Integration by parts and Quasi-invariance of Horizontal Wiener Measure on a foliated compact manifold

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(Based on joint works with Fabrice Baudoin and Maria Gordina)
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Outline of the Talk

- Motivation: Sub-Riemannian Structure $\leftrightarrow$ Math Finance.
  - Greek evaluation with Malliavin calculus.
  - Large Deviation Principle. (variational representation)
- Totally geodesic Riemannian foliations.
- Quasi-invariance of horizontal Wiener measure.
  - General idea.
  - Main theorem.
  - Examples.
- Clark-Ocone formula and Integration by parts formulas.
- Functional inequalities.
- Applications and future work.

Motivation of the talk.

Motivation I: Greek evaluation with Malliavin calculus

For Black-Scholes model: $dS_t^x = rS_t^x dt + \sigma S_t^x dB_t$ (GBM).
Consider the European option with payoff function: $\phi = f(S_T)$.
The option price is: $V_0 = \mathbb{E}^Q(e^{-rT} f(S_T))$.
The Delta is: $\Delta := \frac{\partial V_0}{\partial S_0} = \nabla \mathbb{E}(e^{-rT} f(S_T))$

$$\frac{\partial V_0}{\partial S_0} \underbrace{=}_{\nabla \leftrightarrow \mathbb{E}} \mathbb{E}(e^{-rT} f'(S_t) \frac{dS_T}{dS_0}) \underbrace{=}_{I.B.P} e^{-rT} \mathbb{E}(f(S_T) \delta(\frac{dS_T}{dS_0}(DS_T)^{-1}))$$

$$\mathbb{E}(f'(F)G) = \mathbb{E}(f(F)H(F,G)), \ H(H,G) = \delta(Gh_t(D_\gamma F)^{-1})$$

$$DS_T = \int_0^T D_t S_T dt = \sigma S_T T, \quad \frac{dS_T}{dS_0} = \frac{S_T}{S_0}, \quad \delta(1) = B_T$$

$$\Delta = \frac{e^{-rT}}{S_0 \sigma T} \mathbb{E}(f(S_T) B_T) \Rightarrow \text{Monte Carlo.}$$

Elliptic case (**BS**): Fournié-Lasry-Lebuchoux-Lions-Touzi, 99', 01' Finance and Stochastics.

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Elliptic case (**BS**): Fournié-Lasry-Lebuchoux-Lions-Touzi, 99', 01' Finance and Stochastics.

$$??? \quad dX_t^x = b(X_t^x)dt + \sum_{i=1}^{2} \sigma_i(X_t^x) \circ dB_t^i, \quad X_t^x \in \mathbb{R}^3, [\sigma_1, \sigma_2] = \frac{\partial}{\partial z}.$$

Motivation II: More application with Malliavin calculus

Integration by parts formula $\leftrightarrow$ Clark-Ocone formula.

$$F = \mathbb{E}[F] + \int_0^T \mathbb{E}[D_t F | \mathcal{F}_t] dW(t)$$

$F = f(X_{t_1}, \cdots, X_{t_n})$, what if X_t is the solution of the Purple SDE.?

Application of Clark-Ocone formula.

- Karatzas-Ocone-Li, 1991. extension of Clark-Ocone formula.
- Karatzas-Ocone, 1991. Optimal portfolio.

what about? $\quad dX_t^x = b(X_t^x)dt + \sum_{i=1}^{d} \sigma_i(X_t^x) \circ dB_t^i, \quad X_t^x \in \mathbb{R}^n.$

$span\{\sigma_1, \cdots, \sigma_d, [\sigma_i, \sigma_j], [\sigma_i, [\cdots[\sigma_j, \sigma_k]]]\} = \mathbb{R}^n.$

$d < n.$

Motivation III: large deviation principle (variational representation)

Variational representation: Boué-Dupuis, 98'. Ann. Prob.

$$-\log \mathbb{E} e^{-f(W)} = \inf_{v} \mathbb{E}\{\frac{1}{2}\int_0^1 \|v_s\|^2 ds + f(W + \int_0^{\cdot} v_s ds)\}$$

Replace W with $X \Rightarrow$ we have LDP, explicit rate function obtained.

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Idea of the proof

- Girsanov theorem
- Relative entropy function.
- what about the soluiton to Purple SDE? i.e. hypoelliptic.
- Ben Arous-Deuschel 94' CPAM.(Hypoelliptic).
  Baxendale-Strook, 88' PTRF. (manifold)
- Deuschel-Friz-Jacquier-Violante, 13', 14' CPAM. Stochastic Volatility, hypoelliptic.

sub-Riemannian Geometry.

What is a foliation?

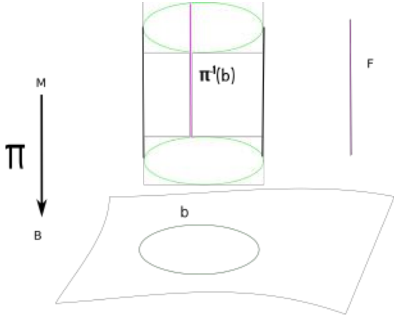
Course description from IHP (07') for Steven Hurder

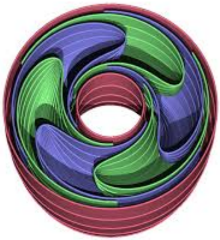
A foliation is like an onion: you just peel space back layer by layer to see what's there. The word foliation is related to the French term feuilletage,as in the very tasty pastry mille feuilles.

Riemannian submersion

Definition

A smooth surjective map $\pi : (\mathbb{M}, g) \to (\mathbb{B}, j)$ is called a Riemannian submersion if its derivative maps $T_x\pi : T_x\mathbb{M} \to T_x\mathbb{B}$ are orthogonal projections, i.e. for every $x \in \mathbb{M}$, the map $(T_{p(x)\pi})^* T_{p(x)}\pi : T_{p(x)}\mathbb{B} \to T_{p(x)}\mathbb{B}$ is the identity. A Riemannian submersion is said to have **totally geodesic fibers** if fore every $b \in \mathbb{B}$, the set $\pi^{-1}(b)$ is a totally geodesic submanifold of $\mathbb{M}$.

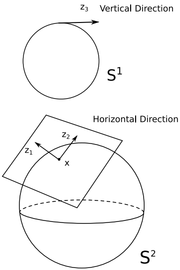




Totally geodesic foliations.

In general, for a manifold $\mathbb{M}^{n+m}$, we can have m-dimensional totally geodesic leaves and n-dimensional horizontal direction. The sub-Riemannian structure is $(g_{\mathcal{H}}, \mathcal{H})$

- Hopf fibration $U(1) \to \mathbb{S}^{2n+1} \to \mathbb{C}P^n$.
- Riemannian submersion, $\pi : \mathbb{M}^{n+m} \to \mathbb{B}^n$.
- K-contact manifold, foliated by Reeb vector field.
- Sasakian manifold
- Generalized Hopf fibration.
- ...



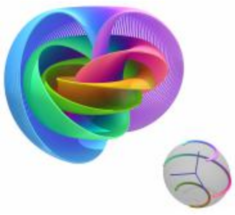


Figure: Hopf fibration: Wikipedia

Brownian motion on totally geodesic foliations.

Construction of BM on a Riemannian manifold.

- Isometry $u : (\mathbb{R}^{n+m}, \langle \cdot, \cdot \rangle) \to (T_x M, g)$: $e_i \longmapsto u(e_i)$
- Orthonormal frame bundle $\mathcal{O}(\mathbb{M}) = \cup_{x \in \mathbb{M}} \mathcal{O}(\mathbb{M})_x$.
- Canonical projection $\pi : \mathcal{O}(\mathbb{M}) \to \mathbb{M}$ with $\pi u = x$.
- Horizontal lift $\pi_* : H_u \mathcal{O}(\mathbb{M}) \to T_x \mathbb{M}$, $\pi_* H_i(x, u) = u(e_i)$.

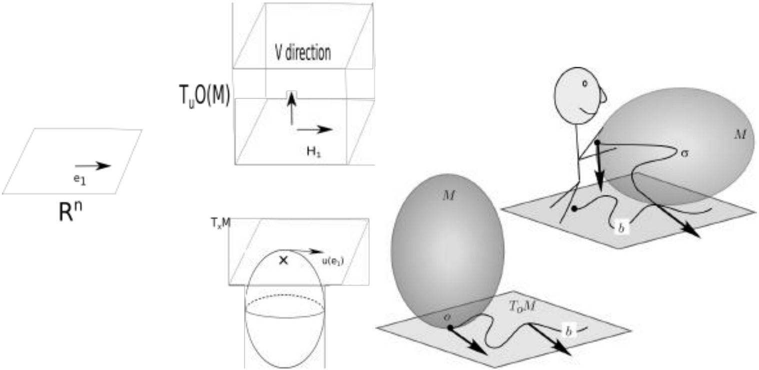


Figure: Rolling without slipping by B. K. Driver

Construction of horizontal Brownian motion on foliations.

$$dU_t = \sum_{i=1}^{n+m} H_i(U_t) \circ dB_t^i, U_0 = (x, u) \in \mathcal{O}(\mathbb{M}), \text{with } \pi u = x$$

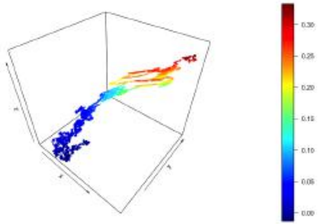
$\pi(U_t) = X_t$: a Brownian motion on Riemannian manifold $\mathbb{M}$. An isometry $u : (\mathbb{R}^{n+m}, \langle \cdot, \cdot \rangle) \longrightarrow (T_x\mathbb{M}, g)$ will be called *horizontal* if

- $u(\mathbb{R}^n \times \{0\}) \subset \mathcal{H}_x; \quad u(\{0\} \times \mathbb{R}^m) \subset \mathcal{V}_x.$
- $e_1, \cdots, e_n, f_1, \cdots, f_m$ canonical basis of $\mathbb{R}^{n+m}$
- $u(e_1), \cdots, u(e_n), u(f_1), \cdots, u(f_m)$ basis for $T\mathbb{M}$.
- $A_1, \cdots, A_n, V_1, \cdots, V_m$: lift of $u(e_i), u(f_i)$ on $T\mathcal{O}(\mathbb{M})$.
- *Horizontal frame bundle*: $\mathcal{O}_{\mathcal{H}}(\mathbb{M})$.
- $$dU_t = \sum_{i=1}^{n} A_i(U_t) \circ dB_t^i, \quad U_0 \in \mathcal{O}_{\mathcal{H}}(\mathbb{M}),$$

$W_t = \pi(U_t)$ is a horizontal Brownian motion on foliation $\mathbb{M}$.

Construction of horizontal BM on Heisenberg group: I.

$$\mathbf{B} = (B^1_{s,t}, B^2_{s,t}, \frac{1}{2}(\int_s^t B^1_{s,r} dB^2_r - \int_s^t B^2_{s,r} dB^1_r))$$



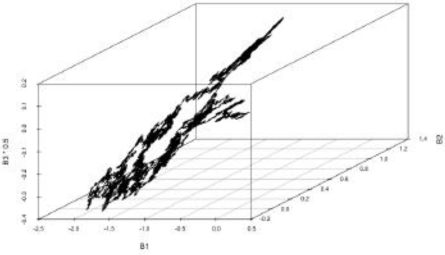
$$d\mathbf{B}_t = X \circ dB^1_t + Y \circ dB^2_t$$

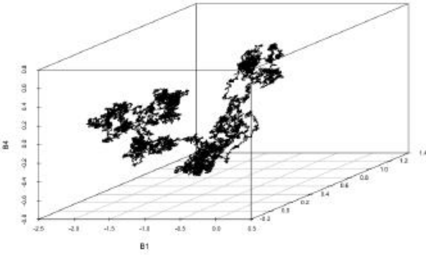
$$X = \partial_x - \frac{1}{2} y \partial_z$$

$$Y = \partial_y + \frac{1}{2} x \partial_z$$

$\sigma_1 = X,\ \sigma_2 = Y$ and $[\sigma_1, \sigma_2] = \sigma_1\sigma_2 - \sigma_2\sigma_2 = \partial_z$. Purple SDE.

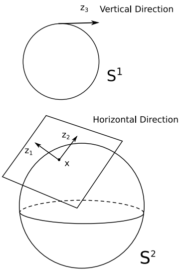
Construction of horizontal BM on Heisenberg group: II.





Horizontal BM on Totally geodesic foliations. (Hopf fibration)

$$X_t = \frac{e^{-i\theta}}{\sqrt{1+|\omega(t)|^2}}(\omega(t),1),$$ (Baudoin-Wang, PTRF17')



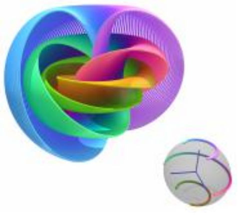


Figure: Hopf fibration: Wikipedia

Quasi-invariance of Horizontal Wiener Measure.

General idea of quasi-invariance of Wiener measure

Let μ be the Wiener measure on the path space W, and $T : W \to W$ a transformation of W. The image measure μ_T is defined by

$$\mu_T(B) = \mu(T^{-1}B)$$

In general, the transformation T can be represented as a flow, and the flow is generated by a vector field Z on the path space W.

$$\frac{dU_t^Z(x)}{dt} = Z(U_t^Z(x)), \quad U_0^Z(x) = x$$

If $(U_t^Z)_*\mu$ and μ are mutually absolutely continuous, we say μ is quasi-invariant under Z.
Radon-Nikodym derivative: $d(U_t^Z)_*\mu/d\mu$=?

Quasi-invariance of Wiener measure: progress

- R.H Cameron, W.T Martin, C-M Thm Ann. of Math. 1944.
- I.V Girsanov Girsanov Thm Theory Probab. Appl, 1960.
- L Gross Hilbert space Trans. AMS, 94 (1960).
- Bruce K. Driver Compact Riemannian manifold for C^1 C-M functions. JFA, (1992), pp. 272-376
- B.K Driver For pinned Brownian motion Trans. AMS 1994.
- Elton. P. Hsu Compact Riemannian manifold for all C-M paths. JFA (1995).
- Bruce K. Driver Heat kernel measures on loop groups, JFA (1997).
- E. P. Hsu Noncompact case JFA (2002).
- E. P. Hsu, C. Ouyang, complete case. JFA (2009).
- heat kernel measure and finite dimensional approximation
- F. Baudoin, M. Gordina and T. Melcher, heat kernel measures on infinite-dimensional Heisenberg groups. (Trans. AMS 13').

Cameron-Martin Theorem

Theorem (CM Thm: we have $U_t^Z(x) = x + tZ$.)

Let μ denote the Wiener measure on the path space $W = C_0([0,1], \mathbb{R}^n)$. Let $h = (h_1, \cdots, h_n)$ be a path in $L^2[0,1]$ and define the translation $T : \mathbb{R}^n \to \mathbb{R}^n$ by

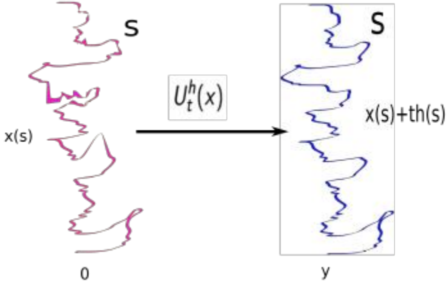
$$T(\omega) = \omega + \int_0^{\cdot} h ds,$$

$$\omega \to (\omega_1 + \int_0^{\cdot} h_1 ds, \cdots, \omega_n + \int_0^{\cdot} h_n ds)$$

Then μ_T is absolutely continuous w.r.t. μ and

$$\frac{d\mu_T}{d\mu}(\omega) = \exp(\sum_{i=1}^{n} \int_0^1 h_i(s) d\omega_i(s) - \frac{1}{2} \sum_{i=1}^{n} \int_0^1 h_i^2(s) ds)$$

Cameron-Martin Theorem and horizontal Wiener measure.



$$H = \{h \in W(\mathbb{R}^n) | \text{h is a.c. and } h(0) = 0, \int_0^T |h'(s)|^2 ds < \infty\}$$

$(W_t)_{0 \leq t \leq 1}$ is the **horizontal Brownian motion** and the law μ_W of W_t on $\mathbb{M}$ will be referred to as the horizontal Wiener measure on $\mathbb{M}$. Therefore, μ_W is a probability measure on the space $C_{x_0}(\mathbb{M})$ of continuous paths $w : [0,1] \to \mathbb{M}$, $w(0) = x_0$.

Quasi-invariance of Wiener measure on a Riemannian manifold.

- μ_X: Wiener measure on $W(\mathbb{M})$.
- $/\!/_t$: stochastic parallel translation along X, $/\!/_t : T_x\mathbb{M} \to T_{X_t}\mathbb{M}$.

$$\begin{array}{ccc} \gamma \in \mathcal{W}(\mathbb{M}) & \xrightarrow{\tilde{p}_h(\gamma)} & \mathcal{W}(\mathbb{M}) \ni \zeta^t \gamma \\ \mathcal{I}\uparrow & & \mathcal{I}\uparrow \\ \omega \in \mathcal{W}_0\left(\mathbb{R}^n\right) & \xrightarrow{p_h(\omega)} & \mathcal{W}_0\left(\mathbb{R}^n\right) \ni \xi^t \omega \end{array}$$

Not Diagram Proof

For Presentation Only!

$$\zeta^t \gamma = \mathcal{I} \circ \xi^t \omega \circ \mathcal{I}^{-1}$$

$$\frac{d\zeta^t \gamma}{dt} = Z(\zeta^t \gamma)$$

what is Z?

Quasi-invariance of Wiener measure on Riemannian manifolds.

Theorem (B. Driver 1992, E.P. Hsu 1995)

Let $h \in H(T_x\mathbb{M})$, and Z^h be the μ_X − a.e. well defined vector field on $W(\mathbb{M})$ given by

$$Z^h_s(\gamma) = /\!/_s(\gamma)h_s, \text{ for } s \in [0,1]$$

Then Z^h admits a flow e^{tZ^h} on $W(\mathbb{M})$ and the Wiener measure μ_X is quasi-invariant under the this flow.

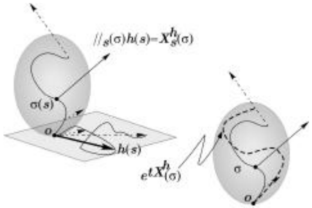


Figure: From: Bruce K. Driver, Curved Wiener Space Analysis

Quasi-invariance on compact foliated manifolds.

Theorem (Baudoin,F. and Gordina, 2017)

Let $h \in \mathcal{CM}_{\mathcal{H}}(\mathbb{R}^{n+m})$. Consider the $\mu_X^{\mathcal{H}}$-a.e. defined process given by

$$v_t^h(\omega) = h(t) + \underbrace{\int_0^t T_{\mathcal{I}_{\mathcal{H}}(\omega^{\mathcal{H}})_s}(A \circ d\omega_s^{\mathcal{H}}, Ah(s)).}_{\text{martingale part in vertical direction.}}$$

and Z^h be the $\mu_X^{\mathcal{H}} -$ a.e. well defined vector field on $W(\mathbb{M}^{n+m})$ given by

$$Z_s^h(\gamma) = \hat{\Theta}_s^{\varepsilon}(\gamma) v_s^h, \text{ for } s \in [0,1]$$

Then Z^h admits a flow e^{tZ^h} on $W(\mathbb{M}^{n+m})$ and the horizontal Wiener measure $\mu_X^{\mathcal{H}}$ is quasi-invariant under this flow.

Outline of the proof

- Recall that: $\zeta^t\gamma = \mathcal{I}\circ\xi^t\omega\circ\mathcal{I}^{-1} : W_{\mathcal{H}}(\mathbb{M}) \to W_{\mathcal{H}}(\mathbb{M})$, is generated by $\mathbf{D}_v :=$ *Directional Derivative*.
- $\xi_v^t\omega = \omega^{\mathcal{H}} + \int_0^t p_v(\xi_v^\lambda(\omega))d\lambda$.
- $p_v(t) := \mathcal{I}_*^{-1}Z^h =$
  $v_{\mathcal{H}}(t) + \frac{1}{2}\int_0^t (\mathfrak{Ric}_{\mathcal{H}})_{U_s}(Av_{\mathcal{H}}(s))ds - \frac{1}{\varepsilon}\int_0^t J_{Vv(s)}(Ad\omega_s^{\mathcal{H}})_{U_s}$
  $+ \int_0^t \left(\int_0^s \widehat{\Omega}^{\varepsilon}_{U_\tau}(A\circ d\omega_\tau^{\mathcal{H}}, Av(\tau)+Vv(\tau))\right) d\omega_s^{\mathcal{H}}$.
- For every fixed $t\in\mathbb{R}$, the law $\mathbb{P}_{\zeta_v^t}$ of ζ_v^t is equivalent to the horizontal Wiener measure $\mathbb{P}_X$ with Radon-Nikodym derivative :

$$\frac{d\mathbb{P}_{\zeta_v^t}}{d\mathbb{P}_X}(w) = \frac{d\mathbb{P}_{\xi_v^t}}{d\mathbb{P}_{\mathcal{H}}}(\mathcal{I}^{-1}w), \quad w\in W_{\mathcal{H}}(\mathbb{M}).$$

- $$\frac{d\mathbb{P}_{\xi_v^t}}{d\mathbb{P}_{\mathcal{H}}}(\omega^{\mathcal{H}}) = \exp\left[\int_0^T a_s^t(\xi^{-t}\omega)^* d\omega_s^{\mathcal{H}} - \frac{1}{2}\int_0^T |a_s^t(\xi^{-t}\omega)|^2 ds\right].$$

  where $a^t = v'_{\mathcal{H}}t + \int_0^t c(\xi^\lambda)d\lambda + \int_0^t b(\xi^\lambda)a^\lambda d\lambda$,

Quasi-invariance on foliated manifold: example

No Diagram Proof At All. For Presentation Only!

$$\begin{array}{ccccc}
\gamma \in \mathcal{W}(\mathbb{M}^{n+m}) & \xrightarrow{\hat{p}_h(\gamma)} & \mathcal{W}(\mathbb{M}^{n+m}) \ni \zeta^t\gamma & & , \hat{p}_h(\gamma) = e^{t\hat{\Theta}_s^{\varepsilon}(\gamma)v_s^h} \\
\mathcal{I}\uparrow & & \mathcal{I}\uparrow & & \\
\sigma \in \mathcal{W}(\mathbb{B}^n) & \xrightarrow{\tilde{p}_h(\sigma)} & \mathcal{W}(\mathbb{B}^n) \ni \eta^t\sigma & & , \tilde{p}_h(\sigma) = e^{t/\!/_s(\sigma)h_s} \\
\mathcal{I}\uparrow & & \mathcal{I}\uparrow & & \\
\omega \in \mathcal{W}_0(\mathbb{R}^n) & \xrightarrow{p_h(\omega)} & \mathcal{W}_0(\mathbb{R}^n) \ni \xi^t\omega & & , p_h(\omega) = \omega + th
\end{array}$$

Heisenberg group : $v_s^h(B) = (h_1(s), h_2(s), \int_0^s h_1(\tau)dB_\tau^2 - \int_0^s h_2(\tau)dB_\tau^1)$.

Hopf fibration : $v_s^h = (h_1(s), h_2(s), \int_0^s ((\widehat{\Theta}_\tau^{\varepsilon})^{-1}R)\langle J\widehat{\Theta}_\tau^{\varepsilon}h(\tau), \widehat{\Theta}_\tau^{\varepsilon}\circ dB_\tau\rangle_{\mathcal{H}})$.

Quasi-invariance of horizontal Wiener measure on Heisenberg group.

$$
\begin{aligned}
&\widetilde{p}_h(w)_t \\
=&w_t + \left(h(t), \sum_{i=1}^{n} h^i(t) w_t^{i+n} - h^{i+n}(t) w_t^i + \right. \\
&\left. \int_0^t (h^i(s) + 2w_s^i) dh^{i+n}(s) - (h^{i+n}(s) + 2w_s^{i+n}) dh^i(s) \right)
\end{aligned}
$$

Let $\mu_{\mathcal{H}}^h$ be the pushforward of $\mu_{\mathcal{H}}$ under $\widetilde{p}_h$. The density is explicitly given by

$$
\frac{d\mu_{\mathcal{H}}^h}{d\mu_{\mathcal{H}}} = \exp\left(\sum_{i=1}^{2n} \int_0^T h_i'(s) dw_s^i - \frac{1}{2} \int_0^T |h'(s)|_{\mathbb{R}^{2n}}^2 ds \right).
$$

Clark-Ocone formula and Integration by Parts formulas.

Various Gradients

Definition

For $F = f(w_{t_1}, ..., w_{t_n}) \in \mathcal{F}C^{\infty}(C_x(\mathbb{M}))$ we define

- **Intrinsic gradient (Intrinsic derivative):**

$$D_t^{\varepsilon} F = \sum_{i=1}^{n} \mathbf{1}_{[0,t_i]}(t) \Theta_{t_i}^{\varepsilon} d_i f(W_{t_1}, \cdots, W_{t_n}), \quad 0 \leq t \leq 1$$

- **Damped gradient (Damped Malliavin derivative):**

$$\tilde{D}_t^{\varepsilon} F = \sum_{i=1}^{n} \mathbf{1}_{[0,t_i]}(t) (\tau_t^{\varepsilon})^{-1} \tau_{t_i}^{\varepsilon} d_i f(X_{t_1}, \cdots, X_{t_n}), \quad 0 \leq t \leq 1.$$

- **Directional Derivative** For an $\mathcal{F}$-adapted, $T_x\mathbb{M}$-valued semimartingale $(v(t))_{0 \leqslant t \leqslant 1}$ with $v(0) = 0$,

$$\mathbf{D}_v F = \sum_{i=1}^{n} \left\langle d_i f(W_{t_1}, \cdots, W_{t_n}), \widehat{\Theta}_{t_i}^{\varepsilon} v(t_i) \right\rangle$$

Clark-Ocone formula

Clark-Ocone formula Baudoin-F.15, Baudoin-F.-Gordina17
Let $\varepsilon > 0$. Let $F = f(W_{t_1}, \cdots, W_{t_n})$, $f \in C^\infty(\mathbb{M})$. Then

$$F = \mathbb{E}_x(F) + \int_0^1 \langle \mathbb{E}_x(\tilde{D}_s^\varepsilon F | \mathcal{F}_s), \widehat{\Theta}_s^\varepsilon dB_s \rangle.$$

Clark-Ocone formula

Clark-Ocone formula Baudoin-F.15, Baudoin-F.-Gordina17

Let $\varepsilon > 0$. Let $F = f(W_{t_1}, \cdots, W_{t_n})$, $f \in C^\infty(\mathbb{M})$. Then

$$F = \mathbb{E}_x(F) + \int_0^1 \langle \mathbb{E}_x(\tilde{D}_s^\varepsilon F | \mathcal{F}_s), \widehat{\Theta}_s^\varepsilon dB_s \rangle.$$

Gradient Representation.

Let $F = f(W_{t_1}, \cdots, W_{t_n}) \in \mathcal{F}C^\infty(C_x(\mathbb{M}))$. We have

$$d\mathbb{E}_x(F) = \mathbb{E}_x\left(\sum_{i=1}^n \tau_{t_i}^\varepsilon d_i f(W_{t_1}, \cdots, W_{t_n}) \right).$$

Baudoin 14, sub-Rie transverse symmetry $dP_t f(x) = \mathbb{E}(\tau_t^\varepsilon df(X_t))$

Integration by parts formula for the damped gradient

Theorem (Baudoin-F.15, Baudoin-F.-Gordina17)

Suppose $F \in \mathcal{F}C^\infty(C_x(\mathbb{M}))$ and $\gamma \in \mathcal{CM}_{\mathcal{H}}(\mathbb{M}, \Omega)$, then

$$\mathbb{E}_x \left(\int_0^1 \langle \widetilde{D}_s^\varepsilon F, \widehat{\Theta}_s^\varepsilon \gamma'(s) \rangle ds \right) = \mathbb{E}_x \left(F \int_0^1 \langle \gamma'(s), dB_s \rangle_{\mathcal{H}} \right). \quad (2.1)$$

Definition

An $\mathcal{F}_t$-adapted absolutely continuous $\mathcal{H}_x$-valued process $(\gamma(t))_{0 \leqslant t \leqslant 1}$ such that $\gamma(0) = 0$ and $\mathbb{E}_x \left(\int_0^1 \|\gamma'(t)\|_{\mathcal{H}}^2 dt \right) < \infty$ will be called a *horizontal Cameron-Martin process*. The space of horizontal Cameron-Martin processes will be denoted by $\mathcal{CM}_{\mathcal{H}}(\mathbb{M}, \Omega)$.

Recall the Cameron-Martin space.

$$H = \{h \in W(\mathbb{R}^n) | \text{h is a.c. and } h(0) = 0, \int_0^T |h'(s)|^2 ds < \infty\}$$

Integration by parts formula for directional derivative

Theorem (Baudoin-F.-Gordina17)

Suppose $F \in \mathcal{F}C^\infty(C_x(\mathbb{M}))$ and $v \in TW_{\mathcal{H}}(\mathbb{M}, \Omega)$, then

$$\mathbb{E}_x(\mathbf{D}_v F) = \mathbb{E}_x\left(F\int_0^1 \left\langle v'_{\mathcal{H}}(t) + \frac{1}{2}(\widehat{\Theta}_t^\varepsilon)^{-1}\mathfrak{Ric}_{\mathcal{H}}\widehat{\Theta}_t^\varepsilon v_{\mathcal{H}}(t), dB_t\right\rangle_{\mathcal{H}}\right).$$

Definition

An $\mathcal{F}_t$-adapted $T_x\mathbb{M}$-valued continuous semimartingale $(v(t))_{0\leqslant t\leqslant 1}$ such that $v(0)=0$ and $\mathbb{E}_x\left(\int_0^1 \|v(t)\|^2 dt\right) < \infty$ will be called a *tangent process* if the process

$$v(t) - \int_0^t (\widehat{\Theta}_s^\varepsilon)^{-1} T(\widehat{\Theta}_s^\varepsilon \circ dB_s, \widehat{\Theta}_s^\varepsilon v(s))$$

is a horizontal Cameron-Martin process. The space of tangent processes will be denoted by $TW_{\mathcal{H}}(\mathbb{M}, \Omega)$.

Functional inequalities.

Functional inequalities

Theorem (Baudoin-F,15',)

For every cylindric function $G \in \mathcal{F}C^{\infty}\left(C_x(\mathbb{M})\right)$ we have the following log-Sobolev inequality.

$$\mathbb{E}_x(G^2 \ln G^2) - \mathbb{E}_x(G^2) \ln \mathbb{E}_x(G^2) \leq 2e^{3T(K+\frac{\kappa}{\varepsilon})}\mathbb{E}_x\left(\int_0^T \|D_s^{\varepsilon} G\|_{\varepsilon}^2 ds\right).$$

Functional inequalities

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Yang-Mills condition$\delta_{\mathcal{H}} T(\cdot) = \sum_{i=1}^n \nabla_{X_i} T(X_i, \cdot) = 0$

Theorem (Baudoin-F,15',)

- *Improved Log-Sobolev inequality*
- *Equivalent conditions: two-sided uniform Ricci curvature bound $\leftrightarrow$ Log-Sobolev inequality $\leftrightarrow$Poincare inequality $\leftrightarrow$ gradient estimates.*

Concentration inequalities.

we denote by d_ε the distance associated with g_ε

Proposition (Baudoin-F 15', under curvature bounds)

Let $\varepsilon > 0$. We have for every $T > 0$ and $r \geq 0$

$$\mathbb{P}_x\left(\sup_{0\leq t\leq T} d_\varepsilon(X_t,x) \geq \mathbb{E}_x\left[\sup_{0\leq t\leq T} d_\varepsilon(X_t,x)\right] + r\right) \leq \exp\left(\frac{-r^2}{2Te^{\left(K+\frac{\kappa}{\varepsilon}\right)T}}\right)$$

$$\Longrightarrow \lim\sup_{r\to\infty} \frac{1}{r^2} \ln \mathbb{P}_x\left(\sup_{0\leq t\leq T} d_\varepsilon(X_t,x) \geq r\right) \leq -\frac{1}{2Te^{\left(K+\frac{\kappa}{\varepsilon}\right)T}}$$

Proposition (Baudoin-F 15', K=0 and bounds.)

$$\lim\sup_{r\to+\infty} \frac{1}{r^2} \ln \mathbb{P}_x\left(\sup_{0\leq t\leq T} d(X_t,x) \geq r\right) \leq -\frac{1}{2T},$$

$$\lim\inf_{r\to+\infty} \frac{1}{r^2} \ln \mathbb{P}_x\left(\sup_{0\leq t\leq T} d(X_t,x) \geq r\right) \geq -\frac{D}{2nT},$$

Future work and applicaitons.

- Compact $\Rightarrow$ complete;
- Rough paths proof of Picard iteration;
- Path space on sub-Riemannian manifold $\Rightarrow$ Loop space on sub-Riemannian manifold;
- Totally geodesic foliations $\Rightarrow$ non totally geodesic foliations;
- Ricci flow on totally geodesic foliations, F. 17', 18'. differential Harnack inequalities, monotonicity formulas.
- Stochastic Ricci flow and random matrices.
- Characterization of Ricci flow using Path space and Wasserstein space on sub-Riemannian manifolds.

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- Stochastic Ricci flow and random matrices.
- Characterization of Ricci flow using Path space and Wasserstein space on sub-Riemannian manifolds.
- Quasi-invariance of horizontal Wiener measure. Girsanov Thm
- Is there any finance model that satisfies a Heisenberg group type stochastic differential equation? control problem.
- LDP, SV model, Malliavin calculus application.

Bibliography

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F. Baudoin, Q. Feng, and M. Gordina, *Quasi-invariance of horizontal wiener measure on a compact foliated riemannian manifold*,arXiv:1706.07040 (2017). Revision for JFA.

Thanks for your attention!

Technical part: various connections.

Bott connection

$$\nabla_X Y = \begin{cases} \pi_{\mathcal{H}}(\nabla^R_X Y), X, Y \in \Gamma^\infty(\mathcal{H}), \\ \pi_{\mathcal{H}}([X, Y]), X \in \Gamma^\infty(\mathcal{V}), Y \in \Gamma^\infty(\mathcal{H}), \\ \pi_{\mathcal{V}}([X, Y]), X \in \Gamma^\infty(\mathcal{H}), Y \in \Gamma^\infty(\mathcal{V}), \\ \pi_{\mathcal{V}}(\nabla^R_X Y), X, Y \in \Gamma^\infty(\mathcal{V}), \end{cases}$$

Isomorphism

$$g_{\mathcal{H}}(J_Z(X), Y)_x = g_{\mathcal{V}}(Z, T(X, Y))_x, Z \in \Gamma^\infty(\mathcal{V}), X, Y \in \Gamma^\infty(\mathcal{H})$$

Damped connection

$$\nabla^\varepsilon_X Y = \nabla_X Y - T(X, Y) + \frac{1}{\varepsilon} J_Y X, \quad X, Y \in \Gamma^\infty(\mathbb{M}).$$

Adjoint connection of the damped connection

$$\widehat{\nabla}^\varepsilon_X Y := \nabla^\varepsilon_X Y - T^\varepsilon(X, Y) = \nabla_X Y + \frac{1}{\varepsilon} J_X Y,$$

Technical part: Bochner-Weitzenböck identity

- $$\square_\varepsilon = -(\nabla_{\mathcal{H}} - \mathfrak{T}^\varepsilon_{\mathcal{H}})^*(\nabla_{\mathcal{H}} - \mathfrak{T}^\varepsilon_{\mathcal{H}}) - \frac{1}{\varepsilon}\mathbf{J}^2 + \frac{1}{\varepsilon}\delta_{\mathcal{H}} T - \mathfrak{Ric}_{\mathcal{H}},$$
- Let $f \in C_0^\infty(\mathbb{M})$, $x \in \mathbb{M}$ and $\varepsilon > 0$, then
  $$dLf(x) = \square_\varepsilon df(x). \quad \text{(Baudoin-Kim-Wang. 16' CGT)}$$
- Consequence
  $$dP_t f = Q_t^\varepsilon df, \; P_t = e^{tL}, \; Q_t^\varepsilon = e^{t\square_\varepsilon}$$
- Clark-Ocone formula
  $$F = \mathbb{E}_x(F) + \int_0^T \langle \mathbb{E}_x(\tilde{D}_s^\epsilon F | \mathcal{F}_s), \widehat{\Theta}^\varepsilon_{0,s} dB_s\rangle_{\mathcal{H}}.$$

Technical part: stochastic parallel translation.

$\widehat{\Theta}_t^\varepsilon : T^*_{X_t}\mathbb{M} \to T^*_x\mathbb{M}$ is a solution to

$$d[\widehat{\Theta}_t^\varepsilon \alpha(X_t)] = \widehat{\Theta}_t^\varepsilon \widehat{\nabla}^\varepsilon_{\circ dX_t}\alpha(X_t),$$

$\tau_t^\varepsilon : T^*_{X_t}\mathbb{M} \to T^*_x\mathbb{M}$ is a solution of

$$\begin{aligned} &d[\tau_t^\varepsilon \alpha(X_t)] \qquad (3.2)\\ &= \tau_t^\varepsilon \left(\nabla_{\circ dX_t} - \mathfrak{T}^\varepsilon_{\circ dX_t} - \frac{1}{2}\left(\frac{1}{\varepsilon}\mathbf{J}^2 - \frac{1}{\varepsilon}\delta_{\mathcal{H}} T + \mathfrak{Ric}_{\mathcal{H}}\right) dt \right) \alpha(X_t),\\ &\tau_0 = \mathbf{Id}, \quad \tau_t^\varepsilon = \mathcal{M}_t^\varepsilon \Theta_t^\varepsilon. \end{aligned}$$

$\mathcal{M}_t^\varepsilon : T^*_x\mathbb{M} \to T^*_x\mathbb{M}$, $t \geqslant 0$, is the solution to ODE

$$\frac{d\mathcal{M}_t^\varepsilon}{dt} = -\frac{1}{2}\mathcal{M}_t^\varepsilon \Theta_t^\varepsilon \left(\frac{1}{\varepsilon}\mathbf{J}^2 - \frac{1}{\varepsilon}\delta_{\mathcal{H}} T + \mathfrak{Ric}_{\mathcal{H}}\right) (\Theta_t^\varepsilon)^{-1}, \qquad (3.3)$$

$$\mathcal{M}_0^\varepsilon = \mathbf{Id}.$$

Technical part: example.

Given a number ρ, suppose that $G(\rho)$ is simply a connected three-dimensional Lie group whose Lie algebra $\mathfrak{g}$ admits a basis $\{X, Y, Z\}$ satisfying

$$[X, Y] = Z,\ [X, Z] = -\rho Y,\ [Y, Z] = \rho X.$$

$$\rho = 0, \varepsilon = \infty \Rightarrow \hat{\Theta}_t^{\varepsilon} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \tau_t^{\infty} = \begin{pmatrix} 1 & 0 & B_t^2 \\ 0 & 1 & -B_t^1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\rho \neq 0 \Rightarrow \tau_t^{\infty} = \begin{pmatrix} e^{-\frac{\rho}{2}t} & 0 & \int_0^t e^{-\frac{\rho}{2}s} dB_s^2 \\ 0 & e^{-\frac{\rho}{2}t} & -\int_0^t e^{-\frac{\rho}{2}s} dB_s^1 \\ 0 & 0 & 1 \end{pmatrix}$$