooth region**: Lions and Sznitman (1984)
  - **Polyhedron**: Ruth (1987), Dai & Ruth (1996), Dupuis & Ishii (1991)
  - **Nonsmooth region**: Taska (1992)
  - **Time-dependent domain**: Burdzy, Kang & Ramanan (2007), Burdzy, Chen & Sylvester (2004)
- **Reflection direction**:
  - **Oblique reflection**: Constantini (1991), Burdzy, Chen, Marshall, Ramanan (2015)
- **Dynamics**:
  - **BSDE**: Ma & Zhang (2005)
  - **Discontinuous dynamic**: Ma (1994)

Step 3: NE via Skorokhod

Ingredient 1: common waiting region (unbounded)

$$\mathcal{W}_{NE}(y) = \{\mathbf{x} \in \mathbb{R}^N : |\tilde{x}^i| < f_N^{-1}(y) \text{ for } 1 \leq i \leq N\}$$

$$F_i(y) = \{\mathbf{x} \in \mathbb{R}^N : \frac{1}{N-1}(-\mathbf{1} + N\mathbf{e}_i) \cdot \mathbf{x} = f_N^{-1}(y)\} \cap \overline{\mathcal{W}_{NE}(y)}$$

$$F_{N+i}(y) = \{\mathbf{x} \in \mathbb{R}^N : \frac{1}{N-1}(-\mathbf{1} + N\mathbf{e}_i) \cdot \mathbf{x} = -f_N^{-1}(y)\} \cap \overline{\mathcal{W}_{NE}(y)}$$

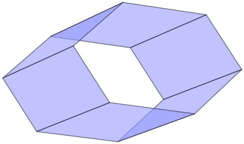


Figure: $\mathcal{W}_{NE}(y)$ when $N = 3$

Step 3: NE via Skorokhod

Ingredient 2: reflection direction

$\gamma(\mathbf{x}) = -\mathbf{e}_i$ on $F_i(y)$
$\gamma(\mathbf{x}) = \mathbf{e}_i$ on $F_{i+N}(y)$, $i = 1, 2, \cdots, N$

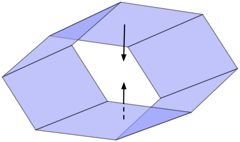


Figure: Reflection direction when $N = 3$

Step 3: NE via Skorokhod

Ingredient 1: common waiting region (unbounded)

$$\mathcal{W}_{NE}(y) = \{\boldsymbol{x} \in \mathbb{R}^N : |\widetilde{x}^i| < f_N^{-1}(y) \text{ for } 1 \leq i \leq N\}$$

$$F_i(y) = \{\boldsymbol{x} \in \mathbb{R}^N : \frac{1}{N-1}(-\boldsymbol{1} + N\boldsymbol{e}_i) \cdot \boldsymbol{x} = f_N^{-1}(y)\} \cap \overline{\mathcal{W}_{NE}(y)}$$

$$F_{N+i}(y) = \{\boldsymbol{x} \in \mathbb{R}^N : \frac{1}{N-1}(-\boldsymbol{1} + N\boldsymbol{e}_i) \cdot \boldsymbol{x} = -f_N^{-1}(y)\} \cap \overline{\mathcal{W}_{NE}(y)}$$

Ingredient 2: reflection direction

$\boldsymbol{\gamma}(\boldsymbol{x}) = -\boldsymbol{e}_i$ on $F_i(y)$
$\boldsymbol{\gamma}(\boldsymbol{x}) = \boldsymbol{e}_i$ on $F_{i+N}(y)$, $i = 1, 2, \cdots, N$

Ingredient 3: dynamic without control

$$\boldsymbol{X}_t = \boldsymbol{B}_t$$

Step 3: NE via Skorokhod

Lemma (Skorokhod solution given y)

For fixed $y > 0$, there exists a reflected process $\boldsymbol{R}_y(t) = (R_y^1(t), \dots, R_y^N(t))$ *with* $\boldsymbol{R}_y(0) = \boldsymbol{x} \in \overline{\mathcal{W}_{NE}(y)}$ *such that*

$$R_y^i(t) = x^i + B^i(t) + \eta_y^i(t) - \eta_y^{i+N}(t) \in \overline{\mathcal{W}_{NE}(y)} \quad \text{for } 1 \le i \le N,$$

where $(j = 1, 2, \cdots, 2N)$

- $(\eta_y^j(t);\ t \ge 0)$ *is the local time process on the boundary*
- η_y^j *increases only at times t such that* $R_y^j(t) \in F_j(y)$

Key idea:

- Skew symmetry condition for bounded polyhedron in Ruth Williams (1987)
- Localization argument

Step 3: NE via Skorokhod

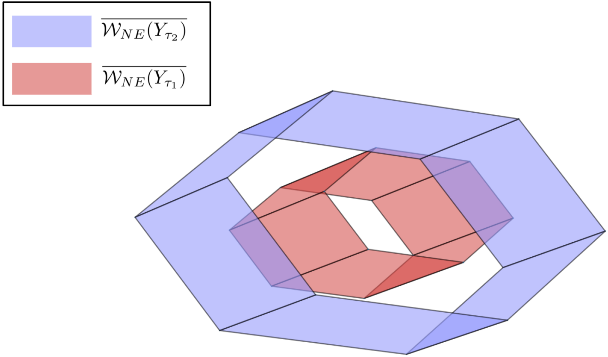


Figure: Pooling: evolving domain when $N = 3$

Step 3: NE via Skorokhod

Theorem (Skorokhod solution (Guo, Tang & X. 2018))

Inductively, for $k \geq 2$, let

$$\tau_k := \inf\left\{t > \tau_{k-1} : \boldsymbol{R}_{Y_{\tau_{k-1}}}(t - \tau_{k-1}) \in \partial\mathcal{W}_{NE}(Y_{\tau_{k-1}})\right\},$$

where $\boldsymbol{R}_{Y_{\tau_{k-1}}}$ is a copy of the reflected process in $\mathcal{W}_{NE}(Y_{\tau_{k-1}})$, starting at $\boldsymbol{X}_{\tau_{k-1}}$ and driven by $\boldsymbol{B}_k = (B_k^1, \ldots, B_k^N)$. Then we have for $\tau_{k-1} \leq t \leq \tau_k$,

$$X_t^i = X_{\tau_{k-1}}^i + B_k^i(t - \tau_{k-1}) + \eta^i_{Y_{\tau_{k-1}}}(t - \tau_{k-1}) - \eta^{i+N}_{Y_{\tau_{k-1}}}(t - \tau_{k-1}),$$

and

$$Y_t = Y_{\tau_{k-1}} - \eta^i_{Y_{\tau_{k-1}}}(t - \tau_{k-1}) - \eta^{i+N}_{Y_{\tau_{k-1}}}(t - \tau_{k-1})$$

are the NE strategies.

Step 3: Sequential Jumps at Time 0

$\mathcal{A}_i$ is defined in the way

- Player who is *furtherest away* controls
- Player with the *largest index* will control if ties occur

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Dividing Game: Set-up

Dividing game

$$
\begin{aligned}
J^i(\mathbf{x}, \mathbf{y}; \xi) &= \mathbb{E}\int_0^\infty e^{-\alpha t} h(X_t^i - \overline{X}_t)dt \\
dX_t^i &= dB_t^i + d\xi_t^{i,+} - d\xi_t^{i-}, \quad X_{0-}^i = x^i \\
dY_t^i &= -d\check{\xi}_t^i, \qquad Y_{0-}^i = y^i
\end{aligned}
$$

- $\overline{X}_t = \frac{\sum_{i=1}^N X_t^i}{N}$: mean position
- h: convex, symmetric, $0 < k \leq h'' < K$
- $\alpha > 0$: discount factor
- **Constraints:**
  - **Zero-borrowing:** $Y_t^i \geq 0$ for all t a.s. and i
  - **No simultaneous jump:** $\mathbb{P}(d\xi_t^i d\xi_t^j \neq 0) = 1$, $i \neq j$

Dividing Game: HJB System

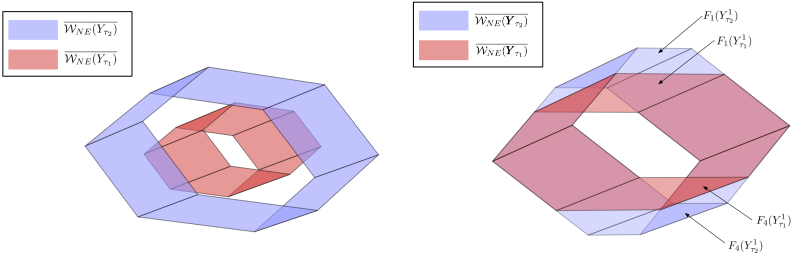
Dividing: HJB system

$$\begin{cases} \min_{(x^i,y^i)\in\mathbb{R}\times\mathbb{R}_+} \left\{ -\alpha v^i + h + \frac{1}{2}\sum_{j=1}^{N} v^i_{x^j x^j}, -v^i_{y^i} + v^i_{x^i}, -v^i_{y^i} - v^i_{x^i} \right\} = 0, \\ \qquad\qquad\qquad\qquad \text{for } (\boldsymbol{x},\boldsymbol{y}) \in \mathcal{W}_{-i}, \\ \min_{(x^j,y^j)\in\mathbb{R}\times\mathbb{R}_+} \left\{ -v^i_{y^j} + v^i_{x^j}, -v^i_{y^j} - v^i_{x^j} \right\} = 0, \\ \qquad\qquad\qquad\qquad \text{for } (\boldsymbol{x},\boldsymbol{y}) \in \mathcal{A}_j, j \neq i, \\ \mathcal{A}_i \cap \mathcal{A}_j = \emptyset. \end{cases}$$

Pooling: HJB system

$$\begin{cases} \min_{(x^i,y)\in\mathbb{R}\times\mathbb{R}_+} \left\{ -\alpha v^i + h + \frac{1}{2}\sum_{j=1}^{N} v^i_{x^j x^j}, -v^i_{y} + v^i_{x^i}, -v^i_{y} - v^i_{x^i} \right\} = 0, \\ \qquad\qquad\qquad\qquad \text{for } (\boldsymbol{x},y) \in \mathcal{W}_{-i}, \\ \min_{(x^j,y)\in\mathbb{R}\times\mathbb{R}_+} \left\{ -v^i_{y} + v^i_{x^j}, -v^i_{y} - v^i_{x^j} \right\} = 0, \\ \qquad\qquad\qquad\qquad \text{for } (\boldsymbol{x},y) \in \mathcal{A}_j, j \neq i, \\ \mathcal{A}_i \cap \mathcal{A}_j = \emptyset. \end{cases}$$

Dividing Game: NE Strategies



(a) Pooling

(b) Dividing

Figure: Comparison of NE Strategies when $N = 3$

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Sharing Game: Set-up

Sharing game

$$
\begin{aligned}
J^i(\boldsymbol{x}, \boldsymbol{y}; \boldsymbol{\xi}) &:= \mathbb{E} \int_0^\infty e^{-\alpha t} h(X_t^i - \overline{X}_t) dt \\
dX_t^i &= dB_t^i + d\xi_t^{i,+} - d\xi_t^{i-}, \qquad X_{0-}^i = x^i \\
dY_t^i &= -\sum_{j=1}^N \frac{a_{ji} Y_{t-}^i}{\sum_{k=1}^N a_{jk} Y_{t-}^k} d\check{\xi}_t^j, \qquad Y_{0-}^i = y^i
\end{aligned}
$$

- $\overline{X}_t = \frac{\sum_{i=1}^N X_t^i}{N}$: mean position
- h: convex, symmetric, $0 < k \leq h'' < K$
- $\alpha > 0$: discount factor
- **Constraints:**
  - **Zero-borrowing:** $Y_t^i \geq 0$ for all t a.s. and i
  - **No simultaneous jump:** $\mathbb{P}(d\xi_t^i d\xi_t^j \neq 0) = 1$, $i \neq j$

Sharing Game: HJB

HJB system

$$\begin{cases} \min_{(x^i,\boldsymbol{y})\in\mathbb{R}\times\mathbb{R}^N_+} \Big\{ -\alpha v^i + h + \frac{1}{2}\sum_{j=1}^N v^i_{x^j x^j}, -\sum_{j=1}^N \frac{a_{ij}y^j}{\sum_{j=1}^N a_{ij}y^j} v^i_{y^j} + v^i_{x^i}, \\ \qquad -\sum_{j=1}^N \frac{a_{ij}y^j}{\sum_{j=1}^N a_{ij}y^j} v^i_y - v^i_{x^i} \Big\} = 0, \\ \qquad \text{for } (\boldsymbol{x},\boldsymbol{y}) \in \mathcal{W}_{-i}, \\ \min_{(x^j,\boldsymbol{y})\in\mathbb{R}\times\mathbb{R}^N_+} \Big\{ -\sum_{k=1}^N \frac{a_{jk}y^k}{\sum_{s=1}^N a_{js}y^s} v^i_{y^k} + v^i_{x^j}, \\ \qquad -\sum_{k=1}^N \frac{a_{jk}y^k}{\sum_{s=1}^N a_{js}y^s} v^i_{y^k} - v^i_{x^j} \Big\} = 0 \\ \qquad \text{for } (\boldsymbol{x},\boldsymbol{y}) \in \mathcal{A}_j, j \neq i, \\ \mathcal{A}_i \cap \mathcal{A}_j = \emptyset. \end{cases}$$

Sharing Game: Game Value

Game value of sharing(Guo, Tang & X. (2018))

$$v^i(\boldsymbol{x}, y) = \begin{cases} p_N(\widetilde{x}^i) + A_N(y)\cosh(\widetilde{x}^i\sqrt{\frac{2(N-1)\alpha}{N}}) & \text{in } \mathcal{W}_i \\ v^i\left(\boldsymbol{x}^{-i}, x^i_+ + \frac{\sum_{k\neq i} x^k}{N-1}, f_N(x^i_+)\right) & \text{in } \mathcal{A}_i^+ \\ v^i\left(\boldsymbol{x}^{-i}, \frac{\sum_{k\neq i} x^k}{N-1} - x^i_-, f_N(x^i_-)\right) & \text{in } \mathcal{A}_i^- \\ v^i\left(\boldsymbol{x}^{-j}, x^j_+ + \frac{\sum_{k\neq j} x^k}{N-1}, f_N(x^j_+)\right) & \text{in } \mathcal{A}_j^+, j \neq i \\ v^i\left(\boldsymbol{x}^{-j}, \frac{\sum_{k\neq j} x^k}{N-1} - x^j_-, f_N(x^j_-)\right) & \text{in } \mathcal{A}_j^-, j \neq i \end{cases}$$

- $\widetilde{x}^i = x^i - \frac{\sum_{j\neq i} x^j}{N-1}$, $x^i_\pm$: unique positive root of $z \mp f_N(z) = \widetilde{x}^i \mp \sum_{j=1}^N a_{ij} y^j$
- $f_N(\cdot)$: threshold function

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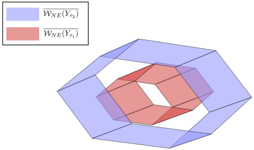
Game Comparison: NE Strategies

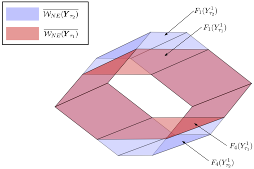
1	1	1
1	1	1
1	1	1

1		
	1	
		1

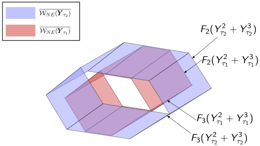
1		
	1	1
	1	1



(a) Pooling



(b) dividing



(c) Sharing

Comparison: Game Values

Proposition (Game value comparison (Guo & Tang and X. 2018))

$\forall(\boldsymbol{x},\boldsymbol{y}) \in \mathbb{R}^N \times \mathbb{R}^N_+$, *when* $y = \sum_{j=1}^N y^j$, $(\boldsymbol{x}, y) \in \mathcal{W}_i^{pool}$, *and* $(\boldsymbol{x},\boldsymbol{y}) \in \mathcal{W}_i^{share} \cap \mathcal{W}_i^{divide}$, *for each* $i = 1, 2, \cdots, N$,

$$v^i_{pool}(\boldsymbol{x}, y) \le v^i_{share}(\boldsymbol{x}, y) \le v^i_{divide}(\boldsymbol{x}, \boldsymbol{y}).$$

- Sharing has lower cost than playing selfishly
- Among all sharing strategies, pooling provides the lowest cost

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Controlled Rank-dependent SDEs

Controlled rank-dependent SDE (Guo, Tang & X. (2018))

$$
\begin{aligned}
dX_t^i &= \sum_{j=1}^{N} 1_{F^i(\boldsymbol{X}_t, \boldsymbol{Y}_t) = F^{(j)}(\boldsymbol{X}_t, \boldsymbol{Y}_t)} \left(b_j dt + \sigma_j dB_t^j + \lambda^{j,+} d\xi_t^{j,+} - \lambda^{j,-} d\xi_t^{j,-} \right) \\
Y_t^i &= Y_0^i - \xi_t^{i,+} - \xi_t^{i,-} \quad \text{for } 1 \le i \le N
\end{aligned}
$$

- Pooling game: $F^i(\boldsymbol{x}, \boldsymbol{y}) = |x_i - \frac{\sum_{j \neq i} x_j}{N-1}|$
- Dividing game: $F^i(\boldsymbol{x}, \boldsymbol{y}) = |x_i - \frac{\sum_{j \neq i} x_j}{N-1} - f_N^{-1}(y^i)|$
- Sharing game: $F^i(\boldsymbol{x}, \boldsymbol{y}) = |x_i - \frac{\sum_{j \neq i} x_j}{N-1} - f_N^{-1}(\sum_{i=1}^{j} a_{ij} y^j)|$
- $F^{(1)} \le \ldots \le F^{(N)}$: the order statistics of $(F^i)_{1 \le i \le N}$

Controlled Rank-dependent SDEs

Controlled rank-dependent SDEs

$$\begin{aligned} dX_t^i &= \sum_{j=1}^{N} 1_{F^i(\mathbf{X}_t,\mathbf{Y}_t)=F^{(j)}(\mathbf{X}_t,\mathbf{Y}_t)} \left(b_j dt + \sigma_j dB_t^j + \lambda^{j,+} d\xi_t^{j,+} - \lambda^{j,-} d\xi_t^{j,-} \right) \\ Y_t^i &= Y_0^i - \xi_t^{i,+} - \xi_t^{i,-} \quad \text{for } 1 \le i \le N \end{aligned}$$

- F^i : rank function depends on both $\boldsymbol{X}$ and $\boldsymbol{Y}$
- $F^{(1)} \le \ldots \le F^{(N)}$: the order statistics of $(F^i)_{1 \le i \le N}$
- $b_i \in \mathbb{R}$, $\sigma_i \ge 0$
- $(\xi^{i,+}, \xi^{i,-})$: the controls

Controlled Rank-dependent SDEs

Controlled rank-dependent SDEs

$$dX_t^i = \sum_{j=1}^{N} 1_{F^i(\boldsymbol{X}_t, \boldsymbol{Y}_t) = F^{(j)}(\boldsymbol{X}_t, \boldsymbol{Y}_t)} \left(\delta_j dt + \sigma_j dB_t^j + d\xi_t^{j,+} - d\xi_t^{j,-} \right)$$

$$Y_t^i = Y_0^i - \xi_t^{i,+} - \xi_t^{i,-} \quad \text{for } 1 \le i \le N$$

- $F^i(\boldsymbol{X}_t, \boldsymbol{Y}_t) = x^i$ and $\lambda^{i,+} = \lambda^{i,-} = 0$: rank-dependent SDE
  - **"Up the River problem":** Aldous (2002)
  - **Stochastic portfolio:** Fernholz (2002)
  - **Atlas model ($\delta^1 = 1$, $\delta^2 = \cdots = \delta^N = 0$):** Banner, Fernholz and Karatzas (2005), Ichiba, Karatzas and Shkolnikov (2013), Pal and Pitman (2008), Cabezas, Dembo and Sarantsev (2017), Tang and Tsai (2018)

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No Resource Alocation Constraint

- **Single agent (fuel follower):** Beneš, Shepp and Witsenhausen (1980), Karatzas (1983), Bayraktar (2007)
- **Stochastic games:**
  - **NE with finite players and MFG:** Guo & X. (2018)
  - **Pareto optimality:** Guo & X. (2018)

Thank you!

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