Pricing Variance Swaps on Time-Changed Markov Processes

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Intro to variance swaps

- Let F be forward price of an asset.
- Assume $F_t > 0$ for all t.
- Define the process $X := \log F$.
- The floating leg of a variance swap pays (to the long side)

$$\sum_{t_i \in [0,T]} \left(X_{t_{i+1}} - X_{t_i} \right)^2 . \tag{1}$$

- As sampling frequency increases: (1) $\xrightarrow{\mathbb{P}} [X]_T$.
- A swap whose floating leg pays $[X]_T$ is called a continuously monitored variance swap.
- Using risk-neutral pricing, the fair strike of a VS is $\mathbb{E}\,[X]_T$
- Question: how to compute $\mathbb{E}\,[X]_T$?

Non-parametric approach (with no jumps)

Suppose $F_t = \exp(X_t)$ with

$$dX_t = -\frac{1}{2}\sigma_t^2 dt + \sigma_t dW_t.$$

In this setting, Neuberger (1990) and Dupire (1993) show VS priced by log contract:

$$\mathbb{E}\,[X]_T = -2\,\mathbb{E}\,(X_T - X_0) = -2\,\mathbb{E}\log(F_T/F_0).$$

Quick proof:

$$\begin{aligned}\mathbb{E}\,[X]_T &= \mathbb{E}\int_0^T \sigma_t^2 dt \\ &= -2\,\mathbb{E}\int_0^T dX_t + 2\,\mathbb{E}\int_0^T \sigma_t dW_t \\ &= -2\,\mathbb{E}\,(X_T - X_0). \qquad \square\end{aligned}$$

Synthetic European contracts

As shown in Carr and Madan (1998), if $h \in C^2(\mathbb{R}^+)$ then for any $\kappa \in \mathbb{R}^+$ we have

$$h(F_T) = h(\kappa) + h'(\kappa)\Big((F_T - \kappa)^+ - (\kappa - F_T)^+\Big) + \int_0^\kappa h''(K)(K - F_T)^+ dK + \int_\kappa^\infty h''(K)(F_T - K)^+ dK.$$

Taking expectations, we have

$$\mathbb{E}\, h(F_T) = h(\kappa) + h'(\kappa)\Big(C(T,\kappa) - P(T,\kappa)\Big) + \int_0^\kappa h''(K) P(T,K) dK + \int_\kappa^\infty h''(K) C(T,K) dK,$$

where $P(T,K)$ and $C(T,K)$ are European put and call prices

Synthetic $\log$ contract and VIX

To price a VS, take $h(F) = -2\log(F/F_0)$.

$$\begin{aligned}\mathbb{E}\,[X]_T &= -2\,\mathbb{E}\,\log(F_T/F_0)\\ &= \int_0^{F_0} \frac{2}{K^2} P(T,K)dK + \int_{F_0}^{\infty} \frac{2}{K^2} C(T,K)dK, \qquad (2)\end{aligned}$$

Discretized version of (2) is used to construct VIX, the CBOE's 30-day forward looking measure of volatility.

Note: equation (2) prices VS correctly only when X experiences no jumps.

Non-parametric pricing of VS with jumps

Suppose $F_t = \exp(Y_{\tau_t})$ where τ is a continuous stochastic clock (possibly correlated with Y) and Y is a Lévy process

$$dY_t = b\,dt + \sigma\,dW_t + \int_{\mathbb{R}} z d\widetilde{N}_t(dz),$$

$$d\widetilde{N}_t(dz) = dN_t(dz) - \mu(dz)dt,$$

$$b = -\frac{1}{2}\sigma^2 - \int_{\mathbb{R}} (e^z - 1 - z)\nu(dz).$$

In this setting Carr, Lee, and Wu (2011) show

$$\mathbb{E}\,[X]_T = -Q\,\mathbb{E}\,(X_T - X_0) = -Q\,\mathbb{E}\,\log(F_T/F_0).$$

The multiplyer Q depends only on the Lévy process Y – not on the clock

$$Q = \frac{\sigma^2 + \int_{-\infty}^{\infty} z^2\nu(dz)}{\sigma^2/2 + \int_{-\infty}^{\infty} (e^z - 1 - z)\,\nu(dz)}$$

Note: if $\nu \equiv 0$ then $Q = 2$ (recover result of Neuberger/Dupire).

Features and limitations of time-changed Lévy processes

Features

- Allows for jumps, stochastic volatility and leverage effect
- Multiplier Q depends only on background Lévy process – not time-change.
- Includes many popular models (e.g., Heston, Exponential Lévy, etc.) in a single framework.

Possible limitations

- In time-changed Lévy approach, the multiplier Q should be constant

$$Q = \frac{-\mathbb{E}\left[\log F\right]_T}{\mathbb{E}\,\log(F_T/F_0)}.$$

Evidence from Carr, Lee, and Wu (2011) suggests Q is not constant in time or across maturities.

Time-changed Markov Processes

Suppose F is modeled by

$$F_t = \exp(Y_{\tau_t})$$

where τ is a continuous stochastic clock possibly correlated with Y and Y is any continuous time scalar Markov process

$$dY_t = b(Y_t)\,dt + a(Y_t)\,dW_t + \int_{\mathbb{R}} z\,d\widetilde{N}_t(Y_{t-}, dz),$$

Here, $d\widetilde{N}_t(Y_{t-}, dz)$ is a compensated Poisson random measure with state-dependent jumps

$$d\widetilde{N}_t(Y_{t-}, dz) = dN_t(Y_{t-}, dz) - \mu(Y_{t-}, dz)dt,$$

and the drift $b(Y_t)$ is fixed by a and μ so that F is a martingale

$$b(Y_t) = -\frac{1}{2}a^2(Y_t) - \int_{\mathbb{R}} \mu(Y_{t-}, dz)\left(e^z - 1 - z\right).$$

The generator of Y

Note that Y has generator

$$\mathcal{A} = \frac{1}{2}a^2(y)\left(\partial^2 - \partial\right) + \int_{\mathbb{R}} \mu(y,dz)\Big(\theta_z - 1 - z\partial\Big) - \int_{\mathbb{R}} \mu(y,dz)\Big(e^z - 1 - z\Big)\partial,$$

where θ_z is the shift operator: $\theta_z f(y) = f(y+z)$.
Note, for analytic f, we have

$$e^{z\partial} f(y) = \sum_{n=0}^{\infty} \frac{z^n}{n!}\partial^n f(y) = f(y+z).$$

Formally, then, we re-write the generator $\mathcal{A}$ as follows:

$$\mathcal{A} = \frac{1}{2}a^2(y)\left(\partial^2 - \partial\right) + \int_{\mathbb{R}} \mu(y,dz)\Big(e^{z\partial} - 1 - z\partial\Big) - \int_{\mathbb{R}} \mu(y,dz)\Big(e^z - 1 - z\Big)\partial.$$

The quadratic variation process $[Y]$

Note that $d[Y]_t$ is given by

$$d[Y]_t = \left(a^2(Y_t) + \int_{\mathbb{R}} z^2 \mu(Y_t, dz) \right) dt + \underbrace{\int_{\mathbb{R}} z^2 d\widetilde{N}(Y_{t-}, dz)}_{\text{martingale}}.$$

Hence

$$\mathbb{E}\, d[Y]_t = \mathbb{E} \left(a^2(Y_t) + \int_{\mathbb{R}} z^2 \mu(Y_t, dz) \right) dt.$$

Likewise, for $G \in \text{dom}(\mathcal{A})$ we have

$$dG(Y_t) = \mathcal{A}G(Y_t)dt + \text{martingale}.$$

Hence

$$\mathbb{E}\, dG(Y_t) = \mathbb{E}\, \mathcal{A}G(Y_t)dt.$$

Variance Swap Pricing

Suppose we can find a function G such that

$$-\mathcal{A}\, G(y) = a^2(y) + \int_{\mathbb{R}} z^2 \mu(y, dz).$$

Then, using results from the previous page, we have

$$\begin{aligned}
\mathbb{E}\,[Y]_{\tau_T} &= \mathbb{E} \int_0^{\tau_T} d[Y]_t \\
&= \mathbb{E} \int_0^{\tau_T} \left(a^2(Y_t) + \int_{\mathbb{R}} z^2 \mu(Y_t, dz) \right) dt \\
&= -\mathbb{E} \int_0^{\tau_T} \mathcal{A} G(Y_t) dt \\
&= -\mathbb{E} \int_0^{\tau_T} dG(Y_t) + \cancel{\mathbb{E}\,\text{martingale}} \\
&= -\mathbb{E}\, G(Y_{\tau_T}) + G(Y_0).
\end{aligned}$$

Recall $F_t = \exp(Y_{\tau_t})$

Continuity of time-change τ implies: $[\log F]_T = [Y]_{\tau_T}$. Hence

$$\underbrace{\mathbb{E}\,[\log F]_T}_{\text{A}} = \underbrace{-\mathbb{E}\,G(\log F_T)}_{\text{B}} + \underbrace{G(\log F_0)}_{\text{C}}.$$

- A: Fair strike of a variance swap.
- B: Value of a European contract with payoff: $-G(\log F_T)$.
- C: Value of $G(\log F_0)$ zero-coupon bonds.

Quantity B can be constructed from T-maturity calls/puts a la Carr and Madan (1998).

- To price a VS, we must solve OIDE:

$$-\mathcal{A}\,G(y) = a^2(y) + \int_{\mathbb{R}} z^2 \mu(y, dz).$$

Simplification

Define: $H := \partial G$ so that $G(y) = \int H(y)dy$. Then H solves

$$-\frac{\mathcal{A}}{\partial} H = a^2(y) + \int_{\mathbb{R}} z^2 \mu(y, dz),$$

where

$$\frac{\mathcal{A}}{\partial} = \frac{1}{2} a^2(y) \left(\partial - 1\right) + \int_{\mathbb{R}} \mu(y, dz) \left(\frac{e^{z\partial} - 1 - z\partial}{\partial} \right) - \int_{\mathbb{R}} \mu(y, dz) \left(e^z - 1 - z \right)$$

and

$$\frac{e^{z\partial} - 1 - z\partial}{\partial} := \sum_{n=2}^{\infty} \frac{1}{n!} z^n \partial^{n-1}.$$

Example 1: jump-intensity proportional to local variance

Introduce $\gamma(y) > 0$. Assume

$$a^2(y) = \gamma^2(y)\,\sigma^2, \qquad \mu(y, dz) = \gamma^2(y)\,\nu(dz),$$

Easy to check that $H(y)$ is given by

$$H = Q := \frac{\sigma^2 + I_2}{\sigma^2/2 + I_0}, \qquad G = Q\,y,$$

where

$$I_0 := \int_{\mathbb{R}} \nu(dz)\,(e^z - 1 - z)\,, \qquad I_n := \int_{\mathbb{R}} \nu(dz) z^n, \qquad n \geq 2.$$

Note: this case includes Time-changed Lévy case (take $\gamma(y) = 1$).
Thus, we recover result of Carr, Lee, and Wu (2011):

$$\mathbb{E}\,[\log F]_T = -\mathbb{E}\,G(\log F_T) + G(\log F_0) = -Q\log(F_T/F_0).$$

Neuberger-Dupire $\subset$ Carr-Lee-Wu $\subset$ Carr-Lee-Lorig.

Limiting cases

The following limiting cases are useful:

$$\begin{aligned} \text{No Jumps}: \quad & \nu \equiv 0, \quad & H = 2, \\ \text{Pure Jumps}: \quad & \sigma = 0, \quad & H = I_2/I_0 =: Q_0. \end{aligned}$$

We can use these limiting cases to build other exact solutions.

Example 2: building around pure jump solution

Introduce $e_c(y) = e^{cy}$ and $\delta \geq 0$. Assume

$$a^2(y) = \delta\, \sigma^2(y), \qquad \mu(y, dz) = \eta(y)\nu(dz), \qquad \frac{\sigma^2(y)}{2\,\eta(y)} = e_c(y).$$

Then H solves

$$0 = \delta e_c \left(\mathcal{A}_1 H + 2\right) + \left(\mathcal{A}_0 H + I_2\right), \tag{3}$$

where we have defined operators $\mathcal{A}_0$ and $\mathcal{A}_1$

$$\mathcal{A}_0 = \int_{\mathbb{R}} \nu(dz) \left(\frac{e^{z\partial} - 1 - z\partial}{\partial} \right) - \int_{\mathbb{R}} \nu(dz) \left(e^z - 1 - z\right),$$
$$\mathcal{A}_1 = \partial - 1.$$

Assume H is power series in δ

$$H = \sum_{n=0}^{\infty} \delta^n H_n. \tag{4}$$

Insert expansion (4) into OIDE (6) and collect terms of like order in δ

$$\begin{aligned}
\mathcal{O}(\delta^0) : \quad & \mathcal{A}_0 H_0 = -I_2, & \\
\mathcal{O}(\delta) : \quad & \mathcal{A}_0 H_1 = -e_c \left(\mathcal{A}_1 H_0 + 2 \right), & \\
\mathcal{O}(\delta^n) : \quad & \mathcal{A}_0 H_n = -e_c \mathcal{A}_1 H_{n-1}, & n \geq 2.
\end{aligned}$$

We need to study the operator $\mathcal{A}_0$ and its inverse

The operator $\mathcal{A}_0$

$\mathcal{A}_0$ is a pseudo-differential operator (ΨDO)

$$\mathcal{A}_0 = \int_{\mathbb{R}} \nu(dz) \left(\frac{e^{z\partial} - 1 - z\partial}{\partial} \right) - \int_{\mathbb{R}} \nu(dz) \left(e^z - 1 - z \right).$$

ΨDO's are characterized by their action on oscillating exponentials

$$\mathcal{A}_0 \psi_\lambda = \phi_\lambda \psi_\lambda, \qquad \psi_\lambda := \tfrac{1}{\sqrt{2\pi}} e^{i\lambda y}.$$

where ϕ_λ, called the symbol of $\mathcal{A}_0$, satisfies ($\partial \to i\lambda$)

$$\phi_\lambda = \int_{\mathbb{R}} \nu(dz) \left(\frac{e^{i\lambda z} - 1 - i\lambda z}{i\lambda} \right) - \int_{\mathbb{R}} \nu(dz) \left(e^z - 1 - z \right).$$

The inverse operator $\mathcal{A}_0^{-1} = \frac{1}{\mathcal{A}_0}$ is given by

$$\frac{1}{\mathcal{A}_0} \cdot = \int_{\mathbb{R}} d\lambda \, \frac{1}{\phi_\lambda} \langle \psi_\lambda, \cdot \rangle \psi_\lambda, \qquad \langle u, v \rangle := \int_{\mathbb{R}} dy \, \overline{u}(y) \, v(y),$$

Using definition of $\frac{1}{\mathcal{A}_0}$ we find $H_0 = Q_0$ and for $n \geq 1$:

$$H_n = (Q_0 - 2) \underbrace{\int \cdots \int}_{n} d\lambda_n \frac{\psi_{\lambda_n}}{\phi_{\lambda_n}} \langle \psi_{\lambda_1}, e_c \rangle \times$$

$$\prod_{k=1}^{n-1} d\lambda_k \frac{-\chi_{\lambda_k}}{\phi_{\lambda_k}} \langle \psi_{\lambda_{k+1}}, e_c \psi_{\lambda_k} \rangle, \tag{5}$$

where $\chi_\lambda = i\lambda - 1$ is symbol of $\mathcal{A}_1 = \partial - 1$.
Noting that

$$\langle \psi_\lambda, e_c \rangle = \sqrt{2\pi}\delta(\lambda + ic) \quad \text{and} \quad \langle \psi_\mu, e_c \psi_\lambda \rangle = \delta(\mu - \lambda + ic),$$

equation (5) becomes

$$H_n = (Q_0 - 2) \frac{\sqrt{2\pi}\, \psi_{-inc}}{\phi_{-inc}} \prod_{k=1}^{n-1} \left(\frac{-\chi_{-ikc}}{\phi_{-ikc}} \right) \qquad n \geq 1.$$

No integrals! ☺

Using $H = \sum_n \delta^n H_n$ we have

$$H = Q_0 + (Q_0 - 2) \sum_{n=1}^{\infty} a_n (\delta e_c)^n, \quad a_n = \frac{1}{\phi_{-inc}} \prod_{k=1}^{n-1} \frac{-\chi_{-inc}}{\phi_{-ikc}}.$$

If the measure ν is such that

1. $\int_{\mathbb{R}} (e^{ncz} - 1 - ncz)\nu(dz) < \infty$ for all $n \in \mathbb{N}$,
2. $\lim_{n\to\infty} \frac{n^2 c^2}{\int_{\mathbb{R}} (e^{ncz} - 1 - ncz)\nu(dz)} = 0$,

then the coefficients a_n satisfy

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 0,$$

and the series converges (and the radius of convergence is $\mathbb{R}$)

Finally, using $G(y) = \int H(y)dy$, the function that prices the variance swap is

$$G = Q_0\, y + \sum_{n=1}^{\infty} \delta^n G_n,$$

$$G_n = (Q_0 - 2)\frac{e_{nc}}{nc \cdot \phi_{-inc}} \prod_{k=0}^{n-1} \left(\frac{-\chi_{-ikc}}{\phi_{-ikc}} \right), \qquad n \geq 1.$$

In figures 1 and 2 we plot $Q_0 \log(F_T/F_0)$ and

$$h(F_T) := -G(\log F_T) + G(\log F_0) + A(F_T - F_0),$$

as a function of F_T

- The constant A is chosen so that $h(F_T)$ has the same slope as $-Q_0 \log(F_T/F_0)$ at $F_T = F_0$.
- Forward contracts $(F_T - F_0)$ have no value since $\mathbb{E}\, F_T = F_0$.

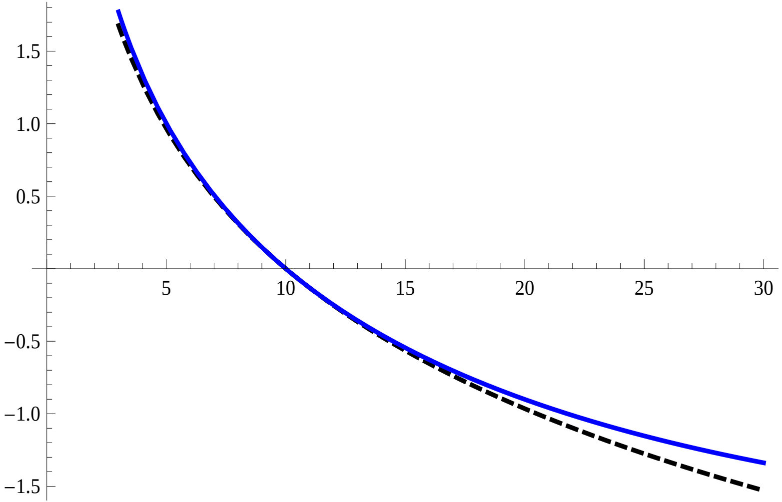

Figure 1: We plot $h(F_T)$ as a function of F_T (solid blue). For comparison we also plot $-Q_0 \log(F_T/F_0)$ (dashed black). In this Figure, $F_0 = 10.0$, $c = 0.23$, $\delta = 0.22$ and jumps are distributed with a Dirac mass $\nu \sim \delta_{z_0}$ with $z_0 = 1.0$.

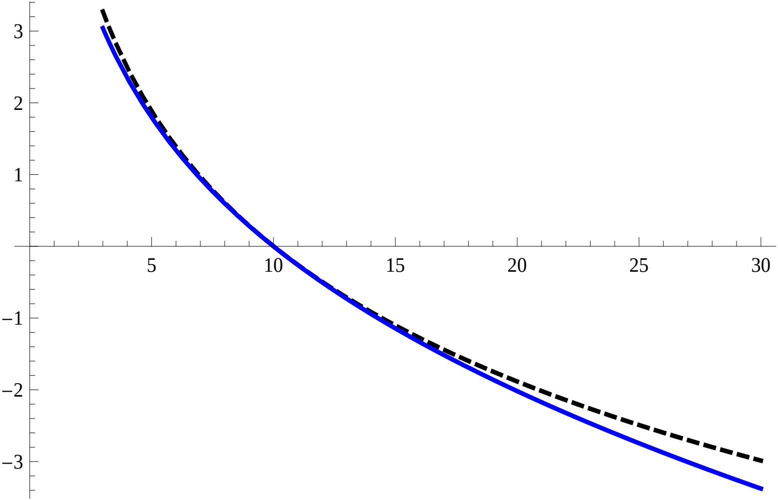

Figure 2: We plot $h(F_T)$ as a function of F_T (solid blue). For comparison we also plot $-Q_0 \log(F_T/F_0)$ (dashed black). In this Figure, $F_0 = 10.0$, $c = -0.21$, $\delta = 1.00$ and jumps are jumps are distributed with a Dirac mass $\nu \sim \delta_{z_0}$ with $z_0 = -1.0$.

Example 3: building around no-jump solution

Introduce $e_c(y) = e^{cy}$ and $\delta \geq 0$. Assume

$$a^2(y) = \sigma^2(y), \qquad \mu(y, dz) = \delta\, \eta(y) \nu(dz), \qquad \frac{2\,\eta(y)}{\sigma^2(y)} = e_c(y).$$

Then H solves

$$\text{around no jump :} \qquad 0 = \delta e_c \left(\mathcal{A}_0 H + I_2\right) + \left(\mathcal{A}_1 H + 2\right), \quad (6)$$

Compare to

$$\text{around pure jump :} \qquad 0 = \left(\mathcal{A}_0 H + I_2\right) + \delta e_c \left(\mathcal{A}_1 H + 2\right)$$

Just reverse roles of:

$$\mathcal{A}_0 \longleftrightarrow \mathcal{A}_1 \qquad \text{and} \qquad 2 \longleftrightarrow I_2,$$

$$\phi_\lambda \longleftrightarrow \chi_\lambda$$

Reversing roles of ϕ and χ (symbols of $\mathcal{A}_0$ and $\mathcal{A}_1$ resp.)

$$G = 2\,y + \sum_{n=1}^{\infty} \delta^n G_n,$$

$$G_n = (2I_0 - I_2)\frac{e_{nc}}{nc \cdot \chi_{-inc}} \prod_{k=0}^{n-1} \left(\frac{-\phi_{-ikc}}{\chi_{-ikc}} \right), \qquad n \geq 1.$$

Conditions for convergence:

- $\int_{\mathbb{R}} (e^{ncz} - 1 - ncz)\nu(dz) < \infty$ for all $n \in \mathbb{N}$,
- $\lim_{n\to\infty} \frac{\int_{\mathbb{R}} (e^{ncz} - 1 - ncz)\nu(dz)}{n^2 c^2} = 0$ (reciprocal of prev. cond.).

In figures 3 and 4 we plot $2\log(F_T/F_0)$ and

$$h(F_T) := -G(\log F_T) + G(\log F_0) + A(F_T - F_0),$$

as a function of F_T. The constant A is chosen so that $h(F_T)$ has the same slope as $-2\log(F_T/F_0)$ at $F_T = F_0$.

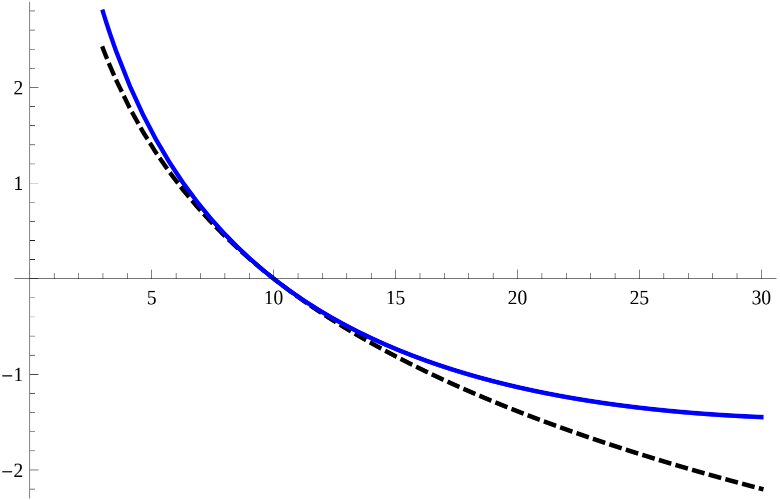

Figure 3: We plot $h(F_T)$ as a function of F_T (solid blue). For comparison we also plot $-2\log(F_T/F_0)$ (dashed black). In this Figure, $F_0 = 10.0$, $c = 0.39$, $\delta = 1.25$ and $\nu = \delta_{z_0}$ (Dirac measure) with $z_0 = -1.50$.

Negative jumps raise value of VS relative to two $\log$ contracts.

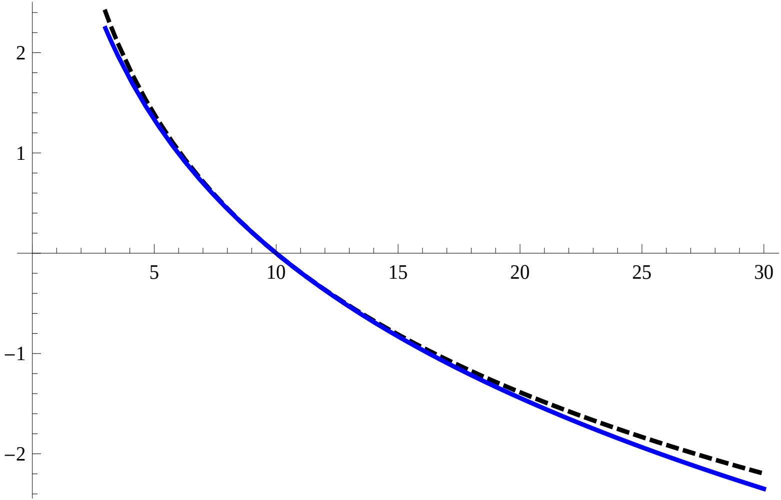

Figure 4: We plot $h(F_T)$ as a function of F_T (solid blue). For comparison we also plot $-2\log(F_T/F_0)$ (dashed black). In this Figure, $F_0 = 10.0$, $c = -1.05$, $\delta = 1.00$ and $\nu = \delta_{z_0}$ (Dirac measure) with $z_0 = 1.75$. Positive jumps lower value of VS relative to two $\log$ contracts.

Example 4: subordinate diffusions

Model forward price as

$$F_t = \exp(Y^{\phi}_{\tau_t}), \qquad Y^{\phi}_t = Y_{T_t}$$

where τ is a continuous time-change, Y is a diffusion absorbed at endpoints $L < R$

$$dY_t = -\frac{1}{2}\sigma^2(Y_t)dt + \sigma(Y_t)dW_t,$$

and T is a Lévy subordinator

$$dT_t = b\,dt + \int_0^\infty z dN^{\rho}_t(dz), \qquad \mathbb{E}dN^{\rho}_t(dz) = \rho(dz)dt.$$

The process Y^{ϕ} experiences jumps because the subordinator T jumps.

Some spectral theory

The generator of Y, given by

$$\mathcal{A} = \frac{1}{2}\sigma^2(y)(\partial^2 - \partial),$$
$$\text{dom}(\mathcal{A}) = \{f \in C^2([L,R]) : f(L) = f(R) = 0\},$$

is self-adjoint on $L^2([L,R],\mathfrak{m})$ where $\mathfrak{m}$ is the speed measure of $\mathcal{A}$

$$\langle f, \mathcal{A}g\rangle_{\mathfrak{m}} = \langle \mathcal{A}f, g\rangle_{\mathfrak{m}}, \qquad \mathfrak{m}(y) = \frac{e^{-y}}{\sigma^2(y)}.$$

By the spectral theorem, the operator $g(\mathcal{A})$ is defined as

$$g(\mathcal{A}) = \sum_n g(\lambda_n)\langle\psi_n, \cdot\rangle_{\mathfrak{m}}\psi_n, \qquad \mathcal{A}\psi_n = \lambda_n\psi_n.$$

In particular

resolvent : $(\mathcal{A} - z)u = h \quad \Rightarrow \quad u = \frac{1}{\mathcal{A} - z}h,$

semigroup : $u(t,y) = \mathbb{E}_y h(Y_t) \quad \Rightarrow \quad u(t,y) = e^{t\mathcal{A}}h(y).$

$\mathcal{A}^\phi$ – the generator of $Y_t^\phi = Y_{T_t}$

The subordinator T is characterized by its Laplace exponent

$$\mathbb{E}e^{\lambda T_t} = e^{t\phi(\lambda)}, \qquad \phi(\lambda) = b\lambda + \int_0^t \rho(ds)(e^{s\lambda} - 1).$$

We compute the semigroup $e^{t\mathcal{A}^\phi}$ of Y^ϕ as follows

$$\begin{aligned} e^{t\mathcal{A}^\phi} h(y) &= \mathbb{E}_y h(Y_t^\phi) = \mathbb{E}\,\mathbb{E}_y[h(Y_{T_t})|T_t] \\ &= \mathbb{E}e^{T_t\mathcal{A}} h(y) = e^{t\phi(\mathcal{A})} h(y). \end{aligned}$$

Therefore, the generator $\mathcal{A}^\phi$ is given by

$$\mathcal{A}^\phi = \lim_{t\to 0} \frac{1}{t}\left(e^{t\mathcal{A}^\phi} - 1\right) = \lim_{t\to 0} \frac{1}{t}\left(e^{t\phi(\mathcal{A})} - 1\right) = \phi(\mathcal{A}),$$

And the resolvent is given by

$$\frac{1}{\mathcal{A}^\phi - z} = \frac{1}{\phi(\mathcal{A}) - z}.$$

VS pricing

The function G that prices the VS solves

$$\mathcal{A}^{\phi}G(y) = h(y),$$
$$h(y) = -b\,\sigma^2(y) - \int_{L-y}^{R-y} \mu(y,dz)\,z^2,$$
$$\mu(y,dz) = \int_0^{\infty} \rho(ds) p_Y(s,y,y+z)dz,$$
$$p_Y(s,y,y+z) = e^{t\mathcal{A}}\delta_{y+z}(y).$$

The solution can be written down directly:

$$G = \frac{1}{\mathcal{A}^{\phi}}\,h = \frac{1}{\phi(\mathcal{A})}\,h = \sum_n \frac{1}{\phi(\lambda_n)}\langle \psi_n, h\rangle_{\mathfrak{m}}\psi_n.$$

Specific solutions are computed by solving $\mathcal{A}\psi_n = \lambda_n\psi_n$ and computing $\phi(\lambda)$.

Simple example

Let background process Y have dynamics

$$dY_t = -\frac{1}{2}\sigma^2 dt + \sigma dW_t \qquad \Rightarrow \qquad \mathcal{A} = \frac{1}{2}\sigma^2(\partial^2 - \partial).$$

We need to solve eigenvalue problem

$$\mathcal{A}\psi_n = \lambda_n \psi_n, \qquad \psi_n(L) = \psi_n(R) = 0.$$

The solution is

$$\psi_n(y) = e^{y/2}\sqrt{\frac{\sigma^2}{R-L}}\sin\Big(\alpha_n(y-L)\Big), \qquad \alpha_n = \frac{n\pi}{R-L},$$

$$\lambda_n = \frac{-\sigma^2}{2}\left(\alpha_n^2 + \frac{1}{4}\right), \qquad n \in \mathbb{N}.$$

Let the Lévy density of the subordinator T be exponential

$$\rho(ds) = Ce^{-\eta s}\,ds, \qquad \Rightarrow \qquad \phi(\lambda) = b\lambda + \frac{C\lambda}{\eta^2 - \eta\lambda},$$

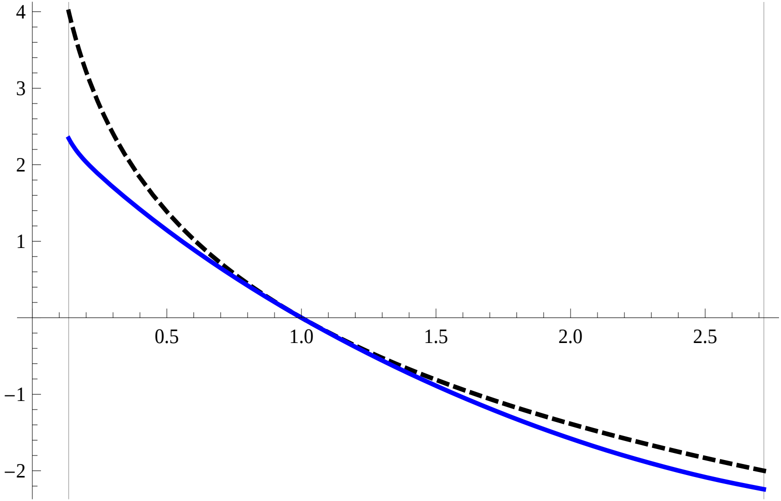

Figure 5: We plot $h(F_T) = -G(\log F_T) + G(\log F_0) + A(F_T - F_0)$, (solid blue) and $-Q\log(F_T/F_0)$ (dashed black) as a function of F_T. The Lévy measure ρ of the subordinator is exponential: $\rho(ds) = Ce^{-\eta s}ds$. In this Figure, $\sigma = 1$, $b = 0$ (i.e., no diffusion component), $C = 1$, $\eta = 1$, $L = -2$, $R = 1$, $F_0 = e^{Y_0^\phi} = 1$, $Q = 2$.

Quick recap

- We have shown that, in the Time-change Markov process setting, a VS has the same value as a European option with payoff $h(F_T) := -G(\log F_T) + G(\log F_0)$.
- One can compute the value of this option $\mathbb{E}\, h(F_T)$ using co-terminal calls and puts

$$\mathbb{E}\, h(F_T) = h(\kappa) + h'(\kappa)\Big(C(T,\kappa) - P(T,\kappa)\Big) + \int_0^\kappa h''(K)P(T,K)dK + \int_\kappa^\infty h''(K)C(T,K)dK,$$

- For certain special cases, we can also compute $\mathbb{E}\, h(F_T)$ directly from model parameters.
- This will allow us to show how the ratio $\frac{-\mathbb{E}\,[\log F]_T}{\mathbb{E}\,\log(F_T/F_0)}$ varies as a function of F_0.

Special case 1: European option pricing

Introduce $\omega > 0$ and $\delta > 0$. Let

$$a^2(y) = 2\,\omega^2, \qquad \mu(y, dz) = \delta\,\omega^2 e_c(y)\,\nu(dz),$$

This model falls under the "building around no jumps" setting of example 3. Thus, we know the function G that prices the VS. We wish to find $u(t, y) := \mathbb{E}_y\, G(Y_t)$. From the KBE we have

$$(-\partial_t + \mathcal{A})u = 0, \qquad u(0, y) = G(y).$$

where $\mathcal{A}$ is the infinitesimal generator of Y.

The generator $\mathcal{A}$ of Y

The process Y has generator

$$\mathcal{A} = \delta e_c \mathcal{L}_0 + \mathcal{L}_1$$

with

$$\mathcal{L}_0 = \omega^2 \int_{\mathbb{R}} \nu(dz)\Big(e^{z\partial} - 1 - z\partial\Big) - \omega^2 \int_{\mathbb{R}} \nu(dz)\Big(e^{z} - 1 - z\Big)\partial,$$
$$\mathcal{L}_1 = \omega^2 \left(\partial^2 - \partial\right).$$

$\mathcal{L}_0$ and $\mathcal{L}_1$ are ΨDOs with symbols Φ_λ and $\mathcal{X}_\lambda$ respectively

$$\Phi_\lambda = \omega^2 \int_{\mathbb{R}} \nu(dz)\Big(e^{i\lambda z} - 1 - i\lambda z\Big) - \omega^2 \int_{\mathbb{R}} \nu(dz)\Big(e^{z} - 1 - z\Big)i\lambda,$$
$$\mathcal{X}_\lambda = \omega^2 \left(-\lambda^2 - i\lambda\right).$$

Assume u is power series in δ (same game as for H)

$$u = \sum_{n \geq 0} \delta^n u_n.$$

Insert expansion for u and $\mathcal{A} = \delta e_c \mathcal{L}_0 + \mathcal{L}_1$ into KBE and collect like powers of δ.

$$\begin{aligned} \mathcal{O}(\delta^0): \quad & (-\partial_t + \mathcal{L}_1) u_0 = 0, & u_0(0,y) &= G(y), \\ \mathcal{O}(\delta^n): \quad & (-\partial_t + \mathcal{L}_1) u_n = -e_c \mathcal{L}_0 u_{n-1}, & u_n(0,y) &= 0. \end{aligned}$$

The formal solution is

$$\begin{aligned} \mathcal{O}(\delta^0): \quad & u_0(t,y) = e^{t\mathcal{L}_1} G(y) \\ \mathcal{O}(\delta^n): \quad & u_n(t,y) = \int_0^t ds\, e^{(t-s)\mathcal{L}_1} e^{cy} \mathcal{L}_0 u_{n-1}(s,y), \end{aligned}$$

where the semigroup of operators $\mathcal{P}_t := e^{t\mathcal{L}_1}$ is given by

$$e^{t\mathcal{L}_1} \cdot = \int_{\mathbb{R}} d\lambda\, e^{t\mathcal{X}_\lambda} \langle \psi_\lambda, \cdot \rangle \psi_\lambda,$$

Using $\langle \psi_\mu, e_c \mathcal{L}_0 \psi_\lambda \rangle = \Phi_\lambda \, \delta(\lambda - \mu - ic)$, one can find an explicit expression for u_n:

$$u_n(t,y) = \int_{\mathbb{R}} d\lambda \left(\sum_{k=0}^{n} \frac{e^{t\mathcal{X}_{\lambda-ikc}}}{\prod_{j\neq k}^{n} (\mathcal{X}_{\lambda-ikc} - \mathcal{X}_{\lambda-ijc})} \right) \cdots$$

$$\times \left(\prod_{k=0}^{n-1} \Phi_{\lambda-ikc} \right) \langle \psi_\lambda, G \rangle \, \psi_{\lambda-inc}(y).$$

- $u(t,y) := \mathbb{E}_y \, G(Y_t) = \sum_n \delta^n u_n$ is price of option with no time-change ($\tau_t = t$)
- To price option with independent time-change τ, simply condition on τ_t

$$v(t,y) := \mathbb{E}_y \, G(Y_{\tau_t}) = \mathbb{E} \, \mathbb{E}_y[G(Y_{\tau_t})|\tau_t] = \mathbb{E} u(\tau_t, y).$$

Analytic formulas result as long as Laplace transform $\mathbb{E} \, e^{\lambda \tau_t}$ is known (e.g. $\tau_t = \int_0^t c_s ds$ where c is CIR).

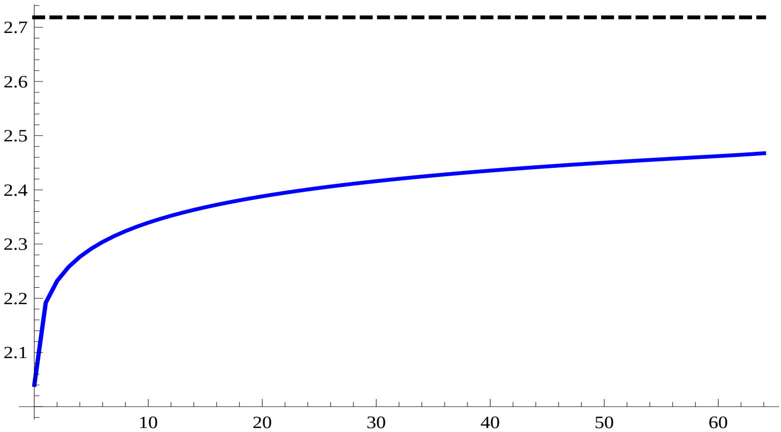

Figure 6: A plot of $\left(\frac{\mathbb{E}_y\,[\log F]_T}{-\mathbb{E}_y\,\log(F_T/F_0)}\right)$ as a function of $F_0 = e^y$ (blue).

$$F_0 \to 0: \quad \frac{\mu(y,\mathbb{R})}{a(y)} \to 0 \quad \left(\frac{\mathbb{E}_y\,[\log F]_T}{-\mathbb{E}_y\,\log(F_T/F_0)}\right) \to 2,$$

$$F_0 \to \infty: \quad \frac{\mu(y,\mathbb{R})}{a(y)} \to \infty \quad \left(\frac{\mathbb{E}_y\,[\log F]_T}{-\mathbb{E}_y\,\log(F_T/F_0)}\right) \to Q_0 (= e!).$$

Special case 2: European option pricing

Return to the subordinated diffusion setting

$$F_t = \exp(Y^{\phi}_{\tau_t}), \qquad Y^{\phi}_t = Y_{T_t}$$

where τ is a continuous time-change, Y is a diffusion absorbed at endpoints $L < R$ and T is a Lévy subordinator.
We know the function G that prices the VS.
We also know how to compute

$$u(t,y) = \mathbb{E}_y G(Y^{\phi}_t) = e^{t\mathcal{A}^{\phi}} G(y) = \sum_n e^{t\phi(\lambda_n)} \langle \psi_n, G \rangle_{\mathfrak{m}} \psi_n(y).$$

If the time-change τ is independent of Y^{ϕ} and we know its Laplace transform $\mathbb{E}\, e^{\lambda \tau_t} = L(t,\lambda)$, then the price can be computed by conditioning on τ_t

$$\begin{aligned} v(t,y) &:= \mathbb{E}_y\, G(Y^{\phi}_{\tau_t}) = \mathbb{E}\,\mathbb{E}_y[G(Y^{\phi}_{\tau_t})|\tau_t] = \mathbb{E} u(\tau_t, y) \\ &= \sum_n L(t, \phi(\lambda_n)) \langle \psi_n, G \rangle_{\mathfrak{m}} \psi_n(y). \end{aligned}$$

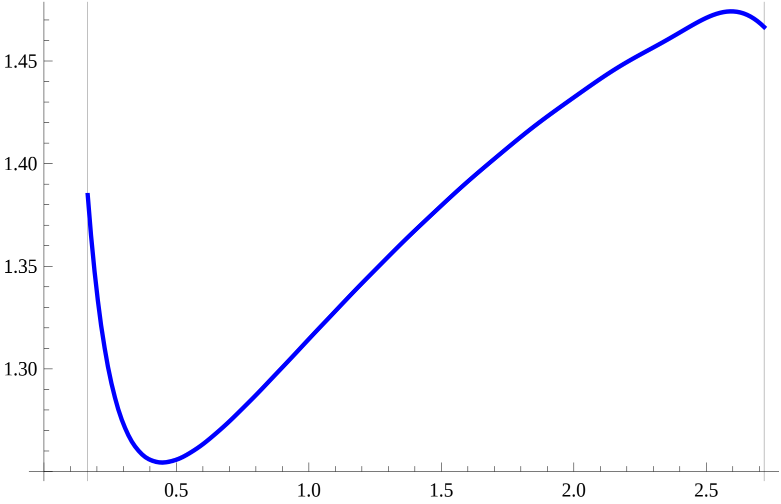

Figure 7: A plot of $\left(\frac{\mathbb{E}_y [\log F]_T}{-\mathbb{E}_y \log(F_T/F_0)}\right)$, as a function of $F_0 = e^y$ for the subordinated diffusion model. We let the Lévy measure ρ of the subordinator be exponential: $\rho(ds) = Ce^{-\eta s}ds$ and we assume a deterministic clock $\tau_t = t$. We use the following parameters: $\sigma = 1$, $b = 1$, $C = 2.0$, $\eta = 1$, $L = -2$, $R = 1$, $T = 3$.

Review

1. We have shown that when F is modeled as the exponential of a time-changed Markov process $F_t = \exp(Y_{\tau_t})$ the VS is priced by a European option whose payoff G depends only on the dynamics of Y – not on the time-change τ.
2. For certain cases, we can explicitly compute the function G that prices the VS.
3. When Y is a Lévy process we recover the results of Carr, Lee, and Wu (2011) ($G(\log F_T) = \log F_T$).

Carr-Lee-Wu $\subset$ Carr-Lee-Lorig

4. When Y is not a Lévy process we find that the ratio $\left(\frac{\mathbb{E}_y \, [\log F]_T}{-\mathbb{E}_y \, \log(F_T/F_0)} \right)$ depends on the value of $F_0 = e^y$.

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