Time-Inconsistent Optimal Control Problems

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Outline

1. Introduction

A General Setting:

$(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ — a complete filtered probability space

$W(\cdot)$ — a one-dimensional standard Brownian motion

$\mathbb{F} = \{\mathcal{F}_t\}_{t \geq 0}$ (natural filtration of $W(\cdot)$,

augmented by all the $\mathbb{P}$-null sets)

$\mathbb{R}^m \supseteq U$ — closed, bounded or unbounded

(could even be a matric space in general)

$T > 0$ — a time horizon

$$\mathcal{U}[t, T] = \Big\{ u : [t, T] \times \Omega \to U \;\Big|\; u(\cdot) \text{ is } \mathbb{F}\text{-adapted} \Big\}.$$

A Classical Problem:

Consider **state equation**:

$$\begin{cases} dX(s) = b(s, X(s), u(s))ds + \sigma(s, X(s), u(s))dW(s), \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad s \in [t, T], \\ X(t) = x, \end{cases}$$

$b : [0, T] \times \mathbb{R}^n \times U \to \mathbb{R}^n$ — *drift*

$\sigma : [0, T] \times \mathbb{R}^n \times U \to \mathbb{R}^{n\times d}$ — *diffusion*

$[0, T) \times \mathbb{R}^n \ni (t, x)$ — *initial pair*

$\mathcal{U}[t, T] \ni u(\cdot)$ — a *control process*

$X : [0, T] \times \Omega \to \mathbb{R}^n$ — a *state process.*

Under mild conditions, $X(\cdot) \equiv X(\cdot\,; t, x, u(\cdot))$ uniquely exists.

Introduce a **cost functional** (disutility)

$$J^0(t,x;u(\cdot)) = \mathbb{E}_t\Big[\int_t^T e^{-\delta(s-t)} g^0(s,X(s),u(s))ds + e^{-\delta(T-t)} h^0(X(T))\Big].$$

$g^0 : [0,T] \times \mathbb{R}^n \times U \to \mathbb{R}$ — *running cost rate*

$h^0 : \mathbb{R}^n \to \mathbb{R}$ — *terminal cost*

$\delta \geq 0$ — *discount rate*, $\quad \mathbb{E}_t[\,\cdot\,] = \mathbb{E}[\,\cdot\,|\,\mathcal{F}_t]$

Problem (C). $\forall (t,x) \in [0,T) \times \mathbb{R}^n$, find a $\bar{u}(\cdot) \in \mathcal{U}[t,T]$ s.t.

$$J^0(t,x;\bar{u}(\cdot)) = \inf_{u(\cdot)\in\mathcal{U}[t,T]} J^0(t,x;u(\cdot)) \equiv V^0(t,x).$$

$\bar{u}(\cdot)$ — *optimal control* for (t,x)

$\bar{X}(\cdot) \equiv X(\cdot\,;t,x,\bar{u}(\cdot))$ — *optimal state process*

$(\bar{X}(\cdot),\bar{u}(\cdot))$ — *optimal pair*

$V^0(\cdot\,,\cdot)$ — *value function*

Bellman's Principle of Optimality: For any $\tau \in [t, T]$,

$$V^0(t,x) = \inf_{u(\cdot)\in\mathcal{U}[t,\tau]} \mathbb{E}_t\Big[\int_t^\tau e^{-\delta(s-t)} g^0(s, X(s), u(s))ds$$
$$+e^{-\delta(\tau-t)} V^0\big(\tau, X(\tau; t, x, u(\cdot))\big)\Big].$$

Let $(\bar{X}(\cdot), \bar{u}(\cdot))$ be optimal for $(t,x) \in [0, T) \times \mathbb{R}^n$.

$$V^0(t,x) = J^0(t,x;\bar{u}(\cdot)) = \mathbb{E}_t\Big[\int_t^\tau e^{-\delta(s-t)} g^0(s, \bar{X}(s), \bar{u}(s))ds$$
$$+e^{-\delta(\tau-t)} J^0\big(\tau, \bar{X}(\tau; t, x, u(\cdot)); \bar{u}(\cdot)\big|_{[\tau,T]}\big)\Big]$$
$$\geq \mathbb{E}_t\Big[\int_t^\tau e^{-\delta(s-t)} g^0(s,\bar{X}(s),\bar{u}(s))ds + e^{-\delta(\tau-t)} V^0\big(\tau,\bar{X}(\tau;t,x,\bar{u}(\cdot))\big)\Big]$$
$$\geq \inf_{u(\cdot)\in\mathcal{U}[t,\tau]} \mathbb{E}_t\Big[\int_t^\tau e^{-\delta(s-t)} g^0(s, X(s), u(s))ds$$
$$+e^{-\delta(\tau-t)} V^0\big(\tau, X(\tau; t, x, u(\cdot))\big)\Big] = V^0(t,x).$$

Thus, all the equalities hold.

Consequently,

$$\mathbb{E}_t\Big[J^0\big(\tau,\bar X(\tau);\bar u(\cdot)\big|_{[\tau,T]}\big)-V^0(\tau,\bar X(\tau))\Big]=0,\qquad \text{a.s.}$$

Since $J^0\big(\tau,\bar X(\tau);\bar u(\cdot)\big|_{[\tau,T]}\big)-V^0(\tau,\bar X(\tau))\geq 0,\ \text{a.s.}$, it follows

$$\begin{aligned}J^0\big(\tau,\bar X(\tau);\bar u(\cdot)\big|_{[\tau,T]}\big)&=V^0(\tau,\bar X(\tau))\\&=\inf_{u(\cdot)\in\mathcal{U}[\tau,T]}J^0\big(\tau,\bar X(\tau);u(\cdot)\big),\quad\text{a.s.}\end{aligned}$$

Hence, $\bar u(\cdot)\big|_{[\tau,T]}\in\mathcal{U}[\tau,T]$ is **optimal** for $\big(\tau,\bar X(\tau;t,x,\bar u(\cdot))\big)$.

This is called the **time-consistency** of Problem (C).

2. Time-Inconsistent Problems

- People keep changing minds (Hard to keep the commitments)

* Promise to quit smoking/Plan to finish a job.
* Consumption habit/living standard is changing.

- Environment is changing

* Advances of technology (computer, internet, new material,...)
* New limits of resources (oil, natural gas, living space,...)

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- It is very **hard** (if not impossible) to make a long-term **time-consistent** plan (without even mentioning **optimality**).

- **Time-Inconsistency:** An optimal policy/strategy made at a moment is **NOT** necessarily optimal at a later time moment.

Time-Inconsistent Preferences:

Scenario 1:

Option A: Receive $5,000 now

Option B: Receive $5,500 a year from now

Most people prefer *A* (Uncertainty-averse).

Scenario 2:

Option C: Receive $5,000 in three years

Option D: Receive $5,500 in four years

Most people prefer D (right now,
and may change at the end of the 3rd year).

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* People have different preferences at different time moments, (which leads to time-inconsistency)

* People's discounting is **subjective**, not necessarily exponential.

Exponential Discounting vs Hyperbolic Discounting:

Exponential discounting: $\lambda(t) = e^{-\delta t}, \quad \delta > 0$ — discount rate

Hyperbolic discounting: $\lambda(t) = \frac{1}{1+kt}$ — a hyperbola

If let $k = e^{\delta} - 1 \sim \delta$, then

$$e^{-\delta t} = \frac{1}{(1+k)^t}$$

More general hyperbolic discounting: $\lambda(t) = \frac{1}{(1+kt)^{\alpha}}$, $\alpha > 0$.

• Strotz (1955): Problem with general discounting

$$\begin{cases} \text{maximize} \int_0^T \lambda(t-\tau) u(C(t), t) dt, \\ \text{subject to} \int_0^T C(t) dt = K. \end{cases}$$

$C(t)$ — consumption rate

$u(C, t)$ — utility (satisfaction level of having C)

$\lambda(t - \tau)$ — general discounting.

In general, the above problem is **time-inconsistent**.

The problem is time-consistent iff $\lambda(s) = e^{-\delta s} I_{[0,\infty)}(s)$.

Some History.

• *Qualitative Aspects:* Hume (1739), Smith (1759), Malthus (1828), Jevons (1871), Marshall (1890), Böhm–Bawerk (1891), Pareto (1909),...

A survey by Palacios-Huerta (2003).

• *Quantitative Aspects:*

Strotz (1955), Pollak (1968), Peleg–Yaari (1973), Goldman (1980), Laibson (1997), ...

• *Recent Works:*

Basak–Chabakauri (2010), Björk–Murgoci, Björk–Murguci–Zhou (2012), Hu–Jin–Zhou

Ekeland–Lazrak (2010), Ekeland–Pirvu (2008), Marin-Solano–Navas (2010), Marin-Solano–Shevkoplyas (2011), Ekeland–Mbodji–Pirvu (2012).

Yong (2011, 2012)

Example 2.1. (General discounting) Consider

$$\begin{cases} dX(s) = u(s)ds + X(s)dW(s), & s \in [t, T], \\ X(t) = x, \end{cases}$$

with cost functional

$$J(t, x; u(\cdot)) = \mathbb{E}_t \Big[\int_t^T \rho(s, t)|u(s)|^2 ds + g(t)|X(T)|^2 \Big].$$

$\rho(s, t), g(t)$ are deterministic non-constant, continuous and positive functions. They represent general discounting.

Recall: Exponential discounting:

$$\rho(s, t) = e^{-\delta(s-t)}, \quad g(t) = e^{-\delta t}$$

Let $P(\cdot\,,t)$ solve Riccati equation:

$$P_s(s,t) - \frac{P(s,t)^2}{\rho(s,t)} = 0, \quad s \in [t,T], \qquad P(T,t) = g(t).$$

Then

$$\begin{aligned} J(t,x;u(\cdot)) &= P(t,t)x^2 + \mathbb{E}_t \int_t^T \rho(s,t)\Big|u(s) + \frac{P(s,t)}{\rho(s,t)}X(s)\Big|^2 ds \\ &\geq P(t,t)x^2 = J(t,x;u^*(\cdot\,;t,x)) = \inf_{u(\cdot)\in\mathcal{U}[t,T]} J(t,x;u(\cdot)), \end{aligned}$$

where

$$u^*(s;t,x) = -\frac{P(s,t)}{\rho(s,t)}X^*(s;t,x), \qquad s \in [t,T],$$

with $X^*(\cdot) \equiv X^*(\cdot\,;t,x)$ being the solution to the following:

$$\begin{cases} dX^*(s) = -\dfrac{P(s,t)}{\rho(s,t)}X^*(s)ds + X^*(s)dW(s), \quad s \in [t,T], \\ X^*(t) = x. \end{cases}$$

If the problem is time-consistent, then $\forall \tau \in (t, T)$,

$$\begin{cases} u^*(s;t,x) = u^*(s;\tau, X^*(\tau;t,x)), \\ X^*(s;t,x) = X^*(s;\tau, X^*(\tau;t,x)), \end{cases} \qquad s \in [\tau, T].$$

Then, one can show that

$$\frac{g(\tau)}{\rho(T,\tau)} = C, \qquad \tau \in [0, T].$$

Thus, if, the above is not true, the problem is not time-consistent. In this case, we actually have

$$J(\tau, X^*(\tau); u^*(\cdot)\big|_{[\tau,T]}) > \inf_{u(\cdot)\in\mathcal{U}[\tau,T]} J(\tau, X^*(\tau); u(\cdot)),$$

for some $\tau \in (t, T)$. This is called the *time-inconsistency* of the problem.

Example 2.2. Consider

$$\begin{cases} dX(s) = u(s)ds + X(s)dW(s), & s \in [t, T], \\ X(t) = x, \end{cases}$$

with cost functional

$$J(t, x; u(\cdot)) = \mathbb{E}_t\Big[\int_t^T |u(s)|^2 ds + |\mathbb{E}_t[X(T)]|^2\Big].$$

Compare with classical case:

$$J(t, x; u(\cdot)) = \mathbb{E}_t\Big[\int_t^T |u(s)|^2 ds + |X(T)|^2\Big].$$

Note: $\mathbb{E}_t[X(T)]$ nonlinearly appears.

Let $\bar{P}(\cdot)$ solves

$$\dot{\bar{P}}(s) - \bar{P}(s)^2 = 0, \qquad s \in [0, T], \qquad \Pi(T) = 1.$$

We can show the following:

$$J(t, x; u(\cdot)) = \bar{P}(0)|x|^2 + \mathbb{E}_t \int_t^T \big| u(s) + \bar{P}(s)\mathbb{E}_t[X(s)] \big|^2 ds \geq \bar{P}(0)|x|^2,$$

with the equality holds when

$$u^*(s) = -\bar{P}(s)\mathbb{E}_t[X^*(s)], \qquad s \in [t, T],$$

where $X^*(\cdot)$ is the solution to the following closed-loop system:

$$\begin{cases} dX^*(s) = -\bar{P}(s)\mathbb{E}_t[X^*(s)]ds + X^*(s)dW(s), \qquad s \in [t, T], \\ X^*(t) = x. \end{cases}$$

This is called a mean-field SDE.

Note:

$$u^*(s) = -\bar{P}(s)\mathbb{E}_t[X^*(s)], \qquad s \in [t, T],$$

is $\mathcal{F}_t$-measurable.

The same argument shows: for any $\tau \in (t, T)$, the optimal control $\widehat{u}(\cdot)$ for $(\tau, X^*(\tau))$ is given by

$$\widehat{u}(s) = -\bar{P}(s)\mathbb{E}_\tau[\widehat{X}(s)], \qquad s \in [\tau, T],$$

where $X^*(\cdot)$ is the solution to the following closed-loop system:

$$\begin{cases} d\widehat{X}(s) = -\bar{P}(s)\mathbb{E}_t[\widehat{X}(s)]ds + \widehat{X}(s)dW(s), & s \in [\tau, T], \\ \widehat{X}(\tau) = X^*(\tau). \end{cases}$$

We can show that the following is **NOT** true:

$$\widehat{u}(s) = u^*(s), \qquad s \in [\tau, T].$$

Thus, the problem is time-inconsistent.

Motivation for problems containing conditional expectation(s) nonlinearly.

Consider an optimal control problem with cost functional

$$J(t,x;u(\cdot)) = \mathbb{E}_t\Big[\int_t^T \Big(|X(s)|^2 + |u(s)|^2\Big)ds + |X(T)|^2\Big].$$

Hope that optimal control and state are not too "random". To this end, we introduce

$$J(t,x;u(\cdot)) = \mathbb{E}_t\Big[\int_t^T \Big(|X(s)|^2 + \lambda \mathrm{var}_t[X(s)] + |u(s)|^2 + \mu \mathrm{var}_t[u(s)]\Big)ds + |X(T)|^2 + \mathrm{var}_t[X(T)]\Big].$$

Note

$$\mathrm{var}_t[X(s)] = \mathbb{E}_t|X(s)|^2 - \big(\mathbb{E}_t[X(s)]\big)^2.$$

Thus, conditional expectations nonlinear present.

Time-inconsistency can be caused by:

- Non-exponential discounting.
- Nonlinear presence of $\mathbb{E}_t[X(T)]$ (and/or $\mathbb{E}_t[X(\cdot)]$, $\mathbb{E}_t[u(\cdot)]$).

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In this talk, we only consider the case of general discounting.

3. Main Results

A General Formulation:

$$\begin{cases} dX(s) = b(s, X(s), u(s))ds + \sigma(s, X(s), u(s))dW(s), & s \in [t,T], \\ X(t) = x, \end{cases}$$

with

$$J(t, x; u(\cdot)) = \mathbb{E}_t \Big[\int_t^T g(t, s, X(s), u(s))ds + h(t, X(T)) \Big].$$

$$\mathcal{U}[t, T] = \Big\{ u : [t, T] \to U \;\Big|\; u(\cdot) \text{ is } \mathbb{F}\text{-adapted} \Big\}.$$

Problem (N). For given $(t, x) \in [0, T) \times \mathbb{R}^n$, find $\bar{u}(\cdot) \in \mathcal{U}[t,T]$ such that

$$J(t, x; \bar{u}(\cdot)) = \inf_{u(\cdot) \in \mathcal{U}[t,T]} J(t, x; u(\cdot)).$$

This problem is **time-inconsistent**.

Definition. $\Psi : [0, T] \times \mathbb{R}^n \to U$ is called a *time-consistent equilibrium strategy* if for any $x \in \mathbb{R}^n$,

$$\begin{cases} d\bar{X}(s) = b(s, \bar{X}(s), \Psi(s, \bar{X}(s)))ds \\ \qquad\qquad +\sigma(s, \bar{X}(s), \Psi(s, \bar{X}(s)))dW(s), \quad s \in [0, T], \\ \bar{X}(0) = x \end{cases}$$

admits a unique solution $\bar{X}(\cdot)$. For some $\Psi^{\Pi} : [0, T] \times \mathbb{R}^n \to U$,

$$\lim_{\|\Pi\| \to 0} d\Big(\Psi^{\Pi}(t, x), \Psi(t, x)\Big) = 0,$$

uniformly for (t, x) in any compact sets, where
$\Pi : 0 = t_0 < t_1 < \cdots < t_{N-1} < t_N = T$, and

$$\begin{aligned} & J^k\big(t_{k-1}, X^{\Pi}(t_{k-1}); \Psi^{\Pi}(\cdot)\big|_{[t_{k-1}, T]}\big) \\ & \leq J^k\big(t_{k-1}, X^{\Pi}(t_{k-1}); u^k(\cdot) \oplus \Psi^{\Pi}(\cdot)\big|_{[t_k, T]}\big), \quad \forall u^k(\cdot) \in \mathcal{U}[t_{k-1}, t_k], \end{aligned}$$

$J^k(\cdot)$ — sophisticated cost functional.

$$
\begin{cases}
dX^{\Pi}(s) = b(s, X^{\Pi}(s), \Psi^{\Pi}(s, X^{\Pi}(s)))ds \\
\qquad\qquad +\sigma(s, X^{\Pi}(s), \Psi^{\Pi}(s, X^{\Pi}(s)))dW(s), \quad s \in [0, T], \\
X^{\Pi}(0) = x
\end{cases}
$$

$$
[u^k(\cdot) \oplus \Psi^{\Pi}(\cdot)\big|_{[t_k, T]}](s) = \begin{cases}
u^k(s), & s \in [t_{k-1}, t_k), \\
\Psi^{\Pi}(s, X^k(s)), & s \in [t_k, T],
\end{cases}
$$

$$
\begin{cases}
dX^k(s) = b(s, X^k(s), u^k(s))ds \\
\qquad\qquad +\sigma(s, X^k(s), u^k(s))dW(s), \quad s \in [t_{k-1}, t_k), \\
dX^k(s) = b(s, X^k(s), \Psi^{\Pi}(s, X^k(s)))ds \\
\qquad\qquad +\sigma(s, X^k(s), \Psi^{\Pi}(s, X^k(s)))dW(s), \quad s \in [t_k, T], \\
X^k(t_{k-1}) = X^{\Pi}(t_{k-1}).
\end{cases}
$$

Equilibrium control:

$$\bar{u}(s) = \Psi(s, \bar{X}(s)), \qquad s \in [0, T].$$

Equilibrium state process $\bar{X}(\cdot)$, satisfying:

$$\begin{cases} d\bar{X}(s) = b(s, \bar{X}(s), \Psi(s, \bar{X}(s)))ds \\ \qquad\qquad +\sigma(s, \bar{X}(s), \Psi(s, \bar{X}(s)))dW(s), \quad s \in [0, T], \\ \bar{X}(0) = x \end{cases}$$

Equilibrium value function:

$$V(t, \bar{X}(t)) = J(t, \bar{X}(t); \bar{u}(\cdot)).$$

Question: How to find $\Psi(\cdot\,,\cdot)$?

Let $D[0,T] = \{(\tau, t) \mid 0 \le \tau \le t \le T\}$. Define

$$a(t,x,u) = \frac{1}{2}\sigma(t,x,u)\sigma(t,x,u)^T, \quad \forall (t,x,u) \in [0,T] \times \mathbb{R}^n \times U,$$

$$\mathbb{H}(\tau,t,x,u,p,P) = \operatorname{tr}\left[a(t,x,u)P\right] + \langle\, b(t,x,u), p\,\rangle + g(\tau,t,x,u),$$
$$\forall (\tau,t,x,u,p,P) \in D[0,T] \times \mathbb{R}^n \times U \times \mathbb{R}^n \times \mathcal{S}^n,$$

Let $\psi : \mathcal{D}(\psi) \subseteq D[0,T] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathcal{S}^n \to U$ such that

$$\mathbb{H}(\tau,t,x,\psi(\tau,t,x,p,P),p,P) = \inf_{u \in U} \mathbb{H}(\tau,t,x,u,p,P) > -\infty,$$
$$\forall (\tau,t,x,p,P) \in \mathcal{D}(\psi).$$

In **classical** case, it just needs

$$H(t,x,p,P) = \inf_{u \in U} \mathbb{H}(t,x,u,p,P) > -\infty,$$
$$\forall (t,x,p,P) \in [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathcal{S}^n.$$

Equilibrium HJB equation:

$$\begin{cases}\Theta_t(\tau,t,x)+\mathrm{tr}\big[a\big(t,x,\psi(t,t,x,\Theta_x(t,t,x),\Theta_{xx}(t,t,x))\big)\Theta_{xx}(\tau,t,x)\big]\\ +\langle\, b\big(t,x,\psi(t,t,x,\Theta_x(t,t,x),\Theta_{xx}(t,t,x))\big),\Theta_x(\tau,t,x)\,\rangle\\ +g\big(\tau,t,x,\psi(t,t,x,\Theta_x(t,t,x),\Theta_{xx}(t,t,x))\big)=0,\ (\tau,t,x)\in D[0,T]\times\mathbb{R}^n,\\ \Theta(\tau,T,x)=h(\tau,x),\qquad (\tau,x)\in[0,T]\times\mathbb{R}^n.\end{cases}$$

Classical HJB Equation:

$$\begin{cases}\Theta_t(t,x)+\mathrm{tr}\big[a\big(t,x,\psi(t,x,\Theta_x(t,x),\Theta_{xx}(t,x))\big)\Theta_{xx}(t,x)\big]\\ +\langle\, b\big(t,x,\psi(t,x,\Theta_x(t,x),\Theta_{xx}(t,x))\big),\Theta_x(t,x)\,\rangle\\ +g\big(t,x,\psi(t,x,\Theta_x(t,x),\Theta_{xx}(t,x))\big)=0,\qquad (t,x)\in[0,T]\times\mathbb{R}^n,\\ \Theta(T,x)=h(x),\quad x\in\mathbb{R}^n.\end{cases}$$

or

$$\begin{cases}\Theta_t(t,x)+H\big(t,x,\Theta_x(t,x),\Theta_{xx}(t,x)\big)=0,\quad (t,x)\in[0,T]\times\mathbb{R}^n,\\ \Theta(T,x)=h(x),\quad x\in\mathbb{R}^n.\end{cases}$$

Equilibrium value function:

$$V(t,x) = \Theta(t,t,x), \qquad \forall (t,x) \in [0,T] \times \mathbb{R}^n.$$

It satisfies

$$V(t,\bar{X}(t;x)) = J\big(t,\bar{X}(t;x);\Psi(\cdot)\big|_{[t,T]}\big), \qquad (t,x) \in [0,T] \times \mathbb{R}^n.$$

Equilibrium strategy:

$$\Psi(t,x) = \psi(t,t,x,V_x(t,x),V_{xx}(t,x)), \qquad (t,x) \in [0,T] \times \mathbb{R}^n.$$

Theorem. *Under proper conditions, the equilibrium HJB equation admits a unique classical solution* $\Theta(\cdot\,,\cdot\,,\cdot)$. *Hence, an equilibrium strategy* $\Psi(\cdot\,,\cdot)$ *exists.*

4. Two Special Cases

1. A Stochastic LQ Problem

For any $(t, x) \in [0, T) \times \mathbb{R}^n$, consider

$$\begin{cases} dX(s) = \big[A(s)X(s) + B(s)u(s)\big]ds \\ \qquad\qquad + \big[A_1(s)X(s) + B_1(s)u(s)\big]dW(s), \quad s \in [t, T], \\ X(t) = x, \end{cases}$$

with the cost functional

$$J(t, x; u(\cdot)) = \frac{1}{2}\mathbb{E}_t\Big[\int_t^T \big(\langle Q(t,s)X(s), X(s)\rangle \\ + \langle R(t,s)u(s), u(s)\rangle\big)ds + \langle G(t)X(T), X(T)\rangle\Big].$$

Function $\Theta(\tau, t, x)$ admits the following representation:

$$\Theta(\tau, t, x) = \frac{1}{2} \langle P(\tau, t)x, x \rangle, \quad (\tau, t, x) \in D[0, T] \times \mathbb{R}^n,$$

with $P(\cdot\,, \cdot)$ satisfying **equilibrium Riccati equation**:

$$\begin{cases} P_t(\tau, t) + P(\tau, t)\widehat{A}(t) + \widehat{A}(t)^T P(\tau, t) \\ \quad + \widehat{A}_1(t)^T P(\tau, t)\widehat{A}_1(t) + \widehat{Q}(\tau, t) = 0, \quad (\tau, t) \in D[0, T], \\ P(\tau, T) = G(\tau), \qquad \tau \in [0, T]. \end{cases}$$

where

$$\begin{cases} \Gamma(t) = \big[R(t, t) + B_1(t)^T P(t, t)B_1(t)\big]^{-1} \\ \qquad \cdot \big[B(t)^T P(t, t) + B_1(t)^T P(t, t)A_1(t)\big], \\ \widehat{A}(t) = A(t) - B(t)\Gamma(t), \quad \widehat{A}_1(t) = A_1(t) - B_1(t)\Gamma(t), \\ \widehat{Q}(\tau, t) = Q(\tau, t) + \Gamma(t)^T R(\tau, t)\Gamma(t). \end{cases}$$

$$V(t, x) = \frac{1}{2} \langle P(t, t)x, x \rangle, \qquad (t, x) \in [0, T] \times \mathbb{R}^n.$$

Let $\Phi(\cdot\,,\cdot)$ satisfy: $\forall (t,s) \in D[0,T]$,

$$\Phi(s,t) = I + \int_t^s \widehat{A}(r)\Phi(r,t)dr + \int_t^s \widehat{A}_1(r)\Phi(r,t)dW(r).$$

Then

$$P(\tau,t) = \mathbb{E}_t\Big[\Phi(T,t)^T G(\tau)\Phi(T,t) + \int_t^T \Phi(s,t)^T\big[Q(\tau,s) + \Gamma(s)^T R(\tau,s)\Gamma(s)\big]\Phi(s,t)ds\Big], \quad 0 \le \tau \le t \le T.$$

Take $\tau = t$, denoting $P(t) = P(t,t)$,

$$P(t) = \mathbb{E}_t\Big[\Phi(T,t)^T G(t)\Phi(T,t) + \int_t^T \Phi(s,t)^T\big[Q(t,s) + \Gamma(s)^T R(t,s)\Gamma(s)\big]\Phi(s,t)ds\Big], \quad t \in [0,T],$$

Hence, we end up with the system for $P(t) \equiv P(t,t)$:

$$\begin{cases} P(t) = \mathbb{E}_t\Big[\Phi(T,t)^T G(t)\Phi(T,t) + \displaystyle\int_t^T \Phi(s,t)^T\big[Q(t,s) \\ \qquad\qquad +\Gamma(s)^T R(t,s)\Gamma(s)\big]\Phi(s,t)ds\Big], \qquad t \in [0,T], \\ \Phi(s,t) = I + \displaystyle\int_t^s \big[A(r) - B(r)\Gamma(r)\big]\Phi(r,t)dr \\ \qquad\qquad + \displaystyle\int_t^s \big[A_1(r) - B_1(r)\Gamma(r)\big]\Phi(r,t)dW(r), \quad (t,s) \in D[0,T], \\ \Gamma(t) = \big[R(t,t) + B_1(t)^T P(t) B_1(t)\big]^{-1} \\ \qquad \cdot\big[B(t)^T P(t) + B_1(t)^T P(t) A_1(t)\big], \quad t \in [0,T]. \end{cases}$$

It is called a **Riccati-Volterra integral equation system**.
Time-consistent equilibrium strategy is given by:

$$\Psi(t,x) = -\Gamma(t)x, \qquad (t,x) \in [0,T] \times \mathbb{R}^n.$$

Let $A_1(\cdot) = 0$ and $B_1(\cdot) = 0$. Then

$$\begin{cases} P(t) = \Phi(T,t)^T G(t)\Phi(T,t) + \int_t^T \Phi(s,t)^T \big[Q(t,s) \\ \qquad\qquad +\Gamma(s)^T R(t,s)\Gamma(s)\big]\Phi(s,t)ds, \qquad t \in [0,T], \\ \Phi(s,t) = I + \int_t^s \big[A(r) - B(r)\Gamma(r)\big]\Phi(r,t)dr, \quad (t,s) \in D[0,T], \\ \Gamma(t) = R(t,t)^{-1}B(t)^T P(t), \quad t \in [0,T]. \end{cases}$$

Time-consistent equilibrium strategy:

$$\Psi(t,x) = -\Gamma(t)x, \qquad (t,x) \in [0,T] \times \mathbb{R}^n.$$

This recovers results of Yong (2011).

Note. Our result also recovers classical LQ problem with exponential discounting.

2. A Generalized Merton's portfolio problem

$$\begin{cases} dX(s) = \big[rX(s) + (\mu - r)u(s) - c(s)\big]ds + \sigma u(s)dW(s), \quad s \in [t,T], \\ X(t) = x, \end{cases}$$

with payoff functional:

$$J(t,x;u(\cdot),c(\cdot)) = \mathbb{E}_t\Big[\int_t^T \nu(t,s)c(s)^\beta ds + \rho(t)X(T)^\beta\Big],$$

$\nu(\cdot\,,\cdot)$ and $\rho(\cdot)$ are given positive functions, and $\beta \in (0,1)$.

Time-consistent equilibrium strategy is given by

$$\Psi(t,x) = \Big(-\frac{(\mu - r)}{\sigma^2(1-\beta)}, \quad \Big[\frac{\nu(t,t)}{\theta(t)}\Big]^{\frac{1}{1-\beta}} \Big) x,$$

where $\theta(\cdot)$ is a solution to the following: (denote $\nu(t) = \nu(t,t)$)

$$\begin{aligned} \theta(t) &= e^{\lambda(T-t) - \beta \int_t^T \left(\frac{\nu(s)}{\theta(s)}\right)^{\frac{1}{1-\beta}} ds} \rho(t) \\ &\quad + \int_t^T e^{\lambda(s-t) - \beta \int_t^\tau \left(\frac{\nu(s)}{\theta(s)}\right)^{\frac{1}{1-\beta}} ds} \Big(\frac{\nu(\tau)}{\theta(\tau)}\Big)^{\frac{\beta}{1-\beta}} \nu(t,\tau) d\tau, \;\; t \in [0,T], \end{aligned}$$

with

$$\lambda = r\beta + \frac{(\mu - r)^2 \beta}{2\sigma^2(1-\beta)}.$$

This recovers a relevant result of Marin-Solano–Navas (2010).

5. Derivation of Equilibrium HJB Equation

Let $\Pi : 0 = t_0 < t_1 < t_2 < \cdots < t_{N-1} < t_N = T$ with

$$\|\Pi\| = \max_{1 \le k \le N} (t_k - t_{k-1}).$$

Consider an N-person differential game:

- N players labeled $1, 2, \cdots, N$
- The k-th player controls on $[t_{k-1}, t_k)$ with state-control pair $(X^k(\cdot), u^k(\cdot))$.
- $X^k(t_k) = X^{k+1}(t_k)$.
- All the players play optimally.
- Each player discounts the future costs in his/her way.

Player N: For any $(t,x)\in[t_{N-1},t_N]\times\mathbb{R}^n$, consider:

$$\begin{cases} dX^N(s)=b\big(s,X^N(s),u^N(s)\big)ds+\sigma\big(s,X^N(s),u^N(s)\big)dW(s), \ \ s\in[t,t_N],\\ X^N(t)=x, \end{cases}$$

with cost functional

$$J^N(t,x;u^N(\cdot))=\mathbb{E}_t\Big[\int_t^{t_N} g\big(t_{N-1},s,X^N(s),u^N(s)\big)ds+h\big(t_{N-1},X^N(T)\big)\Big].$$

Problem (C$_N$). For any $(t,x)\in[t_{N-1},t_N]\times\mathbb{R}^n$, find a $\bar u^N(\cdot)\equiv\bar u^N(\cdot\,;t,x)\in\mathcal{U}[t,t_N]$ such that

$$J^N(t,x;\bar u^N(\cdot))=\inf_{u^N(\cdot)\in\mathcal{U}[t,t_N]} J^N(t,x;u^N(\cdot))\equiv V^\Pi(t,x).$$

Under proper conditions, $V^{\Pi}(\cdot\,,\cdot)$ satisfies

$$\begin{cases} V_t^{\Pi}(t,x) + \mathrm{tr}\big[a(t,x,\psi(t_{N-1},t,x,V_x^{\Pi}(t,x),V_{xx}^{\Pi}(t,x)))V_{xx}^{\Pi}(t,x)\big] \\ \quad + \langle\, b(t,x,\psi(t_{N-1},t,x,V_x^{\Pi}(t,x),V_{xx}^{\Pi}(t,x))), V_x^{\Pi}(t,x)\,\rangle \\ \quad + g(t_{N-1},t,x,\psi(t_{N-1},t,x,V_x^{\Pi}(t,x),V_{xx}^{\Pi}(t,x))) = 0,, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad (t,x) \in [t_{N-1},t_N] \times \mathbb{R}^n, \\ V^{\Pi}(t_N,x) = h(t_{N-1},x), \qquad x \in \mathbb{R}^n. \end{cases}$$

Let $\bar{X}^N(\cdot) \equiv \bar{X}^N(\cdot\,; t_{N-1}, x)$ solve:

$$\begin{cases} d\bar{X}^N(s) = b\big(s, \bar{X}^N(s), \bar{u}^N(s)\big)ds + \sigma\big(s, \bar{X}^N(s), \bar{u}^N(s)\big)dW(s), \\ \hfill s \in [t_{N-1}, t_N], \\ \bar{X}^N(t_{N-1}) = x, \end{cases}$$

where

$$\bar{u}^N(s) = \psi\big(t_{N-1}, s, \bar{X}^N(s), V^\Pi_x(s, \bar{X}^N(s)), V^\Pi_{xx}(s, \bar{X}^N(s))\big), \qquad s \in [t_{N-1}, t_N].$$

By verification theorem, $(\bar{X}^N(\cdot), \bar{u}^N(\cdot))$ is an optimal pair of Problem (C_N) for (t_{N-1}, x).

Player $(N-1)$: For $(t,x)\in[t_{N-2},t_{N-1}]$, consider

$$\begin{cases} dX^{N-1}(s)=b(s,X^{N-1}(s),u^{N-1}(s))ds \\ \qquad\qquad +\sigma(s,X^{N-1}(s),u^{N-1}(s))dW(s), \quad s\in[t,t_{N-1}), \\ X^{N-1}(t)=x. \end{cases}$$

The **sophisticated cost functional**:

$$J^{N-1}(t,x;u^{N-1}(\cdot))=\mathbb{E}_t\Big[\int_t^{t_{N-1}} g(t_{N-2},s,X^{N-1}(s),u^{N-1}(s))ds$$
$$+\int_{t_{N-1}}^{t_N} g\big(t_{N-2},s,\bar{X}^N(s;t_{N-1},X^{N-1}(t_{N-1})),\bar{u}^N(s;t_{N-1},X^{N-1}(t_{N-1}))\big)ds$$
$$+h\big(t_{N-2},\bar{X}^N(t_N;t_{N-1},X^{N-1}(t_{N-1}))\big)\Big].$$

$\big(\bar{X}^N(s;t_{N-1},X^{N-1}(t_{N-1})),\bar{u}^N(s;t_{N-1},X^{N-1}(t_{N-1}))\big)$
— optimal pair of Player N, for $(t_{N-1},X^{N-1}(t_{N-1}))$.

Note: Player $(N-1)$ "discounts" the future costs in his/her own way (t_{N-2} appears). Denote

$$h^{N-1}(x)=\mathbb{E}_{t_{N-1}}\Big[\int_{t_{N-1}}^{t_N} g\big(t_{N-2},s,\bar{X}^N(s;t_{N-1},x),\bar{u}^N(s;t_{N-1},x)\big)ds + h\big(t_{N-2},\bar{X}^N(t_N;t_{N-1},x)\big)\Big].$$

Then

$$J^{N-1}(t,x;u^{N-1}(\cdot))=\mathbb{E}_t\Big[\int_t^{t_{N-1}} g\big(t_{N-2},s,X^{N-1}(s),u^{N-1}(s)\big)ds + h^{N-1}\big(X^{N-1}(t_{N-1})\big)\Big].$$

- Problem has a classical looking.
- The map $x \mapsto h^{N-1}(x)$ is too implicit.

Inspired by the **Four-Step-Scheme**, introduce BSDE:

$$\begin{cases} dY^N(s) = -g\big(t_{N-2}, s, \bar{X}^N(s), \bar{u}^N(s)\big)ds + Z^N(s)dW(s), \ s \in [t_{N-1}, t_N], \\ Y^N(t_N) = h(t_{N-2}, \bar{X}^N(t_N)), \end{cases}$$

which admits a unique adapted solution $(Y^N(\cdot), Z^N(\cdot))$. One has

$$\begin{aligned} Y^N(t_{N-1}) &= \mathbb{E}_{t_{N-1}}\Big[\int_{t_{N-1}}^{t_N} g\big(t_{N-2}, s, \bar{X}^N(s), \bar{u}^N(s)\big)ds + h(t_{N-2}, \bar{X}^N(t_N))\Big] \\ &= h^{N-1}(x). \end{aligned}$$

$$\begin{cases} d\bar{X}^N(s) = b\big(s, \bar{X}^N(s), \bar{u}^N(s)\big)\,ds + \sigma\big(s, \bar{X}^N(s), \bar{u}^N(s))\big)\,dW(s), \\ dY^N(s) = -g\big(t_{N-2}, s, \bar{X}^N(s), \bar{u}^N(s)\big)\,ds + Z^N(s)dW(s), \ \ s \in [t_{N-1}, t_N], \\ \bar{X}^N(t_{N-1}) = x, \qquad Y^N(t_N) = h(t_{N-2}, \bar{X}^N(t_N)), \end{cases}$$

$$\bar{u}^N(s) = \psi(t_{N-1}, s, \bar{X}^N(s), V_x^\Pi(s, \bar{X}^N(s)), V_{xx}^\Pi(s, \bar{X}^N(s)))$$

This is an FBSDE, with deterministic coefficients. Let

$$Y^N(s) = \Theta^N(s, \bar{X}^N(s)), \qquad s \in [t_{N-1}, t_N],$$

with $\Theta^N(\cdot\,,\cdot)$ being a solution to the following PDE:

$$\begin{cases} \Theta_s^N(s,x) + \operatorname{tr}\Big[a\big(s, x, \psi(t_{N-1}, s, x, V_x^\Pi(s,x), V_{xx}^N(s,x))\big)\Theta_{xx}^N(s,x)\Big] \\ + \big\langle\, b\big(s, x, \psi(t_{N-1}, s, x, V_x^\Pi(s,x), V_{xx}^\Pi(s,x))\big), \Theta_x^N(s,x)\,\big\rangle \\ + g\big(t_{N-2}, s, x, \psi(t_{N-1}, s, x, V_x^\Pi(s,x), V_{xx}^\Pi(s,x))\big) = 0, \ \ (s,x) \in [t_{N-1}, t_N] \times \mathbb{R}^n, \\ \Theta^N(t_N, x) = h(t_{N-2}, x), \qquad x \in \mathbb{R}^n, \end{cases}$$

$$\Theta^N(t_{N-1},x) = Y^N(t_{N-1}) = h^{N-1}(x)$$
$$= \mathbb{E}_{t_{N-1}}\Big[\int_{t_{N-1}}^{t_N} g\big(t_{N-2},s,\bar{X}^N(s),\bar{u}^N(s)\big)ds + h(t_{N-2},\bar{X}^N(T))\Big]$$
$$\neq \mathbb{E}_{t_{N-1}}\Big[\int_{t_{N-1}}^{t_N} g\big(t_{N-1},s,\bar{X}^N(s),\bar{u}^N(s)\big)ds + h(t_{N-1},\bar{X}^N(T))\Big] = V^\Pi(t_{N-1},x).$$

With the above representation, we have

$$J^{N-1}(t,x;u^{N-1}(\cdot)) = \mathbb{E}_t\Big[\int_t^{t_{N-1}} g\big(t_{N-2},s,X^{N-1}(s),u^{N-1}(s)\big)ds + \Theta^N\big(t_{N-1},X^{N-1}(t_{N-1})\big)\Big].$$

Problem ($\mathbf{C}_{N-1}$). For $(t,x) \in [t_{N-2},t_{N-1}) \times \mathbb{R}^n$, find a $\bar{u}^{N-1}(\cdot) \equiv \bar{u}^{N-1}(\cdot\,;t,x) \in \mathcal{U}[t_{N-2},t_{N-1}]$ such that

$$J^{N-1}(t,x;\bar{u}^{N-1}(\cdot)) = \inf_{u^{N-1}(\cdot)\in\mathcal{U}[t,t_{N-1}]} J^{N-1}(t,x;u^{N-1}(\cdot)) \equiv V^\Pi(t,x).$$

Under proper conditions, $V^{\Pi}(\cdot\,,\cdot)$ solves

$$\begin{cases} V^{\Pi}_t(t,x)+\inf_{u\in U}\mathbb{H}\big(t_{N-2},t,x,u,V^{\Pi}_x(t,x),V^{\Pi}_{xx}(t,x)\big)=0, \\ \qquad\qquad\qquad\qquad\qquad\qquad (t,x)\in[t_{N-2},t_{N-1})\times\mathbb{R}^n, \\ V^{\Pi}(t_{N-1}-0,x)=\Theta^N(t_{N-1},x), \qquad x\in\mathbb{R}^n. \end{cases}$$

Note

$$V^{\Pi}(t_{N-1}-0,x)=\Theta^N(t_{N-1},x)\neq V^{\Pi}(t_{N-1},x).$$

Recursively, for Player k, $1\le k\le N$, a sophisticated cost functional $J^k(t,x;u^k(\cdot))$ can be constructed, and

$$V^{\Pi}(t,x)=\inf_{u^k(\cdot)\in\mathcal{U}[t,t_k]}J^k(t,x;u^k(\cdot)),\quad (t,x)\in[t_{k-1},t_k)\times\mathbb{R}^n$$

can be defined and corresponding optimal pair $(\bar{X}^k(\cdot),\bar{u}^k(\cdot))$ can be constructed.

Under proper conditions, $V^{\Pi}(\cdot\,,\cdot)$ satisfies

$$\begin{cases} V_t^{\Pi}(t,x)+\mathrm{tr}\big[a(t,x,\psi(\ell^{\Pi}(t),t,x,V_x^{\Pi}(t,x),V_{xx}^{\Pi}(t,x)))V_{xx}^{\Pi}(t,x)\big] \\ \quad +\langle\, b(t,x,\psi(\ell^{\Pi}(t),t,x,V_x^{\Pi}(t,x),V_{xx}^{\Pi}(t,x))),V_x^{\Pi}(t,x)\,\rangle \\ \quad +g(\ell^{\Pi}(t),t,x,\psi(t_{N-1},t,x,V_x^{\Pi}(t,x),V_{xx}^{\Pi}(t,x)))=0, \\ \qquad\qquad\qquad\qquad\qquad\qquad (t,x)\in[t_{k-1},t_k)\times\mathbb{R}^n, \\ V^{\Pi}(t_N,x)=h(t_{N-1},x), \qquad x\in\mathbb{R}^n, \\ V^{\Pi}(t_k-0,x)=\Theta^{k+1}(t_k,x), \qquad x\in\mathbb{R}^n,\ k=1,2,\cdots,N-1, \end{cases}$$

where,

$$\ell^{\Pi}(s)=\sum_{k=1}^{N}t_{k-1}I_{[t_{k-1},t_k)}(s), \qquad s\in[0,T].$$

- $V^{\Pi}(t,x)$ has jumps at $t=t_1,t_2,\cdots,t_{N-1}$.

$\Theta^{k+1}(\cdot\,,\cdot)$ satisfies

$$\begin{cases}\Theta_t^{k+1}(t,x)+\mathrm{tr}\big[a(t,x,\psi(\ell^\Pi(t),t,x,V_x^\Pi(t,x),V_{xx}^\Pi(t,x)))\Theta_{xx}^{k+1}(t,x)\big]\\ \quad+\langle\, b(t,x,\psi(\ell^\Pi(t),t,x,V_x^\Pi(t,x),V_{xx}^\Pi(t,x))),\Theta_x^{k+1}(t,x)\,\rangle\\ \quad+g(t_{k-1},t,x,\psi(\ell^\Pi(t),t,x,V_x^\Pi(t,x),V_{xx}^\Pi(t,x)))=0,\\ \qquad\qquad\qquad\qquad\qquad\qquad (t,x)\in[t_{k-1},T)\times\mathbb{R}^n,\\ \Theta^{k+1}(T,x)=h(t_{k-1},x),\qquad x\in\mathbb{R}^n,\ \ k=1,2.\cdots,N-1.\end{cases}$$

Let us define

$$\Theta^{\Pi}(\tau, t, x) = \sum_{k=1}^{N-1} \Theta^{k+1}(t, x) I_{[t_{k-1}, t_k)}(\tau),$$
$$(\tau, t, x) \in D[0, T] \times \mathbb{R}^n.$$

Then

$$|V^{\Pi}(t, x) - \Theta^{\Pi}(\ell^{\Pi}(t), t, x)| \le K\|\Pi\|, \quad (t, x) \in [0, t_{N-1}) \times \mathbb{R}^n.$$

If

$$\lim_{\|\Pi\|\to 0} \sup_{(\tau,t)\in D[0,T]} \|\Theta^{\Pi}(\tau, t, \cdot) - \Theta(\tau, t, \cdot)\|_{C^2(\mathbb{R}^n)} = 0,$$

then

$$\lim_{\|\Pi\|\to 0} \sup_{t\in[0,T]} \|V^{\Pi}(t, \cdot) - V(t, \cdot)\|_{C^2(\mathbb{R}^n)} = 0,$$

and $\Theta(\cdot\,, \cdot\,, \cdot)$ satisfies the equilibrium HJB equation with

$$V(t, x) = \Theta(t, t, x), \qquad \forall (t, x) \in [0, T] \times \mathbb{R}^n.$$

6. Well-Posedness of the Equilibrium HJB Equation

Consider the following special case:

$$\sigma(t,x,u)=\sigma(t,x), \qquad (t,x,u)\in[0,T]\times\mathbb{R}^n\times U,$$

In this case,

$$\begin{cases}\Theta_t(\tau,t,x)+\frac{1}{2}\mathrm{tr}\left[\sigma(t,x)\sigma(t,x)^T\Theta_{xx}(\tau,t,x)\right]\\ +\langle\, b\big(t,x,\psi(t,t,x,\Theta_x(t,t,x))\big),\Theta_x(\tau,t,x)\,\rangle\\ +g\big(\tau,t,x,\psi(t,t,x,\Theta_x(t,t,x))\big)=0, \quad (\tau,t,x)\in D[0,T]\times\mathbb{R}^n,\\ \Theta(\tau,T,x)=h(\tau,x), \qquad (\tau,x)\in[0,T]\times\mathbb{R}^n.\end{cases}$$

Theorem. *Under some additional mild conditions, The above admits a unique solution $\Theta(\cdot\,,\cdot\,,\cdot)$. Moreover,*

$$\lim_{\|\Pi\|\to 0}\ \sup_{(\tau,t)\in D[0,T]}\|\Theta^\Pi(\tau,t,\cdot)-\Theta(\tau,t,\cdot)\|_{C^2(\mathbb{R}^n)}=0.$$

For any smooth $v(\cdot\,,\cdot)$, denote

$$\begin{aligned}
&\big[\mathcal{L}(t)\varphi(\cdot)\big](x) = \mathrm{tr}\,\big[a(t,x)\varphi_{xx}(x)\big],\\
&\big[\mathcal{B}\big(t,v(t,\cdot)\big)\varphi(\cdot)\big](x) = \langle\, b\big(t,x,\psi(t,t,x,v_x(t,x))\big),\varphi_x(x)\,\rangle,\\
&\mathcal{G}(\tau,t,v(t,\cdot))(x) = g\big(\tau,t,x,\psi(t,t,x,v_x(t,x))\big),\\
&\qquad\qquad\qquad\qquad\qquad\qquad (\tau,t,x)\in D[0,T]\times\mathbb{R}^n.
\end{aligned}$$

Equilibrium HJB equation can be written as

$$\begin{cases}\Theta_t(\tau,t)+\mathcal{L}(t)\Theta(\tau,t)+\mathcal{B}(t,v(t))\Theta(\tau,t)+\mathcal{G}(\tau,t,v(t))=0, \quad t\in[\tau,T],\\ \Theta(\tau,T)=h(\tau).\end{cases}$$

Under some mild conditions, one has

$$\Theta(\tau, t) = \mathcal{E}(T, t; v(\cdot))h(\tau) + \int_t^T \mathcal{E}(s, t; v(\cdot))\mathcal{G}(\tau, s, v(s))ds, \qquad t \in [\tau, T],$$

$\mathcal{E}(\cdot\,,\cdot\,; v(\cdot))$ — the *backward evolution operator* generated by $\mathcal{L}(\cdot) + \mathcal{B}(\cdot\,, v(\cdot))$.

The equilibrium value function $V(t, \cdot) = \Theta(t, t, \cdot)$ satisfies the following **Equilibrium HJB integral equation**

$$V(t) = \mathcal{E}(T, t; V(\cdot))h(t) + \int_t^T \mathcal{E}(s, t; V(\cdot))\mathcal{G}(t, s, V(s))ds, \qquad t \in [0, T].$$

7. Open Problems

1. The well-posedness of the equilibrium HJB equation for the case $\sigma(t, x, u)$ is **not independent** of u.
2. The case that ψ is **not unique**, has **discontinuity**, etc.
3. The case that $\sigma(t, x, u)$ is **degenerate**, viscosity solution?
4. **Random** coefficient case (non-degenerate/degenerate cases).
5. The case involving **conditional expectation**.
6. **Infinite horizon** problems.

Thank You!