Is quantile hedging equivalent to randomized hypothesis testing?

Qingshuo Song

A joint work with Tim Leung and Jie Yang

@ USC Math Finance Colloquium, Los Angeles

Outline

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Quantile hedging problem

Problem setup (QH)

In a market, with initial captical x and strategy π, the wealth $X_t^{x,\pi}$ satisfies

$$X_t^{x,\pi} = x + \int_0^t \pi_u dS_u.$$

Q. What's the price for an option with payoff $F = f(S_T)$?
A. Superhedging price F_0, i.e. the smallest capital x needed for

$$\mathbb{P}\{X_T^{x,\pi} \geq F\} = 100\% \text{ for some } \pi$$

Note, if $x < F_0$, then

$$\mathbb{P}\{X_T^{x,\pi} \geq F\} < 100\% \text{ for any } \pi.$$

(QH). Find a strategy π to maximize the success probability

$$\widetilde{V}(x) = \sup_\pi \mathbb{P}\{X_T^{x,\pi} \geq F\}$$

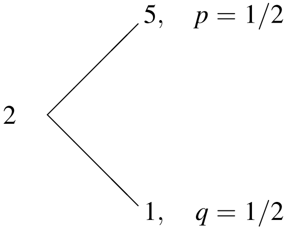
Example (Bin)

(QH) under one-step binomial tree

- The benchmark to beat $F = 1$;
- Find (QH) value

  $$\widetilde{V}(x) = \sup_{\pi} \mathbb{P}\{X_T^{x,\pi} \geq F\};$$

- $\tilde{V}(1) = 1$, What's $\tilde{V}(1/2) = ?$



Example (Meatball)

Equivalent question to (Bin)

Q. Given x dollars, buy meatball as much as possible (kg)?

A. $\tilde{V}(x)$ kg, where $\tilde{V}(x)$ is the maximum success probability of (Bin).
Q. If the meatball is allowed to sold in part, how is the answer different?

Example (BS)

Black-Scholes model with stock benchmark

- Market has single stock with price

$$dS_t = S_t \sigma \left(\theta dt + dW_t \right),$$

- The investor wants to beat the benchmark $F = S_T$, i.e. find

$$\widetilde{V}(x) = \sup_{\pi \in \mathcal{A}(x)} \mathbb{P}\{X_T^{x,\pi} \geq S_T\}.$$

Literatures review

- (QH) was initiated by [Föllmer and Leukert(1999)], and solved to Maximizing *success ratio*,
- *Minimizing shortfall risk* [Cvitanić(2000), Föllmer and Schied(2002), Rudloff(2007), Schied(2004)]
- *Others* [Bouchard et al.(2009)Bouchard, Elie, and Touzi, He and Zhou(2011)]
- Problems are converted to

  Randomized hypothesis testing (RT),

  then solved by

  Neyman-Pearson Lemma (NPLemma).

Note If $\mathbb{P}\{X_T^{x,\pi} \geq F\} = \rho(X_T^{x,\pi})$, then $\rho(\cdot)$ is Not concave.

Two questions in solving quantile hedging

Q1. Quantile hedging = Randomized testing?

Q2. Is NP lemma applicable to Quantile hedging?

Outline

Two kinds of hypothesis testing problems

Mathematical formulation of (PT) and (RT): $V_1(x) \le V(x)$

In a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with given $\mathcal{H} \subset L^{0,+}$

- Pure test space is $\mathcal{I} = \{X : \Omega/\mathcal{F} \mapsto \{0,1\}/2^{\{0,1\}}\}$;
  Pure hypothesis testing (PT) is,

  $$V_1(x) := \sup_{X \in \mathcal{I}} \mathbb{E}[X]$$

  subject to

  $$\sup_{H \in \mathcal{H}} \mathbb{E}[HX] \le x.$$

- Randomized test space is $\mathcal{X} = \{X : \Omega/\mathcal{F} \mapsto [0,1]/\mathcal{B}([0,1])\}$;
  Randomized hypothesis testing (RT) is,

  $$V(x) = \sup_{X \in \mathcal{X}} \mathbb{E}[X]$$

  subject to

  $$\sup_{H \in \mathcal{H}} \mathbb{E}[HX] \le x.$$

Quantile hedging and pure hypothesis testing

Recall (QH) is

$$\widetilde{V}(x) = \sup_\pi \mathbb{P}\{X_T^{y,\pi} \geq F\} \quad \text{subj.} \quad y \leq x.$$

Denote the class of Equivalent Martingale Measures (EMMs) by $\mathcal{Q}$, and

$$\mathcal{Z} := \{\tfrac{d\mathbb{Q}}{d\mathbb{P}} : \mathbb{Q} \in \mathcal{Q}\}, \quad \mathcal{H} = \{ZF : Z \in \mathcal{Z}\}.$$

Then,

$$\widetilde{V}(x) = \sup_{A \in \mathcal{F}_T} \mathbb{P}(A) = \sup_{X \in \mathcal{I}} \mathbb{E}[X] = V_1(x)$$

subjet to

$$\sup_{Z \in \mathcal{Z}} \mathbb{E}[ZFI_A] = \sup_{H \in \mathcal{H}} \mathbb{E}[HX] \leq x.$$

Proposition. (QH) = (PT).
Q. Can we say (QH) = (RT)?

A counter-example for (QH) $\neq$ (RT)

This also gives the solution of (BIN).

Fix $\Omega = \{0,1\}$ and $\mathcal{F} = 2^\Omega$, with $\mathbb{P}\{0\} = \mathbb{P}\{1\} = 1/2$. Define

$$\mathcal{H} = \{H : H(0) = 1/2, H(1) = 3/2\}.$$

1. The value of (RT) $V(x)$ is given by

$$V(x) = \begin{cases} \mathbb{E}[4xI_{\{0\}}] = 2x, & \text{if } 0 \leq x < 1/4; \\ \mathbb{E}[I_{\{0\}} + \frac{4x-1}{3} I_{\{1\}}] = \frac{2x+1}{3}, & \text{if } 1/4 \leq x < 1; \\ \mathbb{E}[1] = 1, & \text{if } x \geq 1. \end{cases}$$

2. The value of (PT) $V_1(x)$ is given by

$$V_1(x) = \begin{cases} \mathbb{E}[0] = 0, & \text{if } 0 \leq x < 1/4; \\ \mathbb{E}[I_{\{0\}}] = \frac{1}{2}, & \text{if } 1/4 \leq x < 1; \\ \mathbb{E}[1] = 1, & \text{if } x \geq 1. \end{cases}$$

Q1. (QH) = (PT) < (RT) in this example. When do we have equality?

Q2. $V(x)$ is the smallest concave envelope of $V_1(x)$. Is it always true?

(RT) formulation

In a probability space $(\Omega, \mathcal{F}, \mathbb{P})$,

- Randomized test space is $\mathcal{X} = \{X : \Omega/\mathcal{F} \mapsto [0,1]/\mathcal{B}([0,1])\}$;
- By $\mathcal{X}_x^{\mathcal{H}}$ denote the collection of $X \in \mathcal{X}$ satisfying $\sup_{H\in\mathcal{H}} \mathbb{E}[HX] \leq x$. Then, the value of (RT) is

$$V(x) = \sup_{X\in\mathcal{X}_x^{\mathcal{H}}} \mathbb{E}[X]$$

- $V(x)$ stays invariant if $\mathcal{H}$ is replaced by $co(\mathcal{H})$ in (RT)

Duality formulation of (RT)

Optimality condition

For any admissible $X \in \mathcal{X}_x^{\mathcal{H}}$, $H \in \mathcal{H}$, and $a \geq 0$

$$
\begin{aligned}
V(x) = \sup_X \mathbb{E}[X] \quad & \leq \sup_X \{\mathbb{E}[X] + a(x - \mathbb{E}[HX])\} \\
& = \sup_X \mathbb{E}[X(1 - aH)] + ax \\
& \leq \mathbb{E}[(1 - aH)^+] + ax := g(H, a).
\end{aligned}
$$

- $\inf\limits_{(H,a)\in\mathcal{H}\times[0,\infty)} g(H,a)$ gives upper bound.
- Strong duality (equality) holds, if $\exists(\hat{H}, \hat{a}, \hat{X}) \in \mathcal{H} \times [0,\infty) \times \mathcal{X}_x^{\mathcal{H}}$ s.t.

$$
\text{(OC)} \begin{cases} \inf_{(a,H)} g(H,a) = g(\hat{H}, \hat{a}), \\ \hat{X} = I_{\{1 > \hat{a}\hat{H}\}} + B I_{\{1 = \hat{a}\hat{H}\}}, \quad \text{for some } B \in \mathcal{X} \\ \mathbb{E}[H\hat{X}] \leq \mathbb{E}[\hat{H}\hat{X}] = x, \quad \forall H \in \mathcal{H}, \end{cases}
$$

Q. (Hard!) Does the optimal triple exist in $\mathcal{H} \times [0,\infty) \times \mathcal{X}_x^{\mathcal{H}}$?

Generalized NPLemma

Some remarks on the existing result

Let $\mathcal{H}$ be L^1-bounded, and $\mathcal{H}_x := \{H \in L^{0,+} : \mathbb{E}[HX] \le x, \ \forall X \in \mathcal{X}_x^{\mathcal{H}}\}$.

- $\mathcal{H} \subset co(\mathcal{H}) \subset \mathcal{H}^{oo} \subset \mathcal{H}_x$
- $\mathcal{H}_x$ is convex in $L^{0,+}$, and closed w.r.t in-probability-convergence.

Theorem [Cvitanić and Karatzas(2001)]

$$V(x) = \inf_{(H,a) \in \mathcal{H}_x \times [0,\infty)} g(H,a) \text{ and } (\hat{H}, \hat{a}, \hat{X}) \in \mathcal{H}_x \times [0,\infty) \times \mathcal{X}_x^{\mathcal{H}}.$$

However, it's hard to to characterize $\mathcal{H}_x$ in (QH), where $\mathcal{H}$ is

$$\mathcal{H} = \{ZF : Z \in \mathcal{Z}\}.$$

Q. If A is a closed set w.r.t. in-probability-convergence, is it closed w.r.t. a.s.-convergence, in the space $L^{0,+}$?

Generalized NP Lemma

Modified result for the use of (QH)

Theorem 1

$$V(x) = \inf_{(H,a)\in co(\mathcal{H})\times[0,\infty)} g(H,a) \text{ and } (\hat{H},\hat{a},\hat{X}) \in \overline{co(\mathcal{H})} \times [0,\infty) \times \mathcal{X}_x^{\mathcal{H}}.$$

Above Theorem resolves (RT) associated to (QH), since $\mathcal{H} = co(\mathcal{H})$.

Q1. Recall $\mathcal{H} \subset co(\mathcal{H}) \subset \mathcal{H}_x$. Can we replace $co(\mathcal{H})$ by $\mathcal{H}$ in Theorem 1?
Q2. Can we replace inf over $[0,\infty)$ by $(0,\infty)$ as of [Cvitanić and Karatzas(2001)]?

The sufficient conditions for (QH) = (PT) = (RT)

By careful examination of (OC), in particular the structure of

$$\hat{X} = I_{\{1>\hat{a}\hat{H}\}} + BI_{\{1=\hat{a}\hat{H}\}}$$

we obtain

Theorem 2 (QH) = (PT) = (RT) under one of the following conditions:

1. $\mathcal{Z}$ is a singleton, and there exists $\mathcal{F}_T$-measurable random variable with continuous cumulative distribution function under $\mathbb{P}$;
2. For all $a \in (0, \infty)$, the minimizer $\hat{Z}_a := \arg\min \mathbb{E}[xa + (1 - aZF)^+]$ satisfies $\mathbb{P}\{a\hat{Z}_a F = 1\} = 0$.

In addition, $\widetilde{V}(x)$ is continuous, concave, and non-decreasing in $x \in [0, \infty)$, and admits the representation:

$$\widetilde{V}(x) = \inf_{a \geq 0, Z \in \mathcal{Z}} \mathbb{E}[xa + (1 - aZF)^+].$$

Outline

Black-Scholes model with stock benchmark

Explicit solution

- Market has single stock with price

$$dS_t = S_t\sigma\,(\,\theta dt + dW_t\,)\,,$$

- The investor wants to beat the benchmark $F = S_T$, i.e. find

$$\widetilde{V}(x) = \sup_{\pi\in\mathcal{A}(x)} \mathbb{P}\{X_T^{x,\pi} \geq S_T\}.$$

- $\mathcal{Z}$ is singleton with element

$$Z_t := \exp\{-\tfrac{1}{2}\int_0^t \theta^2(S_u)du - \int_0^t \theta(S_u)dW_u\}.$$

- W_T has a continuous cdf; Thus, (QH) = (RT) .
- By Corollary 1, $\widetilde{V}(x)$ is a continuous, non-decreasing, and concave, and

$$\widetilde{V}(x) = \inf_{a\geq 0}\{xa + \mathbb{E}[(1 - aZ_TS_T)^+]\}.$$

- Some calculations leads to explicit solution,
  - If $p\sigma = \theta$, then ...
  - If $p\sigma \neq \theta$, then ...

Stochastic factor model

Stochastic control problem

- Market has single stock with price

$$dS_t = S_t\sigma(Y_t)(\theta(Y_t)dt + dW_t),$$

  and the stochastic factor Y follows

$$dY_t = b(Y_t)dt + c(Y_t)(\rho dW_t + \sqrt{1-\rho^2}d\hat{W}_t).$$

- The investor wants to find

$$\widetilde{V}(t,s,x,y) = \sup_{\pi\in\mathcal{A}} \mathbb{P}^{t,s,x,y}\{X_T^{x,\pi} \geq f(S_T, Y_T)\}.$$

- $\mathcal{Z} = \{\tilde{Z}_T^{z,\lambda} : \int_0^T \lambda^2 dt < \infty\}$ where

$$Z_u^{z,\lambda} = z + \int_t^u Z_\nu^{z,\lambda}(-\theta(Y_\nu)dW_\nu - \lambda_\nu d\hat{W}_\nu).$$

- If $\mathbb{P}\{Z_T^{a,\lambda}f(S_T,Y_T) = 1\} = 0, \forall a,$
  $\widetilde{V}(t,s,x,y) = \inf_{a\geq 0}\{xa + U(t,s,y,a)\}$ where

$$U(t,s,y,z) := \inf_{\lambda\in\Lambda_t} \mathbb{E}^{t,s,y}[(1 - Z_T^{z,\lambda}f(S_T,Y_T))^+].$$

Stochastic factor model with general benchmark

Bellman Equation

Define, for any scalar $\lambda \in \mathbb{R}$,

$$
\begin{aligned}
\mathcal{L}^{\lambda} w \quad & = s\theta(y)\sigma(y)w_s + \frac{1}{2}s^2\sigma^2(y)w_{ss} + b(y)w_y + \frac{1}{2}c^2(y)w_{yy} + \\
& \frac{1}{2}(\theta^2(y) + \lambda^2)z^2 w_{zz} + s\sigma(y)c(y)\rho w_{sy} - sz\sigma(y)\theta(y)w_{sz} + \\
& zc(y)(-\theta(y)\rho - \lambda\sqrt{1-\rho^2})w_{yz}.
\end{aligned}
$$

Define $\mathcal{O} = (0, \infty) \times (-\infty, \infty) \times (0, \infty)$

$$
(HJB) \begin{cases} w_t + \inf_{\lambda \in \mathbb{R}} \mathcal{L}^{\lambda} w = 0, & \text{on } (0, T) \times \mathcal{O} \\ w(T, s, y, z) = (1 - zf(s, y))^+, & \text{on } \mathcal{O}. \end{cases}
$$

Proposition. If $\theta(\cdot)$, $\mu(\cdot)$, $b(\cdot)$, $\sigma(\cdot)$, $f(\cdot, \cdot)$ and $c(\cdot)$ are Lipschitz, and

$$
\sup_{y \in \mathbb{R}} \{|\theta(y)| + |\sigma(y)| + |b(y)|\} < \infty,
$$

then U is the unique bounded continuous viscosity solution.

Note Non-unqiueness holds if we drop conditions on coefficients, see counter-example in [Bayraktar et al.(2012)Bayraktar, Huang, and Song].

Outline

Summary

In this work, we consider a more generalized randomized composite hypothesis testing problem. For $x > 0$, define

$$V(x) := \sup_{X\in\mathcal{X}} \inf_{G\in\mathcal{G}} \mathbb{E}[GX] \tag{1}$$

$$\text{subject to } \sup_{H\in\mathcal{H}} \mathbb{E}[HX] \leq x. \tag{2}$$

- Improved Neyman-Pearson Lemma
- Provide the sufficient condition of equivalence on pure testing and randomized testing
- Identify quantile hedging by Neyman-Pearson Lemma

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