

Risk-minimization for life insurance liabilities

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Introduction

- A large number of life insurance and pension products have mortality and longevity as a primary source of risk.
- *(Systematic) mortality risk* denotes here all forms of deviations in aggregate mortality rates from those anticipated at different times and over different times horizons.
- *Longevity risk* refers to the risk that aggregate survival rates for given cohorts are higher than anticipated.
- *Short-term, catastrophic mortality risk* is the risk, that over short period of time, mortality rates are very much higher than would be normally experienced (such as for example in the case of a pandemic influenza or a natural catastrophe).
- *Unsystematic mortality risk*: risk associated with the status of an individual life.

- Although mortality and longevity risk can be re-insured, there is inadequate reinsurance capacity on a global basis to address effectively these risks.
- In addition systematic mortality risk cannot be diversified away by pooling, but on the contrary its impact increases for larger portfolios of insurers.
- Hence several new instruments having mortality and longevity indexes as basis factors have been introduced on the financial markets as alternative source of risk diversification.

Mortality derivatives

- *Longevity bonds*, where coupon payments are linked to the number of survivors in a given cohort.
- *Short-dated, mortality securities*: market traded securities, whose payments are linked to a mortality index. They allow the issuer to reduce its exposure to short-term catastrophic mortality risk.
- *Survivor swaps*, where counterparties swap a fixed series of payments for a series of payments linked to the number of survivors in a given cohort.
- *Mortality options*: financial contracts with mortality rate as underlying.

- The resulting (hybrid) financial market is then incomplete.
- Local risk-minimization naturally appears as suitable hedging method for the new financial instruments recently introduced to hedge against systematic mortality risk in life insurance contracts, where the market incompleteness is due to the presence of an **additional source of randomness**, that is "orthogonal" to the asset price dynamics, but not necessarily independent of them and vice versa. See for example Barbarin [7], B., Rheinländer, and Widenmann [6], Dahl and Møller [13], Møller [19], Møller [20], Riesner [21].

- Objective: study the problem of pricing and hedging life insurance liabilities by means of the risk-minimization approach.
- 3 scenarios:
 - ▶ Single life case: B. and Schreiber [3].
 - ▶ Homogeneous portfolio with basis risk: B., Rheinländer, and Schreiber [5].
 - ▶ Portfolio consisting of different age cohorts: B., Botero, and Schreiber [4].
- Main tools:
 - ▶ Progressive enlargement of filtration, reduced-form modeling from credit risk, see Bielecki and Rutkowski [8].
 - ▶ Quadratic hedging: risk-minimization, introduced by Föllmer and Sondermann [16].
 - ▶ Affine processes, see Duffie et al. [14], Duffie et al. [15].
 - ▶ Random field theory, see Adler [1], Goldstein [17], Kennedy [18].

Risk-minimization with basis risk

- Joint work with Thorsten Rheinländer and Irene Schreiber.
- Objective: study the problem of pricing and hedging life insurance liabilities of a **homogeneous insurance portfolio** (all individuals are of the same age at time 0) by means of the risk-minimization approach and basis risk into account.
- Model the dependency between the index and the insurance portfolio by means of a multidimensional affine mean-reverting diffusion process with stochastic drift.
- Additional tool: affine processes.
 - ▶ Duffie, Pan, and Singleton [14], Duffie, Filipović, and Schachermayer [15]
 - ▶ Biffis [9], Dahl [12]

The setting: insurance portfolio and mortality intensities

- Finite time horizon $T > 0$, probability space $(\Omega, \mathcal{G}, \mathbb{P})$.
- Background filtration $\mathbb{F} = (\mathcal{F}_t)_{t \in [0, T]}$.
- Insurance portfolio: n individuals belonging to the **same age cohort** with remaining lifetimes $\tau^j : \Omega \rightarrow [0, T] \cup \{\infty\}$, $j = 1, \dots, n$.
- $H_t^j = \mathbb{1}_{\{\tau^j \leq t\}}$, $j = 1, \dots, n$
- $N_t = \sum_{j=1}^n \mathbb{1}_{\{\tau^j \leq t\}}$, $t \in [0, T]$.

- $\mathbb{H} = (\mathcal{H}_t)_{t \in [0, T]}$, $\mathcal{H}_t = \mathcal{H}_t^1 \vee \dots \vee \mathcal{H}_t^n$, where

$$\mathcal{H}_t^j = \sigma\{H_s^j : 0 \leq s \leq t\}.$$

- Hazard process Γ^j of τ^j under \mathbb{P} : $\Gamma_t^j = -\ln \mathbb{P}[\tau^j > t \mid \mathcal{F}_t]$.
- **Homogeneity**: set $\Gamma^j = \Gamma$, where

$$\Gamma_t = \int_0^t \mu_s ds, \quad t \in [0, T].$$

- Similarly as in Biffis [9] we assume that the mortality intensity μ is given by the following set of stochastic differential equations:

$$\begin{aligned}d\mu_t &= \gamma_1(\bar{\mu}_t - \mu_t)dt + \sigma_1\sqrt{\mu_t}dW_t^\mu, \\d\bar{\mu}_t &= \gamma_2(m(t) - \bar{\mu}_t)dt + \sigma_2\sqrt{\bar{\mu}_t}dW_t^{\bar{\mu}},\end{aligned}$$

for $t \in [0, T]$ where W^μ and $W^{\bar{\mu}}$ are independent Brownian motions and $\mu_0 = \bar{\mu}_0 = 0$, where $\gamma_1, \gamma_2, \sigma_1, \sigma_2 > 0$, and $m : [0, T] \rightarrow \mathbb{R}_+$ is a continuous deterministic function.

- The process $\bar{\mu}$ represents the **mortality intensity of the equivalent age cohort of the population.**
- Survivor/longevity index: $S_t^{\bar{\mu}} = \exp\left(-\int_0^t \bar{\mu}_s ds\right)$, $t \in [0, T]$.

The setting: financial market

- **Bank account** B : $B_t = \exp(rt)$, $t \in [0, T]$, $r > 0$.
- **Risky asset** S :

$$dS_t = S_t (r dt + \sigma(t, S_t) dW_t),$$

where $S_0 = s$, and the Brownian motion W is independent of $(W^\mu, W^{\bar{\mu}})$.

- **Longevity bond** P with maturity T (Cairns et al. [11]): pays out the value of the survivor index at T , with discounted value $Y_t = \mathbb{E} \left[\frac{S_T^{\bar{\mu}}}{B_T} \mid \mathcal{F}_t \right]$, $t \in [0, T]$.
- $X = S/B$, $Y = P/B$ are continuous (local) (\mathbb{P}, \mathbb{F}) -martingales.

The setting: combined model

- Background filtration $\mathbb{F} = (\mathcal{F}_t)_{t \in [0, T]}$, where

$$\mathcal{F}_t = \sigma\{(W_s, W_s^\mu, W_s^{\bar{\mu}}) : 0 \leq s \leq t\}, t \in [0, T].$$

- Enlarged filtration $\mathbb{G} = \mathbb{F} \vee \mathbb{H}$.
- Hypothesis (H): all \mathbb{F} -(local) martingales are \mathbb{G} -(local) martingales.

- For $i \neq j$, τ^i, τ^j are conditionally independent given \mathcal{F}_T , i.e.

$$\mathbb{E}[\mathbb{1}_{\{\tau^i > t\}} \mathbb{1}_{\{\tau^j > s\}} \mid \mathcal{F}_T] = \mathbb{E}[\mathbb{1}_{\{\tau^i > t\}} \mid \mathcal{F}_T] \mathbb{E}[\mathbb{1}_{\{\tau^j > s\}} \mid \mathcal{F}_T],$$

for $0 \leq s, t \leq T$.

- Fundamental martingales: the compensated process $M_t^j = H_t^j - \Gamma_{t \wedge \tau^j}$, $t \in [0, T]$ follows a \mathbb{G} -martingale for each $j = 1, \dots, n$.
- Define $M_t = \sum_{j=1}^n M_t^j$.

Lemma

For the longevity bond we have the dynamics:

$$Y_t = \mathbb{E} \left[\frac{\exp \left(- \int_0^T \bar{\mu}_s ds \right)}{B_T} \mid \mathcal{G}_t \right] = Y_0 + \int_0^t Y_s e^{-rT} \beta^T(s) \sigma_2 \sqrt{\bar{\mu}_s} dW_s^{\bar{\mu}},$$

for $t \in [0, T]$, where β^T is given by the following partial differential equation:

$$\partial_t \beta^T(t) = 1 + \gamma_2 \beta^T(t) - \frac{1}{2} \sigma_2^2 (\beta^T(t))^2, \quad \beta^T(T) = 0. \quad (1)$$

- We consider the following discounted life insurance payment streams:

▶ **Pure endowment:** $A_t^{pe} = (n - N_t) \frac{C^{pe}}{B_t} \mathbb{1}_{\{t=T\}}$, where $C^{pe} \geq 0$ and belongs to $L^2(\mathcal{F}_T)$,

▶ **Term insurance:** $A_t^{ti} = \int_0^t \frac{C_s^{ti}}{B_s} dN_s = \sum_{j=1}^n \mathbb{1}_{\{\tau^j \leq t\}} \frac{C_{\tau^j}^{ti}}{B_{\tau^j}}$, where $C^{ti} \geq 0$ \mathbb{F} -predictable, $\mathbb{E} \left[\sup_{t \in [0, T]} (C_t^{ti})^2 \right] < \infty$,

▶ **Annuity:** $A_t^a = \int_0^t (n - N_s) \frac{1}{B_s} dC_s^a = \sum_{j=1}^n \int_0^t \mathbb{1}_{\{\tau^j > s\}} \frac{1}{B_s} dC_s^a$, where $C^a \geq 0$ increasing \mathbb{F} -adapted, $\mathbb{E} \left[\sup_{t \in [0, T]} (C_t^a)^2 \right] < \infty$.

Risk-minimization for payment streams (Møller [20])

- Consider a discounted *payment process* $A = (A_t)_{t \in [0, T]}$, which is \mathbb{G} -adapted and such that $\mathbb{E} \left[\sup_{t \in [0, T]} A_t^2 \right] < \infty$.
- An L^2 -strategy is a pair $\varphi = (\xi, \eta)$, such that η is a \mathbb{G} -adapted process and ξ is a \mathbb{G} -predictable process belonging to

$$L^2(X) := \left\{ \xi : \xi \text{ } \mathbb{G}\text{-predictable, } \left(\mathbb{E} \left[\int_0^T \xi_s^2 d[X]_s \right] \right)^{1/2} < \infty \right\},$$

such that the discounted value process $V_t(\varphi) = \xi_t X_t + \eta_t$, $t \in [0, T]$, is right-continuous and square integrable.

- *Cumulative cost process* $C(\varphi)$:

$$C_t(\varphi) = V_t(\varphi) - \int_0^t \xi_s dX_s + A_t, \quad t \in [0, T].$$

- The cost process C describes the accumulated costs of the trading strategy φ during $[0, t]$ including the payments A_t . The portfolio process $V_t(\varphi)$ should therefore be interpreted as the discounted value of the portfolio φ_t held at time t *after* the payments A_t have been made.
- In particular we focus on *0-admissible strategies* such that

$$V_T(\varphi) = 0.$$

- *Risk process* $R(\varphi)$:

$$R_t(\varphi) = \mathbb{E}[(C_T(\varphi) - C_t(\varphi))^2 | \mathcal{G}_t], \quad t \in [0, T].$$

- An L^2 -strategy $\varphi = (\xi, \eta)$ is called *risk-minimizing* (RM), if for any L^2 -strategy $\tilde{\varphi} = (\tilde{\xi}, \tilde{\eta})$ such that $V_T(\tilde{\varphi}) = V_T(\varphi) = 0$ \mathbb{P} -a.s., $\tilde{\xi}_s = \xi_s$ for $s \leq t$ and $\tilde{\eta}_s = \eta_s$ for $s < t$, we have

$$R_t(\varphi) \leq R_t(\tilde{\varphi}), \quad t \in [0, T].$$

- *Martingale decomposition:*

$$\mathbb{E}[A_T | \mathcal{G}_t] = \mathbb{E}[A_T | \mathcal{G}_0] + \int_{]0,t]} \xi_s^A dX_s + L_t^A, \quad t \in [0, T], \quad (2)$$

where $\xi^A \in L^2(X)$ and L^A is a square integrable martingale null at 0 that is strongly orthogonal to the space of stochastic integrals

$$\mathcal{I}^2(X) := \left\{ \int \xi dX \mid \xi \in L^2(X) \right\}.$$

Theorem (Møller [20])

The unique risk-minimizing L^2 -strategy $\varphi^* = (\xi^*, \eta^*)$ for A is given by

$$\xi_t^* = \xi_t^A,$$

$$\eta_t^* = \mathbb{E}[A_T | \mathcal{G}_t] - A_t - \xi_t^A X_t = V_t(\varphi^*) - \xi_t^A X_t,$$

for $t \in [0, T]$ with cumulative cost and risk processes

$$C_t(\varphi^*) = \mathbb{E}[A_T | \mathcal{G}_0] + L_t^A,$$

$$R_t(\varphi^*) = \mathbb{E} \left[\left(L_T^A - L_t^A \right)^2 \middle| \mathcal{G}_t \right],$$

where $V_t(\varphi^*) = \mathbb{E}[A_T | \mathcal{G}_t] - A_t$ and ξ^A, L^A are given by decomposition (2) of $\mathbb{E}[A_T | \mathcal{G}_t]$.

RM for term insurance contracts with basis risk

Theorem

The term insurance process A^{ti} admits a RM strategy

$\varphi = (\xi, \xi^0) = (\xi^X, \xi^Y, \xi^0)$ given by

$$\xi_t = (\xi_t^X, \xi_t^Y) = \left(\frac{(n - N_t)e^{\Gamma_t} \psi_t}{\sigma(t, X_t) X_t}, \frac{(n - N_t)e^{\Gamma_t + rT} \psi_t^{\bar{\mu}}}{Y_t \beta^T(t) \sigma_2 \sqrt{\bar{\mu}_t}} \right)$$

$$\xi_t^0 = V_t^{ti}(\varphi) - \xi_t^X X_t - \xi_t^Y Y_t$$

for $t \in [0, T]$, with discounted value process

$$V_t^{ti}(\varphi) = nU_0^{ti} + \int_0^t \xi_s^X dX_s + \int_0^t \xi_s^Y dY_s + L_t^{ti} - A_t^{ti},$$

where

$$L_t^{ti} = \int_0^t (n - N_s) e^{\Gamma_s} \psi_s^\mu dW_s^\mu + \int_{]0,t]} \left(\frac{C_s^{ti}}{B_s} - \mathbb{E} \left[\int_s^T \frac{C_u^{ti}}{B_u} e^{\Gamma_s - \Gamma_u} d\Gamma_u \mid \mathcal{F}_s \right] \right) dM_s,$$

$$U_t^{ti} = \mathbb{E} \left[\int_0^T \frac{C_s^{ti}}{B_s} e^{-\Gamma_s} d\Gamma_s \mid \mathcal{F}_t \right] = U_0^{ti} + \int_0^t \psi_s dW_s + \int_0^t \psi_s^\mu dW_s^\mu + \int_0^t \psi_s^{\bar{\mu}} dW_s^{\bar{\mu}}.$$

- Optimal cost: $C_t^{ti}(\varphi) = nU_0^{ti} + L_t^{ti}$ for $t \in [0, T]$.
- Risk process: $R_t^{ti}(\varphi) = \mathbb{E}[(L_T^{ti} - L_t^{ti})^2 \mid \mathcal{G}_t]$ for $t \in [0, T]$.

Consider a unit-linked term insurance contract:

$$A_t^{ti,f} = \int_0^t \frac{f(S_s)}{B_s} dN_s = \sum_{i=1}^n \mathbb{1}_{\{\tau^i \leq t\}} \frac{f(S_{\tau^i})}{B_{\tau^i}},$$

for $t \in [0, T]$, where $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a Borel measurable function such that $\mathbb{E} \left[\sup_{t \in [0, T]} f(S_t)^2 \right] < \infty$

Corollary

The process $A^{ti,f}$ admits a RM strategy $\varphi = (\xi, \xi^0) = (\xi^X, \xi^Y, \xi^0)$ given by

$$\xi_t^X = (n - N_t) e^{\Gamma t} \int_t^T F_s^u(t, S_t) Z_t^{\mu, u} du,$$

$$\xi_t^Y = (n - N_t) e^{\Gamma t + r(T-t)} Y_t^{-1} \beta^T(t)^{-1} \int_t^T F^u(t, S_t) Z_t^u (\hat{\beta}_2^u(t) + \beta_2^u(t) \hat{Z}_t^u) du,$$

$$\xi_t^0 = V_t^{ti}(\varphi) - \xi_t^X X_t - \xi_t^Y Y_t,$$

for $t \in [0, T]$.

- Discounted value process

$$V_t^{ti}(\varphi) = nU_0^{ti} + \int_0^t \xi_s^X dX_s + \int_0^t \xi_s^Y dY_s + L_t^{ti} - A_t^{ti,f}.$$

- Optimal cost and risk process

$$C_t^{ti}(\varphi) = nU_0^{ti} + L_t^{ti}$$

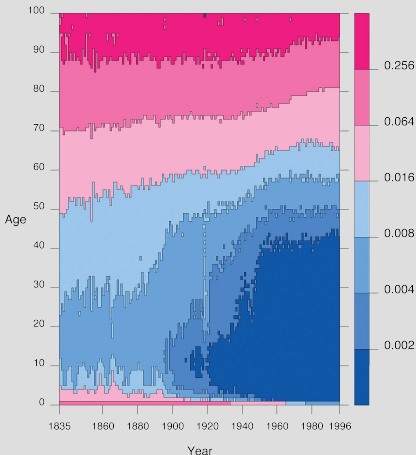
$$R_t^{ti}(\varphi) = \mathbb{E}[(L_T^{ti} - L_t^{ti})^2 | \mathcal{G}_t]$$

for $t \in [0, T]$.

Risk-minimization with dependent mortality risk

- Joint work with Camila Botero and Irene Schreiber.
- Objective: study the problem of pricing and hedging life insurance liabilities with *dependent* mortality risk by means of the risk-minimization approach.
- Consider a portfolio consisting of individuals of **different age cohorts** and take into account the **cross-generational dependency structure**.
- Introduce a model for the mortality intensities that is consistent with typical characteristics of historical mortality data.
- Additional tool: random field theory, in order to model mortality intensities as a **surface** by considering both time and age direction.
 - ▶ Adler [1]
 - ▶ Goldstein [17], Kennedy [18]
 - ▶ Biffis and Millossovich [10]

Andreev [2]: Danish Female Mortality



Typical characteristics of the Mortality Surface

- For fixed point in time: increasing in age
- For fixed age: decreasing in time
- Downward mortality trend is *not* uniform over age and time

→ Use random fields to model the mortality intensity

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