Stationary Solutions and Random Periodic Solutions of Stochastic Equations

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Fixed Point of ODE

Consider ODE

$$\begin{cases} \frac{dv(t)}{dt} = -v(t) \\ v(0) = y \in \mathbb{R}^1. \end{cases}$$

For fixed $t \geq 0$, regard v as a mapping

$$v^y(t) : \mathbb{R}^1 \mapsto \mathbb{R}^1.$$

Then a fixed point is an initial value of the ODE satisfying

$$v^y(t) = y \quad for\ all\ t \geq 0.$$

It is easy to check that $y = 0$ satisfies the requirement.

A Nontrivial Example: Ornstein-Uhlenbeck Process

Consider Ornstein-Uhlenbeck process:

$$\begin{cases} dv(t) = -v(t)dt + dW_t \\ v(0) = Y(\omega) \in L^2(\Omega). \end{cases}$$

For fixed $t \geq 0$, regard v as a mapping

$$v^{Y(\omega)}(t) : L^2(\Omega) \mapsto L^2(\Omega).$$

Almost impossible to find a fixed point like

$$v^{Y(\omega)}(t) = Y(\omega) \quad for\ t \geq 0\ a.s.$$

A Nontrivial Example: Ornstein-Uhlenbeck Process (Continued)

Define the "stochastic fixed point" like

$$v^{Y(\omega)}(t) = Y(\theta_t \omega) \quad for\ t \geq 0\ a.s.,$$

where

$$(\theta_t W)(s) = W(t+s) - W(t) \quad for\ any\ s \in (-\infty, +\infty).$$

You can verify that the "stochastic fixed point" is

$$Y(\omega) = \int_{-\infty}^{0} \mathrm{e}^{s} dW_s.$$

Definition for Stationary Solution (Stochastic Fixed Point)

A measurable space: $(V, \mathscr{B}(V))$.

A metric dynamical system $(\Omega, \mathscr{F}, P, (\theta_t)_{t\geq 0})$, $(\theta_t)_{t\geq 0} : \Omega \to \Omega$ satisfies:

- $P \cdot \theta_t^{-1} = P$;
- $\theta_0 = I$, where I is the identity transformation on Ω;
- $\theta_s \circ \theta_t = \theta_{s+t}$ for all $s, t \geq 0$.

For a measurable random dynamical system $v : [0, \infty) \times V \times \Omega \to V$, the stationary solution is a $\mathscr{F}$ measurable r.v. $Y : \Omega \to V$ such that (Arnold 1998)

$$v^{Y(\omega)}(t, \omega) = Y(\theta_t \omega) \quad for\ t \geq 0\ a.s.$$

Some Existing Results

- Sinai 1991, 1996 Stochastic Burgers equations with C^3 noise under strong smooth conditions
- Mattingly 1999, CMP 2D Stochastic Navier-Stokes equation with additive noise
- E & Khanin & Mazel & Sinai 2000, AM Stochastic inviscid Burgers equations with additive C^3 noise
- Caraballo & Kloeden & Schmalfuss 2004, AMO Stochastic evolution equations with small Lipschitz constant and linear noise

A Basic Assumption in Invariant Manifold Theory: There Exists Stationary Solution

- Arnold 1998
- Duan & Lu & Schmalfuss 2003, AP
- Mohammed & T. Zhang & Zhao 2008, Memoirs of AMS
- Lian & Lu 2010, Memoirs of AMS

Stationary Solutions of Parabolic SPDEs

We use the correspondence between SPDEs and BDSDEs to construct the stationary solutions of SPDEs:

$$
\begin{aligned}
v(t,x) &= v(0,x) + \int_0^t [\mathscr{L} v(s,x) + f\big(x, v(s,x)\big)]ds \\
&\quad + \int_0^t g\big(x, v(s,x)\big) dB_s.
\end{aligned}
$$

$f : \mathbb{R}^d \times \mathbb{R}^1 \to \mathbb{R}^1$, $g : \mathbb{R}^d \times \mathbb{R}^1 \to \mathcal{L}^2_{U_0}(\mathbb{R}^1)$;

B: Wiener process with values in a Hilbert space;

$\mathscr{L}$: a second order differential operator given by

$$
\mathscr{L} = \frac{1}{2} \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x) \frac{\partial}{\partial x_i}
$$

with $\big(a_{ij}(x)\big) = \sigma\sigma^*(x)$.

Results

- Zhang & Zhao 2007, JFA Lipschitz coefficients
- Zhang & Zhao 2010, JDE linear growth coefficients
- Zhang & Zhao 2013, SPA polynomial growth coefficients
- Zhang 2011, SD stationary stochastic viscosity solutions

Using BDSDEs, we construct the stationary solutions of non-linear SPDEs with non-additive noise.

Time Reverse Version of SPDE

For arbitrary $T > 0$, define $\hat{B}_s = B_{T-s} - B_T$.

By the integral transformation, $u(t,x) \triangleq v(T-t,x)$ satisfies terminal-value SPDE

$$\begin{aligned} u(t,x) &= u(T,x) + \int_t^T [\mathscr{L}u(s,x) + f\big(x,u(s,x)\big)]ds \\ &\quad - \int_t^T g\big(x,u(s,x)\big)d^\dagger \hat{B}_s. \end{aligned}$$

Infinite Horizon BDSDEs

Infinite horizon BDSDE:

$$\begin{aligned} \mathrm{e}^{-Ks}Y_s^{t,x} &= \int_s^\infty \mathrm{e}^{-Kr} f(X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})dr + \int_s^\infty K\mathrm{e}^{-Kr}Y_r^{t,x}dr \\ &\quad - \int_s^\infty \mathrm{e}^{-Kr} g(X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})d^\dagger \hat{B}_r \\ &\quad - \int_s^\infty \mathrm{e}^{-Kr}\langle Z_r^{t,x}, dW_r\rangle, \end{aligned}$$

where

$$X_s^{t,x} = x + \int_t^s b(X_r^{t,x})dr + \int_t^s \sigma(X_r^{t,x})dW_r.$$

Infinite horizon BDSDE has a unique solution $(Y_\cdot^{t,\cdot}, Z_\cdot^{t,\cdot}) \in S^{2p,-K}_{\mathscr{F}^{\hat{B},W}} \bigcap L^{2p,-K}_{\mathscr{F}^{\hat{B},W}}([t,\infty]; L^{2p}_\rho(\mathbb{R}^d;\mathbb{R}^1)) \times L^{2,-K}_{\mathscr{F}^{\hat{B},W}}([t,\infty]; L^2_\rho(\mathbb{R}^d;\mathbb{R}^d))$

The Correspondence between SPDEs and BDSDEs on Finite Time Interval

$$
\begin{aligned}
\text{BDSDE}: \ Y_s^{t,x} &= h(X_T^{t,x}) + \int_s^T f(r, X_r^{t,x}, Y_r^{t,x}) dr \\
&\quad - \int_s^T \langle g(r, X_r^{t,x}, Y_r^{t,x}), d^\dagger \hat{B}_r \rangle - \int_s^T \langle Z_r^{t,x}, dW_r \rangle.
\end{aligned}
$$

$$
\begin{aligned}
\text{SPDE}: \ u(t,x) &= h(x) + \int_t^T \{\mathscr{L} u(s,x) + f_n\big(s, x, u(s,x)\big)\} ds \\
&\quad - \int_t^T \langle g\big(s, x, u(s,x)\big), d^\dagger \hat{B}_s \rangle, \qquad 0 \le t \le T.
\end{aligned}
$$

Correspondence: $u(t,x) \triangleq Y_t^{t,x}$ is a solution of SPDE, and

$$
(Y_s^{t,x}, Z_s^{t,x}) = \big(u(s, X_s^{t,x}), (\sigma \nabla u)(s, X_s^{t,x})\big).
$$

Existing Results for Correspondence Between SPDEs and BDSDEs

Based on different smooth requirements for coefficients, this correspondence was established for differential types of solutions of SPDEs.

- Pardoux & Peng 1994, PTRF smooth solution
- Buckdahn & Ma 2001, SPA stochastic viscosity solution
- Bally & Matoussi 2001, JTP weak solution

Metric Dynamical System

Define $\hat{\theta}_t : \Omega \longrightarrow \Omega$, $t \geq 0$, by

$$\hat{\theta}_t \begin{pmatrix} \hat{B}_s \\ W_s \end{pmatrix} = \begin{pmatrix} \hat{B}_{s+t} - \hat{B}_t \\ W_{s+t} - W_t \end{pmatrix}$$

Then for any s, $t \geq 0$,

- $P \cdot \hat{\theta}_t^{-1} = P$;
- $\hat{\theta}_0 = I$, where I is the identity transformation on Ω;
- $\hat{\theta}_s \circ \hat{\theta}_t = \hat{\theta}_{s+t}$.

Also for an arbitrary $\mathscr{F}$ measurable ϕ, set

$$\hat{\theta} \circ \phi(\omega) = \phi\big(\hat{\theta}(\omega)\big).$$

Stationary Property of Infinite Horizon BDSDEs

By uniqueness of solution, the solution of infinite horizon BDSDE $(Y_\cdot^{t,\cdot}, Z_\cdot^{t,\cdot})$ satisfies the stationary property: for any $t \geq 0$,

$$\hat{\theta}_r \circ Y_s^{t,\cdot} = Y_{s+r}^{t+r,\cdot} \quad \hat{\theta}_r \circ Z_s^{t,\cdot} = Z_{s+r}^{t+r,\cdot} \quad for\ r \geq 0,\ \ s \geq t\ a.s.$$

In particular, for any $t \geq 0$,

$$\hat{\theta}_r \circ Y_t^{t,\cdot} = Y_{t+r}^{t+r,\cdot} \quad for\ r \geq 0\ a.s.$$

Transferring the Stationary Property from BDSDEs to Terminal-Value SPDEs

So, for any $t \geq 0$,

$$\hat{\theta}_r \circ u(t, \cdot) = u(t + r, \cdot) \quad for\ r \geq 0\ a.s.$$

By Kolmogorov's continuity lemma, $u(t, \cdot)$ is continuous w.r.t. t. Thus

$$\hat{\theta}_r \circ u(t, \cdot) = u(t + r, \cdot) \quad for\ t,\ r \geq 0\ \ a.s.$$

Time Reverse Transformation

For arbitrary $T > 0$, choose $\hat{B}$ in terminal-value SPDEs as $\hat{B}_s = B_{T-s} - B_T$. We see that $v(t,x) \triangleq u(T-t,x)$ satisfies initial-value SPDE

$$\begin{aligned} v(t,x) &= v(0,x) + \int_0^t [\mathscr{L}v(s,x) + f\big(x, v(s,x)\big)]ds \\ &\quad + \int_0^t g\big(x, v(s,x)\big)dB_s,\ t \geq 0. \end{aligned}$$

In fact, we can prove that $v(t,x,\omega) \triangleq Y_{T-t}^{T-t,x}(\hat{\omega}) = Y_0^{0,x}(\hat{\theta}_{T-t}\hat{\omega})$ is independent of the choice of T as follows:

$$\begin{aligned} \hat{\theta}_{T-t}\hat{\omega} &= \hat{\omega}(T-t+s) - \hat{\omega}(T-t) \\ &= (B_{T-(T-t+s)} - B_T) - (B_{T-(T-t)} - B_T) \\ &= B_{t-s} - B_t. \end{aligned}$$

Transferring the Stationary Property from Terminal-Value to Initial-Value SPDEs

Define $\theta_t = (\hat{\theta}_t)^{-1}$, $t \geq 0$, then θ_t is a shift w.r.t. B satisfying

$$\theta_t \circ B_s = B_{s+t} - B_t.$$

So

$$\theta_r v(t, \cdot, \omega) = v(t + r, \cdot, \omega) \quad for\ r \geq 0\ a.s.$$

In particular, let $Y(\cdot, \omega) = v(0, \cdot, \omega) = Y_T^{T,\cdot}(\hat{\omega})$.

Then the above implies that $Y(\cdot, \omega)$ satisfies the definition of stationary solution:

$$v^{Y(\cdot,\omega)}(t, \cdot, \omega) = Y(\cdot, \theta_t\omega) \quad for\ t \geq 0\ a.s.$$

Main Results for Stationary Solution

Assume that we had known that

- the correspondence between SPDE and BDSDE in some reasonable sense
- the existence and uniqueness of solution of infinite horizon BDSDE

Theorem

For arbitrary T and $t \in [0,T]$, let $v(t,x) \triangleq Y_{T-t}^{T-t,x}$, where $(Y_\cdot^{t,\cdot}, Z_\cdot^{t,\cdot})$ is the solution of the infinite horizon BDSDE with $\hat{B}_s = B_{T-s} - B_T$ for all $s \geq 0$. Then $v(t,\cdot)$ is a "perfect" stationary solution of SPDE.

Assumptions

(H.1). $\exists\ p \geq 2$ and f_0 with $\int_0^\infty \int_{\mathbb{R}^d} |f_0(s,x)|^{8p} \rho^{-1}(x) dx ds < \infty$ s.t.
$|f(s,x,y)| \leq L(|f_0(s,x)| + |y|^p)$;
$|\partial_y f(s,x,y)| \leq L(1 + |y|^{p-1})$.

(H.2). $|f(s,x_1,y) - f(s,x_2,y)| \leq L(1 + |y|^p)|x_1 - x_2|$,
$|\partial_y f(s,x_1,y) - \partial_y f(s,x_2,y)| \leq L(1 + |y|^{p-1})|x_1 - x_2|$,
$|\partial_y f(s,x,y_1) - \partial_y f(s,x,y_2)| \leq L(1 + |y_1|^{p-2} + |y_2|^{p-2})|y_1 - y_2|$,

$g(s,x,y)$: Lipschitz condition on (s,x,y),
$\partial_y g(s,x,y)$: bounded and Lipschitz condition on (x,y).

(H.3). $\exists\ \mu > 0$ with $2\mu - K - p(2p-1)\sum_{j=1}^{\infty} L_j > 0$ s.t.

$$(y_1 - y_2)\big(f(s,x,y_1) - f(s,x,y_2)\big) \leq -\mu|y_1 - y_2|^2.$$

Assumptions (Continued)

(H.4). Diffusion coefficients $b \in C^2_{l,b}(\mathbb{R}^d;\mathbb{R}^d)$, $\sigma \in C^3_b(\mathbb{R}^d;\mathbb{R}^d \times \mathbb{R}^d)$.

(H.5). Matrix $\sigma(x)$ is uniformly elliptic, i.e. $\exists\ \varepsilon > 0$ s.t.

$$\sigma\sigma^*(x) \geq \varepsilon I_d.$$

Approximating Sequences

Step 1. To approximate infinite horizon BDSDEs:

$$\begin{aligned} Y_s^{t,x,m} &= \int_s^m f(r, X_r^{t,x}, Y_r^{t,x,m})dr - \int_s^m g(r, X_r^{t,x}, Y_r^{t,x,m})d^\dagger \hat{B}_r \\ &\quad - \int_s^m \langle Z_r^{t,x,m}, dW_r\rangle. \end{aligned}$$

Step 2. To approximate the polynomial growth generator:

$$f_n(s,x,y) = f(s,x,y)I_{\{|y|\leq n\}} + \partial_y f(s,x,\frac{n}{|y|}y)(y - \frac{n}{|y|}y)I_{\{|y|>n\}}.$$

$$f_n(s,x,y) \longrightarrow f(s,x,y), \quad \text{as } n \to \infty.$$

We need

(i) strongly convergent subsequence in $L^2(\Omega \times [0,T]; L^2_\rho(\mathbb{R}^d;\mathbb{R}^1))$

(ii) $L^p(\Omega \times [0,T]; L^p_\rho(\mathbb{R}^d;\mathbb{R}^1))$, $p \geq 1$, estimate.

Weak Convergence

Define $(U_\cdot^{t,\cdot,n}, V_\cdot^{t,\cdot,n}) \triangleq (f_n(r, X_r^{t,x}, Y_r^{t,x,n}), g_n(r, X_r^{t,x}, Y_r^{t,x,n}))$.

By Alaoglu lemma, a subsequence $(Y_\cdot^{t,\cdot,n}, Z_\cdot^{t,\cdot,n}, U_\cdot^{t,\cdot,n}, V_\cdot^{t,\cdot,n})$ converges weakly to a limit $(Y_\cdot^{t,\cdot}, Z_\cdot^{t,\cdot}, U_\cdot^{t,\cdot}, V_\cdot^{t,\cdot})$ in $L^2(\Omega \times [t,T]; L^2_\rho(\mathbb{R}^d;\mathbb{R}^1) \times L^2_\rho(\mathbb{R}^d;\mathbb{R}^d) \times L^2_\rho(\mathbb{R}^d;\mathbb{R}^1) \times L^2_\rho(\mathbb{R}^d;\mathbb{R}^l))$.

$$Y_s^{t,x} = h(X_T^{t,x}) + \int_s^T U_r^{t,x} dr - \int_s^T \langle V_r^{t,x}, d^\dagger \hat{B}_r \rangle - \int_s^T \langle Z_r^{t,x}, dW_r \rangle.$$

Key: finding a strongly convergent subsequence of $(Y_\cdot^{t,\cdot,n}, Z_\cdot^{t,\cdot,n})$ to get $(U_r^{t,x}, V_r^{t,x}) = (f(r, X_r^{t,x}, Y_r^{t,x}), g(r, X_r^{t,x}, Y_r^{t,x}))$.

The Correspondence between SPDEs and BDSDEs with Coefficients f_n

BDSDEs:

$$\begin{aligned} Y_s^{t,x,n} &= h(X_T^{t,x}) + \int_s^T f_n(r, X_r^{t,x}, Y_r^{t,x,n})dr \\ &\quad - \int_s^T \langle g(r, X_r^{t,x}, Y_r^{t,x,n}), d^\dagger \hat{B}_r \rangle - \int_s^T \langle Z_r^{t,x,n}, dW_r \rangle. \end{aligned}$$

SPDEs:

$$\begin{aligned} u_n(t,x) &= h(x) + \int_t^T \{\mathscr{L} u_n(s,x) + f_n\big(s, x, u_n(s,x)\big)\}ds \\ &\quad - \int_t^T \langle g\big(s, x, u_n(s,x)\big), d^\dagger \hat{B}_s \rangle, \qquad 0 \leq t \leq T. \end{aligned}$$

Correspondence:

$$u_n(t,x) \triangleq Y_t^{t,x,n}, u_n(s, X_s^{t,x}) = Y_s^{t,x,n}, \ (\sigma \nabla u_n)(s, X_s^{t,x}) = Z_s^{t,x,n}.$$

PDEs with Polynomial Growth Coefficients

We apply Rellich-Kondrachov Compactness Theorem to approximating PDEs to derive a strongly convergent subsequence of u_n in Zhang & Zhao 2012, JTP.

Theorem

Let $X \subset\subset H \subset Y$ be Banach spaces, with X reflexive. Here $X \subset\subset H$ means X is compactly embedded in H. Suppose that u_n is a sequence which is uniformly bounded in $L^2([0,T];X)$, and du_n/dt is uniformly bounded in $L^p([0,T];Y)$, for some $p > 1$. Then there is a subsequence which converges strongly in $L^2([0,T];H)$.

But this method does not work for the SPDE/BDSDE as Rellich-Kondrachov Compactness Theorem stands for PDEs and for fixed $\omega \in \Omega$ the subsequence choice may depend on ω.

SPDEs with Polynomial Growth Coefficients

Instead, we use Sobolev-Wiener Compactness Theorem, which is an extension of Rellich-Kondrachov compactness theorem to stochastic case with the help of Malliavin derivatives, proved in Bally & Saussereau 2004, JFA.

The time and space independent case was considered by Da Prato & Malliavin & Nualart 1992 and Peszat 1993

Sobolev-Wiener Compactness Theorem

Let $(u_n)_{n\in\mathbb{N}}$ be a sequence in $L^2([0,T]\times\Omega; H^1(\mathcal{O}))$. Define $u_n^\varphi(s,\omega) \triangleq \int_{\mathcal{O}} u_n(s,x,\omega)\varphi(x)dx$. Suppose that

(1) $\sup_n E[\int_0^T \|u_n(s,\cdot)\|^2_{H^1(\mathcal{O})} ds] < \infty$.

(2) For all $\varphi \in C_c^k(\mathcal{O})$ and $t \in [0,T]$, $u_n^\varphi(s) \in \mathbb{D}^{1,2}$ and $\sup_n \int_0^T \|u_n^\varphi(s)\|^2_{\mathbb{D}^{1,2}} ds < \infty$.

(3) For all $\varphi \in C_c^k(\mathcal{O})$, $(E[u_n^\varphi])_{n\in\mathbb{N}}$ of $L^2([0,T])$ satisfies

(3i) For any $\varepsilon > 0$, there exists $0 < \alpha < \beta < T$ s.t.

$$\sup_n \int_{[0,T]\setminus(\alpha,\beta)} |E[u_n^\varphi(s)]|^2 ds < \varepsilon.$$

(3ii) For any $0 < \alpha < \beta < T$ and $h \in \mathbb{R}^1$ s.t. $|h| < min(\alpha, T-\beta)$,

$$\sup_n \int_\alpha^\beta |E[u_n^\varphi(s+h)] - E[u_n^\varphi(s)]|^2 ds < C_p|h|.$$

Sobolev-Wiener Compactness Theorem

(4) For all $\varphi \in C_c^k(\mathcal{O})$, the following conditions are satisfied:
(4i) For any $\varepsilon > 0$, $\exists\ 0 < \alpha < \beta < T$ and $0 < \alpha' < \beta' < T$ s.t.

$$\sup_n E[\int_{[0,T]^2 \setminus (\alpha,\beta)\times(\alpha',\beta')} |D_\theta u_n^\varphi(s)|^2 d\theta ds] < \varepsilon.$$

(4ii) For any $0 < \alpha < \beta < T$, $0 < \alpha' < \beta' < T$ and $h, h' \in \mathbb{R}^1$ s.t.
$max(|h|, |h'|) < min(\alpha, \alpha', T-\beta, T-\beta')$,

$$\sup_n E[\int_\alpha^\beta \int_{\alpha'}^{\beta'} |D_{\theta+h} u_n^\varphi(s+h') - D_\theta u_n^\varphi(s)|^2 d\theta ds] < C_p(|h|+|h'|).$$

Then $(u_n)_{n\in\mathbb{N}}$ is relatively compact in $L^2(\Omega \times [0,T] \times \mathcal{O}; \mathbb{R}^1)$.

Generalized Equivalence of Norm Principle

The generalized equivalence of norm principle (based on Barles & Lesigne 1997, Bally & Matoussi 2001, JTP) is used to establish the equivalence of norm between the solutions of terminal-value SPDEs and the solutions of BDSDEs. Consider stochastic flows:

$$X_s^{t,x} = x + \int_t^s b(X_r^{t,x})dr + \int_t^s \sigma(X_r^{t,x})dW_r \quad s \geq t,$$

where $b \in C_{l,b}^2(\mathbb{R}^d;\mathbb{R}^d)$, $\sigma \in C_{l,b}^3(\mathbb{R}^d;\mathbb{R}^d \times \mathbb{R}^d)$.

Lemma

If $s \in [t,T]$, $\varphi : \Omega \times \mathbb{R}^d \to \mathbb{R}^1$ is independent of $\mathscr{F}_{t,s}^W$ and $\varphi\rho^{-1} \in L^1(\Omega \times \mathbb{R}^d;\mathbb{R}^1)$, then $\exists\ c, C > 0$ s.t.

$$\begin{aligned} cE[\int_{\mathbb{R}^d} |\varphi(x)|\rho^{-1}(x)dx] &\leq E[\int_{\mathbb{R}^d} |\varphi(X_s^{t,x})|\rho^{-1}(x)dx] \\ &\leq CE[\int_{\mathbb{R}^d} |\varphi(x)|\rho^{-1}(x)dx]. \end{aligned}$$

Definition for Random Periodic Solutions

A measurable space: $(V, \mathscr{B}(V))$.

A metric dynamical system $(\Omega, \mathscr{F}, P, (\theta_t)_{t\geq 0})$.

For a measurable random dynamical system $v : \mathbb{R}^1 \times \mathbb{R}^1 \times V \times \Omega \to V$, the random periodic solution with period $\tau > 0$ is an $\mathscr{F}$ measurable r.v. $Y : \mathbb{R}^1 \times \Omega \to V$ such that

$$v^{t,Y(t,\omega)}(t+\tau,\omega) = Y(t+\tau,\omega) = Y(t,\theta_\tau\omega), \quad t \geq 0 \ a.s.$$

Some Existing Results

- Zhao & Zheng 2009, JDE Random periodic solutions for C^1-cocycles
- Feng & Zhao & Zhou 2011, JDE Random periodic solutions of SDEs with additive noise
- Feng & Zhao 2012, JFA Random periodic solutions of SPDEs with additive noise

Random Periodic Solutions of SDEs with Non-Additive Noise

We study the following SDE valued in $\mathbb{R}^d$:

$$u^{t,\xi}(s) = \xi + \int_t^s \big[- Au^{t,\xi}(r) + b\big(r, u^{t,\xi}(r)\big)\big] dr + \int_t^s \sigma\big(r, u^{t,\xi}(r)\big) dB_r.$$

A is an invertible matrix satisfying

$$\delta \triangleq \inf \big\{ \mathrm{Re}(\lambda) : (\lambda_i)_{i=1,\cdots,d} \text{ are the eigenvalues of } A \big\} > 0;$$

$b : \mathbb{R}^1 \times \mathbb{R}^d \to \mathbb{R}^d, \quad \sigma : \mathbb{R}^1 \times \mathbb{R}^d \to \mathbb{R}^{d\times d};$

$b(t,x) = b(t+\tau, x), \quad \sigma(t,x) = \sigma(t+\tau, x).$

Infinite Horizon Integral Equation

By Duhamel's formula,

$$\begin{aligned} u^{t,\xi}(s) &= e^{-A(s-t)}\xi + \int_t^s e^{-A(s-r)} b\big(r, u^{t,\xi}(r)\big) dr \\ &\quad + \int_t^s e^{-A(s-r)} \sigma\big(r, u^{t,\xi}(r)\big) dB_r. \end{aligned}$$

Introduce the infinite horizon integral equation:

$$X_s = \int_{-\infty}^s e^{-A(s-r)} b(r, X_r) dr + \int_{-\infty}^s e^{-A(s-r)} \sigma(r, X_r) dB_r.$$

Then

$$X_s = e^{-A(s-t)} X_t + \int_t^s e^{-A(s-r)} b(r, X_r) dr + \int_t^s e^{-A(s-r)} \sigma(r, X_r) dB_r.$$

Random Periodic Property of X_s

If the original SDE admits a unique solution $u^{t,\xi}(s)$, then

$$u^{t,X_t}(s) = X_s.$$

X_s is a random periodic solution of the original SDE if

$$X_{s+\tau}(\omega) = X_s(\theta_\tau \omega).$$

Recursive sequence (Qiao & Zhang & X. Zhang, Preprint):

$$X_s^{n+1} = \int_{-\infty}^{s} \mathrm{e}^{-A(s-r)} b(r, X_r^n) dr + \int_{-\infty}^{s} \mathrm{e}^{-A(s-r)} \sigma(r, X_r^n) dB_r.$$

By recursion, for all n,

$$X_{s+\tau}^n(\omega) = X_s^n(\theta_\tau \omega).$$

Main Results for Random Periodic Solution

Theorem

Assume $b, \sigma, \nabla b, \nabla \sigma$ are bounded and $2\|\nabla b\|_\infty^2 \delta^{-2} + 2\|\nabla \sigma\|_\infty^2 (2\delta)^{-1} < 1$. Then the infinite horizon integral equation has a unique solution X_s which is a random periodic solution of SDE.

Sketch of Proof: 1. X^n is a Cauchy sequence in $C(\mathbb{R}^1; L^2(\Omega))$.

2. Take X such that

$$\lim_{n\to\infty} \sup_{s\in\mathbb{R}^1} E|X_s^n - X_s|^2 = 0.$$

3. As $n \to \infty$, it appears that X satisfies the infinite horizon integral equation and

$$X_{s+\tau}(\omega) = X_s(\theta_\tau \omega).$$

Thank You