Robust Optimal Stopping under Volatility Uncertainty

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Our Problem

We consider a robust optimal stopping problem w.r.t. a set $\mathcal{P}$ of mutually singular probabilities on the canonical space Ω:

$$\sup_{\tau \in \mathcal{S}} \inf_{\mathbb{P} \in \mathcal{P}} \mathbb{E}_{\mathbb{P}}\big[Y_\tau\big] = \inf_{\mathbb{P} \in \mathcal{P}} \mathbb{E}_{\mathbb{P}}\big[Y_{\tau^*}\big]. \tag{1}$$

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- $\Omega:=\big\{\omega\in\mathbb{C}([0,\,T];\mathbb{R}):\omega(0)=0\big\}$ equipped with uniform norm

$$\|\omega\|=\|\omega\|_{0,T}:=\sup_{t\in[0,T]}|\omega(t)|;$$

- The *coordinator* process $B_t(\omega):=\omega(t),\ \forall\,(t,\omega)\in[0,\,T]\times\Omega$ is a Brownian motion under the *Wiener* measure $\mathbb{P}_0$ on $\big(\Omega,\mathscr{B}(\Omega)\big)$;
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- $\mathbf{F}:=\big\{\mathcal{F}_t\big\}_{t\in[0,T]}$ be the natural filtration of B (no augmentation);
- $\mathcal{S}$ denotes the set of $\mathbf{F}-$stopping times;
- The reward process Y will be specified later.

Application 1: Risk measures

The worst-case risk measure is defined by

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Applying the theorem to a reward Y yields:

$$\inf_{\tau\in\mathcal{S}} \mathfrak{R}(Y_\tau) = -\sup_{\tau\in\mathcal{S}} \inf_{\mathbb{P}\in\mathcal{P}} \mathbb{E}_{\mathbb{P}}\left[Y_\tau\right] = -\inf_{\mathbb{P}\in\mathcal{P}} \mathbb{E}_{\mathbb{P}}\left[Y_{\tau^*}\right] = \mathfrak{R}\left(Y_{\tau^*}\right).$$

So τ^* is an optimal stopping time for the optimal stopping problem of $\mathfrak{R}$.

Application 2: American Options

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- If Y is an American-style option, then the buyer's subhedging price is

$$a_*(Y) := \sup\Big\{y \in \mathbb{R} : \exists\ \tau \in \mathcal{S} \text{ and } H \in \mathcal{H} \text{ such that } Y_\tau + \int_0^\tau H_r dB_r \geq y,\ \mathbb{P} - a.s. \text{ for any } \mathbb{P} \in \mathcal{P}\Big\}.$$

Proposition

$$a_*(Y) = \sup_{\tau\in\mathcal{S}} \inf_{\mathbb{P}\in\mathcal{P}} \mathbb{E}_{\mathbb{P}}(Y_\tau) = \inf_{\mathbb{P}\in\mathcal{P}} \mathbb{E}_{\mathbb{P}}(Y_{\tau^*})$$

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and there is $H \in \mathcal{H}$ s.t. $Y_{\tau^*} + \int_0^{\tau^*} H_r dB_r \geq a_*(Y)$ P-a.s. for all $\mathbb{P} \in \mathcal{P}$. In particular, the supremum defining $a_*(Y)$ is attained.

Classical Optimal Stopping

Given a proba. $\mathbb{P}$, to solve

$$\sup_{\tau \in \mathcal{S}} \mathbb{E}_{\mathbb{P}}\big[Y_\tau\big] = \mathbb{E}_{\mathbb{P}}\big[Y_{\tau^*}\big], \tag{2}$$

one define the *Snell envelope* of Y:

$$Z_t^{\mathbb{P}} := \operatorname*{esssup}_{\tau \in \mathcal{S}} \mathbb{E}_{\mathbb{P}}\big[Y_\tau | \mathcal{F}_t\big].$$

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- $Z^{\mathbb{P}}$ is an $\mathbb{P}-$supermartingale and $\big\{Z^{\mathbb{P}}_{t \wedge \tau^*}\big\}_{t \in [0,T]}$ is an $\mathbb{P}-$martingale.

the Upper Snell Envelope

The key to solving the problem is the upper Snell envelope of Y:

$$\overline{Z}_t(\omega) := \inf_{\mathbb{P}\in\mathcal{P}(t,\omega)} \sup_{\tau\in\mathcal{S}^t} \mathbb{E}_{\mathbb{P}}\big[Y_\tau^{t,\omega}\big] \geq Y_t(\omega), \ (t,\omega)\in[0,T]\times\Omega,$$

where $\mathcal{P}(t,\omega) \subset \mathfrak{P}_t^Y$ is a path-dependent probability set.

- In particular, $\overline{Z}_0 = \inf_{\mathbb{P}\in\mathcal{P}} \sup_{\tau\in\mathcal{S}} \mathbb{E}_{\mathbb{P}}\big[Y_\tau\big]$.

Main Result

Similar to the classic theory, the first time $\overline{Z}$ meets Y

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is an optimal stopping time for (1), and $\overline{Z}$ has a martingale property w.r.t. the nonlinear expectation $\underline{\mathscr{E}}_t[\xi](\omega) := \inf_{\mathbb{P} \in \mathcal{P}(t,\omega)} \mathbb{E}_{\mathbb{P}}[\xi^{t,\omega}]$:

Theorem

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In particular, τ^* satisfies

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the Lower Snell Envelope

Meanwhile, "Optimal Stopping under Adverse Nonlinear Expectation and Related Games" by Nutz and Zhang addressed the same problem by a different approach:

- They derived the $\underline{\mathscr{E}}$-martingale property of the *discrete time version* of the lower Snell envelope

$$\underline{Z}_t(\omega) := \sup_{\tau \in \mathcal{S}^t} \inf_{\mathbb{P} \in \mathcal{P}(t,\omega)} \mathbb{E}_{\mathbb{P}}\left[Y_\tau^{t,\omega}\right], \quad (t,\omega) \in [0,T] \times \Omega$$

  by using a "*tower property*" of $\underline{\mathscr{E}}$ which relies on another stability of pasting, then they passed to the limit.

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  by using a "*tower property*" of $\underline{\mathscr{E}}$ which relies on another stability of pasting, then they passed to the limit.

- While our method relies on $\overline{Z} \le Z^{\mathbb{P}}$ as well as the martingale property of $Z^{\mathbb{P}}$.

An alternative: a controller-stopper game

From a perspective of a zero-sum controller-stopper game in which the stopper is trying to maximize Y while the controller wants to minimize Y by selecting the distribution law from $\mathcal{P}$, the theorem shows that the controller-stopper game has a value

$$V = \inf_{\mathbb{P}\in\mathcal{P}} \mathbb{E}_{\mathbb{P}}\left[Y_{\tau^*}\right]$$

as its lower value $\underline{V} = \sup_{\tau\in\mathcal{S}} \inf_{\mathbb{P}\in\mathcal{P}} \mathbb{E}_{\mathbb{P}}\left[Y_{\tau}\right]$ coincides with the upper one $\overline{V} = \inf_{\mathbb{P}\in\mathcal{P}} \sup_{\tau\in\mathcal{S}} \mathbb{E}_{\mathbb{P}}\left[Y_{\tau}\right]$.

What's the difficulty?

Although this result seems similar to the one obtained in the classical optimal stopping theory, we have encountered major technical hurdles:

- The lack of a dominating probability in $\mathcal{P}$ deprives $\underline{\mathscr{E}}$ of possessing a dominated convergence theorem, and thus restricts us from taking the classic approach by El Karoui to obtain the martingale property of $\overline{Z}$.

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- we do not have a measurable selection theorem for stopping strategies, which complicates the proof of the dynamic programming principle of $\overline{Z}$.
- The local approach by Fleming and Souganidis that uses comparison principle of viscosity solutions to show the existence of game value does not work for our path-dependent setting.

Shifted Canonical Probability Spaces

- $\Omega^t := \{\omega \in \mathbb{C}([t, T]; \mathbb{R}) : \omega(t) = 0\}$ equipped with uniform norm

$$\|\omega\|_{t,T} := \sup_{s \in [t,T]} |\omega(s)|;$$

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- $\mathbf{F}^t := \{\mathcal{F}^t_s\}_{s \in [t,T]}$ be the natural filtration of B^t;
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For any $\mathcal{F}^t_s$−measurable r.v. η, if $\omega = \omega'$ over $[t, s]$, then $\eta(\omega) = \eta(\omega')$.

Shifted Random Variables and Shifted Processes

Let $0 \le t \le s \le T$ and $\omega \in \Omega^t$. Given a r.v. ξ and a process $X = \{X_r\}_{r\in[t,T]}$ on Ω^t, define their shifted versions on Ω^s by

$$\xi^{s,\omega}(\widetilde{\omega}) := \xi(\omega \otimes_s \widetilde{\omega}), \qquad \forall\, \widetilde{\omega} \in \Omega^s;$$

$$\text{and} \quad X_r^{s,\omega}(\widetilde{\omega}) := X(r, \omega \otimes_s \widetilde{\omega}), \quad \forall\,(r,\widetilde{\omega}) \in [s,T] \times \Omega^s,$$

where the concatenation $\omega \otimes_s \widetilde{\omega} \in \Omega^t$ is defined by

$$\big(\omega \otimes_s \widetilde{\omega}\big)(r) := \omega(r)\,\mathbf{1}_{\{r\in[t,s)\}} + \big(\omega(s) + \widetilde{\omega}(r)\big)\,\mathbf{1}_{\{r\in[s,T]\}}, \quad \forall\, r \in [t,T].$$

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The shifted r.v.'s/processes "inherit" measurability as follows:

Proposition

Let $\mathbb{M}$ be a generic metric space.
1) If an $\mathbb{M}-$valued r.v. ξ on Ω^t is $\mathcal{F}^t_r-$measurable for some $r \in [s, T]$, then $\xi^{s,\omega}$ is $\mathcal{F}^s_r-$measurable.
2) If an $\mathbb{M}-$valued process $\{X_r\}_{r\in[t,T]}$ is $\mathbf{F}^t-$adapted (resp. $\mathbf{F}^t-$progressively measurable), then the shifted process $\left\{X^{s,\omega}_r\right\}_{r\in[s,T]}$ is $\mathbf{F}^s-$adapted (resp. $\mathbf{F}^s-$progressively measurable).

Regular Conditional Probability Distributions

Let $0 \le t \le s \le T$ and $\mathbb{P} \in \mathfrak{P}_t$. There exists a family $\{\mathbb{P}_s^\omega\}_{\omega \in \Omega^t}$ of probabilities on $\big(\Omega^t, \mathscr{B}(\Omega^t)\big)$, called the *regular conditional probability distribution* (r.c.p.d.) of $\mathbb{P}$ w.r.t. $\mathcal{F}_s^t$ such that

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Given $\omega \in \Omega^t$, we can deduce from (iii) that

$$\mathbb{P}^{s,\omega}\big(\widetilde{A}\big) := \mathbb{P}^{\omega}_s\big(\omega \otimes_s \widetilde{A}\big), \quad \forall\, \widetilde{A} \in \mathcal{F}^s_T \tag{3}$$

defines a probability of $\mathfrak{P}_s$.

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The Wiener measures, however, are invariant under path shift:

Lemma

It holds for $\mathbb{P}^t_0-$*a.s.* $\omega \in \Omega^t$ *that* $\big(\mathbb{P}^t_0\big)^{s,\omega} = \mathbb{P}^s_0$.

For a $\mathbb{P}-$integrable r.v. ξ, its shift $\xi^{s,\omega}$ is $\mathbb{P}^{s,\omega}-$integrable:

Proposition

If $\xi \in L^1\big(\mathcal{F}_T^t, \mathbb{P}\big)$ *for some* $\mathbb{P} \in \mathfrak{P}_t$, *it holds for* $\mathbb{P}-$*a.s.* $\omega \in \Omega^t$ *that* $\xi^{s,\omega} \in L^1\big(\mathcal{F}_T^s, \mathbb{P}^{s,\omega}\big)$ *and*

$$\mathbb{E}_{\mathbb{P}^{s,\omega}}\left[\xi^{s,\omega}\right] = \mathbb{E}_{\mathbb{P}}\left[\xi \middle| \mathcal{F}_s^t\right](\omega) \in \mathbb{R}.$$

Regularity Assumptions on reward Y

(**Y1**) For some $\mathbb{P}_\sharp \in \mathfrak{P}_0$, $Y \in \widehat{\mathbb{D}}(\mathbf{F}, \mathbb{P}_\sharp)$, i.e., Y is a $\mathbf{F}-$adapted, RCLL process with $\mathbb{E}_{\mathbb{P}_\sharp}\big[Y_* \ln^+(Y_*)\big] < \infty$;

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(**Y2**) (An one-sided continuity) For any $0 \le t_1 \le t_2 \le T$ and $\omega_1, \omega_2 \in \Omega$

$$Y_{t_1}(\omega_1) - Y_{t_2}(\omega_2) \lesssim (t_2 - t_1) + \|\omega_1(\cdot \wedge t_1) - \omega_2(\cdot \wedge t_2)\|.$$

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For any $t \in [0, T]$ and any $\mathbb{P} \in \mathfrak{P}_t$, if $Y^{t,\omega} \in \widehat{\mathbb{D}}(\mathbf{F}^t, \mathbb{P})$ for some $\omega \in \Omega$, then $Y^{t,\omega'} \in \widehat{\mathbb{D}}(\mathbf{F}^t, \mathbb{P})$ for all $\omega' \in \Omega$.

Regularity Assumptions on reward Y

(**Y1**) For some $\mathbb{P}_\sharp \in \mathfrak{P}_0$, $Y \in \widehat{\mathbb{D}}(\mathbf{F}, \mathbb{P}_\sharp)$, i.e., Y is a $\mathbf{F}-$adapted, RCLL process with $\mathbb{E}_{\mathbb{P}_\sharp}\big[Y_* \ln^+(Y_*)\big] < \infty$;
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(P0) (Non-anticipative) If $\omega_1|_{[0,t]}=\omega_2|_{[0,t]}$, $\mathcal{P}(t,\omega_1)=\mathcal{P}(t,\omega_2)\subset\mathfrak{P}_t^Y$.

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- (P2) are implied by the "stability under finite pasting":

$$\widehat{\mathbb{P}}(A)=\mathbb{P}(A\cap\mathcal{A}_0)+\sum_{j=1}^{\lambda}\mathbb{E}_{\mathbb{P}}\Big[\mathbf{1}_{\{\widetilde{\omega}\in\mathcal{A}_j\}}\mathbb{P}_j\big(A^{s,\widetilde{\omega}}\big)\Big],\quad\forall\,A\in\mathcal{F}_T^t.$$

Assumptions on $\overline{Z}$

(Z1) $\left|\overline{Z}_t(\omega_1) - \overline{Z}_t(\omega_2)\right| \lesssim \|\omega_1 - \omega_2\|_{0,t}, \quad \forall\, \omega_1, \omega_2 \in \Omega, \;\; \forall\, t \in [0, T].$

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Remark: If $\mathcal{P}(t,\omega)$ does not depend on ω, then (Y2) implies (Z1).

(Z2) For any $\alpha > 0$, there is $C_\alpha > 0$ s.t. for any $t \in [0, T)$

$$\sup_{\omega \in O^t_\alpha(\mathbf{0})} \; \sup_{\mathbb{P} \in \mathcal{P}(t,\omega)} \mathbb{E}_{\mathbb{P}}\left[\sup_{r \in [t,(t+\delta)\wedge T]} |B^t_r| \right] \le C_\alpha \delta, \quad \forall\, \delta \in (0, T].$$

Example: Path-dependent SDEs with Controls

Fix $C>0$. For any $t\in[0,T]$, let $\mathcal{U}_t$ collect all $\mathbf{F}^t-$progressively measurable processes $\{\mu_s\}_{s\in[t,T]}$ such that $|\mu_\cdot|\le C$.

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We impose sufficient regularity conditions on $b\colon [0,T]\times\Omega\times\mathbb{R}\to\mathbb{R}$ s.t. for any $\mu\in\mathcal{U}_t$ the path-dependent SDE

$$X_s=\int_t^s b^{t,\omega}(r,X,\mu_r)dr+\int_t^s \mu_r\, dB_r^t,\quad s\in[t,T], \tag{4}$$

admits a unique solution $X^{t,\omega,\mu}$ on $\left(\Omega^t,\mathcal{F}_T^t,\mathbb{P}_0^t\right)$, where

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Let $\mathbb{P}^{t,\omega,\mu}$ be the law of $X^{t,\omega,\mu}$ under $\mathbb{P}_0^t$:

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Example: Path-dependent SDEs with Controls (Cont'd)

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Dynamic Programming Principle (DPP) of $\overline{Z}$

Lemma

1) $\overline{Z}$ *is* $\mathbf{F}-$*adapted.*

2) For any $(t,\omega)\in[0,T]\times\Omega$, $\mathbb{P}\in\mathcal{P}(t,\omega)$ *and* $s\in[t,T]$, $\mathbb{E}_{\mathbb{P}}\left[\left|\overline{Z}_s^{t,\omega}\right|\right]<\infty$.

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We first have a basic dynamic programming principle of $\overline{Z}$:

Proposition

For any $0\le t\le s\le T$ *and* $\omega\in\Omega$,

$$\overline{Z}_t(\omega)=\inf_{\mathbb{P}\in\mathcal{P}(t,\omega)}\sup_{\tau\in\mathcal{S}^t}\mathbb{E}_{\mathbb{P}}\Big[\mathbf{1}_{\{\tau<s\}}Y^{t,\omega}_\tau+\mathbf{1}_{\{\tau\ge s\}}\overline{Z}^{t,\omega}_s\Big].$$

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Consequently, all paths of $\overline{Z}$ are continuous:

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For any $(t,\omega) \in [0,T] \times \Omega$ *and* $\mathbb{P} \in \mathcal{P}(t,\omega)$, $\overline{Z}^{t,\omega} \in \mathbb{C}^1(\mathbf{F}^t, \mathbb{P})$.

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The continuity of $\overline{Z}$ allows us to derive a general version of dynamic programming principle with random horizons.

Proposition

For any $(t, \omega) \in [0, T] \times \Omega$ *and* $\nu \in \mathcal{S}^t$,

$$\overline{Z}_t(\omega) = \inf_{\mathbb{P} \in \mathcal{P}(t,\omega)} \sup_{\tau \in \mathcal{S}^t} \mathbb{E}_{\mathbb{P}} \left[\mathbf{1}_{\{\tau < \nu\}} Y_\tau^{t,\omega} + \mathbf{1}_{\{\tau \geq \nu\}} \overline{Z}_\nu^{t,\omega} \right].$$

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$$\sup_{\tau\in\mathcal{S}^s} \mathbb{E}_{\mathbb{P}_{\widetilde{\omega}}}\left[\mathcal{Y}_\tau^{s,\widetilde{\omega}'}\right] \leq \mathcal{Z}(\widetilde{\omega}')+\frac{2}{3}\varepsilon, \quad \forall\,\widetilde{\omega}' \in O_\delta^s(\widetilde{\omega}). \tag{6}$$

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$$\mathcal{Z}(\widetilde{\omega}) \geq \sup_{\tau \in \mathcal{S}^s} \mathbb{E}_{\mathbb{P}_{\widetilde{\omega}}} \left[\mathcal{Y}_\tau^{s,\widetilde{\omega}} \right] - \varepsilon/3. \tag{5}$$

By the continuity of Y and $\overline{Z}$,

$$\sup_{\tau \in \mathcal{S}^s} \mathbb{E}_{\mathbb{P}_{\widetilde{\omega}}} \left[\mathcal{Y}_\tau^{s,\widetilde{\omega}'} \right] \leq \mathcal{Z}(\widetilde{\omega}') + \frac{2}{3}\varepsilon, \quad \forall\, \widetilde{\omega}' \in O_\delta^s(\widetilde{\omega}). \tag{6}$$

- Let $\left\{\widehat{\omega}_j^t\right\}_{j \in \mathbb{N}}$ be dense in Ω^t and fix $\mathbb{P} \in \mathcal{P}(t,\omega)$.
- For $j = 1, \cdots, \lambda$, set $\mathcal{A}_j := \Big(O_\delta^s(\widehat{\omega}_j^t) \backslash \big(\underset{j' < j}{\cup} O_\delta^s(\widehat{\omega}_{j'}^t) \big) \Big) \in \mathcal{F}_s^t$ and $\mathbb{P}_j := \mathbb{P}_{\widehat{\omega}_j^t}$ via (5).
- Let $\mathbb{P}_\lambda$ be the probability of $\mathcal{P}(t,\omega)$ in (P2) that corresponds to the partition $\{\mathcal{A}_j\}$ and the probabilities $\{\mathbb{P}_j\}$.

Sketchy Proof of DPP "$\le$" (Cont'd)

Using (6),

$$\mathbb{E}_{\mathbb{P}_\lambda}\big[\mathcal{Y}_\tau\big] \le \mathbb{E}_{\mathbb{P}}\Big[\mathbf{1}_{\{\tau<s\}}\mathcal{Y}_\tau + \mathbf{1}_{\{\tau\ge s\}}\mathcal{Z}_s\Big] + \varepsilon.$$

Taking supremum over $\tau \in \mathcal{S}^t$ yields that

$$\overline{Z}_t(\omega) \le \sup_{\tau\in\mathcal{S}^t} \mathbb{E}_{\mathbb{P}}\Big[\mathbf{1}_{\{\tau<s\}}\mathcal{Y}_\tau + \mathbf{1}_{\{\tau\ge s\}}\mathcal{Z}_s\Big].$$

Sketchy Proof of DPP "$\geq$"

- Fix $\mathbb{P} \in \mathcal{P}(t,\omega)$, note $\mathcal{Z}_s(\widetilde{\omega}) \leq \sup_{\zeta \in \mathcal{S}^s} \mathbb{E}_{\mathbb{P}^{s,\widetilde{\omega}}}\Big[\mathcal{Y}_\zeta^{s,\widetilde{\omega}}\Big]$, $\forall \widetilde{\omega} \in \Omega^t$. Given $\zeta \in \mathcal{S}^s$,

$$\mathbb{E}_{\mathbb{P}^{s,\widetilde{\omega}}}\Big[\mathcal{Y}_\zeta^{s,\widetilde{\omega}}\Big] = \mathbb{E}_{\mathbb{P}}\big[\mathcal{Y}_{\zeta(\Pi_s^t)}\big|\mathcal{F}_s^t\big](\widetilde{\omega}) \leq \mathbb{E}_{\mathbb{P}}\big[\mathcal{Y}_{\widehat{\tau}}\big|\mathcal{F}_s^t\big](\widetilde{\omega}) \tag{7}$$

  holds for any $\widetilde{\omega} \in \Omega^t$ except on a $\mathbb{P}-$null set $\mathcal{N}_\zeta$, where $\widehat{\tau}$ is an optimal stopping time.

- Since $\mathcal{S}^s$ is an uncountable set, we can not take supremum over $\zeta \in \mathcal{S}^s$ for $\mathbb{P}-$a.s. $\widetilde{\omega} \in \Omega^t$ in (7) to directly obtain

$$\mathcal{Z}_s \leq \mathbb{E}_{\mathbb{P}}\big[\mathcal{Y}_{\widehat{\tau}}\big|\mathcal{F}_s^{\mathbb{P}}\big] = Z_s^{\mathbb{P}}, \quad \mathbb{P} - a.s.$$

Sketchy Proof of DPP "$\geq$" (Cont'd)

To overcome this difficulty, we shall consider a "dense" countable subset Γ of $\mathcal{S}^s$ in sense that for any $\widehat{\tau} \in \mathcal{S}^s$, we can a $\widehat{\tau}' \in \Gamma$ s.t.

$$\mathbb{E}_{\mathbb{P}}\left[\left|\mathcal{Y}_{\widehat{\tau}'} - \mathcal{Y}_{\widehat{\tau}}\right|\right] < \varepsilon/4.$$

Sketchy Proof of DPP "$\geq$" (Cont'd)

To overcome this difficulty, we shall consider a "dense" countable subset Γ of $\mathcal{S}^s$ in sense that for any $\widehat{\tau} \in \mathcal{S}^s$, we can a $\widehat{\tau}' \in \Gamma$ s.t.

$$\mathbb{E}_{\mathbb{P}}\left[\left|\mathcal{Y}_{\widehat{\tau}'} - \mathcal{Y}_{\widehat{\tau}}\right|\right] < \varepsilon/4.$$

Then using (7),

$$\mathbb{E}_{\mathbb{P}}\left[\mathbf{1}_{\{\tau<s\}}\mathcal{Y}_\tau + \mathbf{1}_{\{\tau\geq s\}}\mathcal{Z}_s\right] \leq \mathbb{E}_{\mathbb{P}}\left[\mathbf{1}_{\{\tau<s\}}\mathcal{Y}_{\tau\wedge s} + \mathbf{1}_{\{\tau\geq s\}}\mathbb{E}_{\mathbb{P}}\left[\mathcal{Y}_{\widehat{\tau}}\middle|\mathcal{F}^t_s\right]\right]$$
$$= \mathbb{E}_{\mathbb{P}}\left[\mathbf{1}_{\{\tau<s\}}\mathcal{Y}_{\tau\wedge s} + \mathbf{1}_{\{\tau\geq s\}}\mathcal{Y}_{\widehat{\tau}}\right] = \mathbb{E}_{\mathbb{P}}\left[\mathcal{Y}_{\overline{\tau}}\right] \leq \sup_{\tau\in\mathcal{S}^t}\mathbb{E}_{\mathbb{P}}\left[\mathcal{Y}_\tau\right].$$

Taking supremum over $\tau\in\mathcal{S}^t$ yields "$\geq$".

Sketchy Proof of the Theorem

- The dynamic programming principle plays two roles in the demonstration of (1): First, it is used to show the continuity of $\overline{Z}$, which is crucial in our construction of approximating stopping times for τ^*. Second, a random-horizon version of DPP directly gives rise to the supermartingale property of $\overline{Z}$.
- The submartingale property of $\overline{Z}$ until τ^*, however, requires a delicate approximation scheme that involves carefully pasting probabilities, $\overline{Z} \leq Z^{\mathbb{P}}$ and the martingale property of $Z^{\mathbb{P}}$.

Thank you for your attention.