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"On the Multi-Dimensional Controller and Stopper Games"

Abstract: We consider a zero-sum stochastic differential controller-and-stopper game in which the state process is a controlled diffusion evolving in a multi-dimensional Euclidean space. In this game, the controller affects both the drift and the volatility terms of the state process. Under appropriate conditions, we show that the game has a value and the value function is the unique viscosity solution to an obstacle problem for a Hamilton-Jacobi-Bellman equation.

Available at http://arxiv.org/abs/1009.0932. Joint work with Yu-Jui Huang.

On the Multi-Dimensional Controller and Stopper Games

Erhan Bayraktar

Joint work with Yu-Jui Huang
University of Michigan, Ann Arbor

September 17, USC Math Finance Colloquium

Outline

- Introduction.
- The setup.
- The sub-solution property of the of the upper value function.
- The super-solution property of the lower value function.
- Comparison.

Introduction

Consider a zero-sum controller-and-stopper game:

- Two players: the "controller" and the "stopper".
- A state process X^α: can be manipulated by the controller through the selection of α.
- Given a time horizon $T > 0$. The stopper has
  - the right to choose the duration of the game, in the form of a stopping time τ in $[0, T]$ a.s.
  - the obligation to pay the controller the running reward $f(s, X_s^\alpha, \alpha_s)$ at every moment $0 \leq s < \tau$, and the terminal reward $g(X_\tau^\alpha)$ at time τ.
- Instantaneous discount rate: $c(s, X_s^\alpha)$, $0 \leq s \leq T$.

We consider the robust (worst-case) optimal stopping problem:

$$V(t,x) := \sup_{\alpha \in \mathcal{A}_t} \inf_{\tau \in \mathcal{T}^t_{t,T}} \mathbb{E}\Bigg[\int_t^\tau e^{-\int_t^s c(u, X_u^{t,x,\alpha})du} f(s, X_s^{t,x,\alpha}, \alpha_s) ds + e^{-\int_t^\tau c(u, X_u^{t,x,\alpha})du} g(X_\tau^{t,x,\alpha})\Bigg],$$

where $\mathcal{A}_t$: set of controls, $\mathcal{T}^t_{t,T}$: set of stopping times.

- $f(s, X_s^\alpha, \alpha_s)$: running cost at time s.
- $g(X_\tau^\alpha)$: terminal cost at time τ.
- $c(s, X_s^\alpha)$: discount rate at time s.
- X^α: a controlled state process.

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us (stopper) and nature (controller)!

If "Stopper" acts first: Instead of choosing one single stopping time, he would like to employ a strategy $\pi : \mathcal{A}_t \mapsto \mathcal{T}^t_{t,T}$.

$$U(t,x) := \inf_{\pi \in \Pi^t_{t,T}} \sup_{\alpha \in \mathcal{A}_t} \mathbb{E}\Bigg[\int_t^{\pi[\alpha]} e^{-\int_t^s c(u, X_u^{t,x,\alpha})du} f(s, X_s^{t,x,\alpha}, \alpha_s) ds + e^{-\int_t^{\pi[\alpha]} c(u, X_u^{t,x,\alpha})du} g(X_{\pi[\alpha]}^{t,x,\alpha}) \Bigg],$$

where Π is the set of strategies $\pi : \mathcal{A}_t \mapsto \mathcal{T}^t_{t,T}$.

Fix $t \in [0, T]$. For any function $\pi : \mathcal{A}_t \mapsto \mathcal{T}^t_{t,T}$, $\pi \in \Pi^t_{t,T}$ if

$$\begin{aligned}&\text{For any } \alpha, \beta \in \mathcal{A}_t \text{ and } s \in [t, T],\ 1_{\{\pi[\alpha] \le s\}} = 1_{\{\pi[\beta] \le s\}} \\ &\text{for } \overline{\mathbb{P}}\text{-a.e. } \omega \in \{\alpha =_{[t,s)} \beta\},\end{aligned} \tag{1}$$

where $\{\alpha =_{[t,s)} \beta\} := \{\omega \in \Omega \mid \alpha_r(\omega) = \beta_r(\omega) \text{ for } s \in [t, s)\}$.

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Non-anticipative strategies for the controller

Fix $t \in [0, T]$. Let $\gamma : \mathcal{T} \mapsto \mathcal{A}_t$ satisfy the following non-anticipativity condition: for any $\tau_1, \tau_2 \in \mathcal{T}$ and $s \in [t, T]$, it holds for $\overline{\mathbb{P}}$-a.e. $\omega \in \Omega$ that

if $\min\{\tau_1(\omega), \tau_2(\omega)\} > s$, then $(\gamma[\tau_1])_r(\omega) = (\gamma[\tau_2])_r(\omega)$ for $r \in [t, s)$.

Then, observe that $\gamma[\tau](\omega) = \gamma[T](\omega)$ on $[t, \tau(\omega))$ $\overline{\mathbb{P}}$-a.s. for any $\tau \in \mathcal{T}$. This implies that employing the strategy γ has the same effect as employing the control $\gamma[T]$. In other words, the controller would not benefit from using non-anticipating strategies.

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First comparison between value functions

If "Controller" acts first: nature does NOT use strategies.

$$V(t,x) = \sup_{\alpha \in \mathcal{A}_t} \inf_{\tau \in \mathcal{T}^t_{t,T}} \mathbb{E}[\cdots].$$

Now it follows that $V \leq U$. We say **the game has a value** if $U = V$. (Not at all clear for the the players both use controls.)

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Related Work

The controller-stopper game is closely related to some common problems in mathematical finance:

- pricing American contingent claims, see e.g. Karatzas & Kou [1998], and Karatzas & Zamfirescu [2005].
- minimizing the probability of lifetime ruin, see Bayraktar & Young [2011].

But, the game itself has been studied to a great extent only in some special cases.

Related Work

One-dimensional case: Karatzas and Sudderth [2001] study the case where X^α moves along an interval on $\mathbb{R}$.

- they show that the game has a value;
- they construct a saddle-point of optimal strategies (α^*, τ^*).

Difficult to extend their results to higher dimensions (their techniques rely heavily on optimal stopping theorems for one-dimensional diffusions).

Multi-dimensional case: Karatzas and Zamfirescu [2008] develop a martingale approach to deal with this. But, require some **strong** assumptions:

- the diffusion term of X^α has to be non-degenerate, and it cannot be controlled!

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Our Goal

We intend to investigate a much more general multi-dimensional controller-stopper game in which

- both the drift and the diffusion terms of X^α can be controlled;
- the diffusion term can be degenerate.

Main Result: Under appropriate conditions,

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$$U^* \geq U \geq V \geq V_*$$

1. Weak DPP for U
$\Rightarrow$ **subsolution** property of U^*

2. Weak DPP for V
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3. A comparison result $\Rightarrow V_* \geq U^*$ (supersol. $\geq$ subsol.)
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Consider a fixed time horizon $T > 0$.

- $\Omega := C([0,T];\mathbb{R}^d)$.
- $W = \{W_t\}_{t\in[0,T]}$: the canonical process, i.e. $W_t(\omega) = \omega_t$.
- $\mathbb{P}$: the Wiener measure defined on Ω.
- $\mathbb{F} = \{\mathcal{F}_t\}_{t\in[0,T]}$: the $\mathbb{P}$-augmentation of $\sigma(W_s, s \in [0,T])$.

For each $t \in [0,T]$, consider

- $\mathbb{F}^t$: the $\mathbb{P}$-augmentation of $\sigma(W_{t\vee s} - W_t, s \in [0,T])$.
- $\mathcal{T}^t$:={$\mathbb{F}^t$-stopping times valued in $[0,T]$ $\mathbb{P}$-a.s.}.
- $\mathcal{A}_t$:={$\mathbb{F}^t$-progressively measurable M-valued processes}, where M is a separable metric space.
- Given $\mathbb{F}$-stopping times τ_1, τ_2 with $\tau_1 \le \tau_2$ $\mathbb{P}$-a.s., define $\mathcal{T}^t_{\tau_1,\tau_2}$:={$\tau \in \mathcal{T}^t$ valued in $[\tau_1,\tau_2]$ $\mathbb{P}$-a.s.}.

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Given $\omega, \omega' \in \Omega$ and $\theta \in \mathcal{T}$, we define the concatenation of ω and ω' at time θ as

$$(\omega \otimes_\theta \omega')_s := \omega_r 1_{[0,\theta(\omega)]}(s) + (\omega'_s - \omega'_{\theta(\omega)} + \omega_{\theta(\omega)}) 1_{(\theta(\omega),T]}(s), \; s \in [0, T].$$

For each $\alpha \in \mathcal{A}$ and $\tau \in \mathcal{T}$, we define the shifted versions:

$$\begin{aligned} \alpha^{\theta,\omega}(\omega') &:= \alpha(\omega \otimes_\theta \omega') \\ \tau^{\theta,\omega}(\omega') &:= \tau(\omega \otimes_\theta \omega'). \end{aligned}$$

Assumptions on b and σ

Given $\tau \in \mathcal{T}$, $\xi \in \mathcal{L}_d^p$ which is $\mathcal{F}_\tau$-measurable, and $\alpha \in \mathcal{A}$, let $X^{\tau,\xi,\alpha}$ denote a $\mathbb{R}^d$-valued process satisfying the SDE:

$$dX_t^{\tau,\xi,\alpha} = b(t, X_t^{\tau,\xi,\alpha}, \alpha_t)dt + \sigma(t, X_t^{\tau,\xi,\alpha}, \alpha_t)dW_t, \qquad (2)$$

with the initial condition $X_\tau^{\tau,\xi,\alpha} = \xi$ a.s.

Assume: $b(t,x,u)$ and $\sigma(t,x,u)$ are deterministic Borel functions, and continuous in (x,u); moreover, $\exists\ K > 0$ s.t. for $t \in [0,T]$, $x, y \in \mathbb{R}^d$, and $u \in M$

$$\begin{aligned} &|b(t,x,u) - b(t,y,u)| + |\sigma(t,x,u) - \sigma(t,y,u)| \le K|x-y|, \\ &|b(t,x,u)| + |\sigma(t,x,u)| \le K(1+|x|), \end{aligned} \qquad (3)$$

This implies for any $(t,x) \in [0,T] \times \mathbb{R}^d$ and $\alpha \in \mathcal{A}$, (2) admits a unique strong solution $X_\cdot^{t,x,\alpha}$.

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This implies for any $(t,x) \in [0,T] \times \mathbb{R}^d$ and $\alpha \in \mathcal{A}$, (2) admits a unique strong solution $X_\cdot^{t,x,\alpha}$.

Assumptions on f, g, and c

f and g are rewards, c is the discount rate $\Rightarrow$ assume $f, g, c \geq 0$.

In addition, Assume:

- $f : [0, T] \times \mathbb{R}^d \times M \mapsto \mathbb{R}$ is Borel measurable, and $f(t, x, u)$ continuous in (x, u), and continuous in x uniformly in $u \in M$.
- $g : \mathbb{R}^d \mapsto \mathbb{R}$ is continuous,
- $c : [0, T] \times \mathbb{R}^d \mapsto \mathbb{R}$ is continuous and bounded above by some real number $\bar{c} > 0$.
- f and g satisfy a polynomial growth condition

$$|f(t, x, u)| + |g(x)| \leq K(1 + |x|^{\bar{p}}) \text{ for some } \bar{p} \geq 1. \tag{4}$$

Set $F(x,y,z) := z + yg(x)$. Observe that

$$V(t,x) = \sup_{\alpha\in\mathcal{A}_t} \inf_{\tau\in\mathcal{T}_{t,T}^t} \mathbb{E}\left[Z_\tau^{t,x,1,0,\alpha} + Y_\tau^{t,x,1,\alpha} g(X_\tau^{t,x,\alpha})\right]$$
$$= \sup_{\alpha\in\mathcal{A}_t} \inf_{\tau\in\mathcal{T}_{t,T}^t} \mathbb{E}\left[F(\mathbf{X}_\tau^{t,x,1,0,\alpha})\right],$$

where $\mathbf{X}_\tau^{t,x,y,z,\alpha} := (X_\tau^{t,x,\alpha}, Y_\tau^{t,x,y,\alpha}, Z_\tau^{t,x,y,z,\alpha})$. Similarly,

$$U(t,x) = \inf_{\pi\in\Pi_{t,T}^t} \sup_{\alpha\in\mathcal{A}_t} \mathbb{E}\left[F(\mathbf{X}_{\pi[\alpha]}^{t,x,1,0,\alpha})\right].$$

More generally, for any $(x,y,z)\in\mathcal{S} := \mathbb{R}^d \times \mathbb{R}_+^2$, define

$$\bar{V}(t,x,y,z) := \sup_{\alpha\in\mathcal{A}_t} \inf_{\tau\in\mathcal{T}_{t,T}^t} \mathbb{E}\left[F(\mathbf{X}_\tau^{t,x,y,z,\alpha})\right].$$

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Reduction to the Mayer form

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Conditional expectation

Lemma

Fix $(t, \mathbf{x}) \in [0, T] \times \mathcal{S}$ and $\alpha \in \mathcal{A}$. For any $\theta \in \mathcal{T}_{t,T}$ and $\tau \in \mathcal{T}_{\theta,T}$,

$$\mathbb{E}[F(\mathbf{X}_\tau^{t,\mathbf{x},\alpha}) \mid \mathcal{F}_\theta](\omega) = J\left(\theta(\omega), \mathbf{X}_\theta^{t,\mathbf{x},\alpha}(\omega); \alpha^{\theta,\omega}, \tau^{\theta,\omega}\right) \ \mathbb{P}\text{-a.s.}$$
$$\left(= \mathbb{E}\left[F\left(\mathbf{X}_{\tau^{\theta,\omega}}^{\theta(\omega),\mathbf{X}_\theta^{t,\mathbf{x},\alpha}(\omega),\alpha^{\theta,\omega}}\right)\right]\right)$$

Subsolution Property of U^*

For $(t, x, p, A) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{M}^d$, define

$$H^a(t, x, p, A) := -b(t, x, a) - \frac{1}{2} Tr[\sigma\sigma'(t, x, a)A] - f(t, x, a),$$

and set

$$H(t, x, p, A) := \inf_{a \in M} H^a(t, x, p, A).$$

Proposition

The function U^* is a viscosity subsolution on $[0, T) \times \mathbb{R}^d$ to the obstacle problem of an HJB equation

$$\max\left\{c(t, x)w - \frac{\partial w}{\partial t} + H_*(t, x, D_x w, D_x^2 w), w - g(x)\right\} \leq 0.$$

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Subsolution Property of U^*

Sketch of proof:

Assume the contrary, i.e. $\exists\, h \in C^{1,2}([0,T) \times \mathbb{R}^d)$ and $(t_0, x_0) \in [0,T) \times \mathbb{R}^d$ s.t.

$$0 = (U^* - h)(t_0, x_0) > (U^* - h)(t, x),\ \forall\, (t, x) \in [0,T) \times \mathbb{R}^d \setminus (t_0, x_0),$$

and

$$\max\left\{ c(t_0, x_0)h - \frac{\partial h}{\partial t} + H_*(t_0, x_0, D_x h, D_x^2 h), h - g(x_0) \right\} (t_0, x_0) > 0. \tag{5}$$

Since by definition $U \leq g$, the USC of g implies $h(t_0, x_0) = U^*(t_0, x_0) \leq g(x_0)$. Then, we see from (5) that

$$c(t_0, x_0)h(t_0, x_0) - \frac{\partial h}{\partial t}(t_0, x_0) + H_*(\cdot, D_x h, D_x^2 h)(t_0, x_0) > 0.$$

Define the function $\tilde{h}(t, x) := h(t, x) + \varepsilon(|t - t_0|^2 + |x - x_0|)^4$. Note that $(\tilde{h}, \partial_t \tilde{h}, D_x \tilde{h}, D_x^2 \tilde{h})(t_0, x_0) = (h, \partial_t h, D_x h, D_x^2 h)(t_0, x_0)$. Then, by LSC of H_*, $\exists\ r > 0$, $\varepsilon > 0$ such that $t_0 + r < T$ and

$$c(t, x)\tilde{h}(t, x) - \frac{\partial \tilde{h}}{\partial t}(t, x) + H^a(\cdot, D_x \tilde{h}, D_x^2 \tilde{h})(t, x) > 0, \qquad (6)$$

for all $a \in M$ and $(t, x) \in \overline{B_r(t_0, x_0)}$.

Proof (continued)

Define $\eta > 0$ by $\eta e^{\bar{c}T} := \min_{\partial B_r(t_0,x_0)}(\tilde{h} - h) > 0$.
Take $(\hat{t}, \hat{x}) \in B_r(t_0, x_0)$ s.t. $|(U - \tilde{h})(\hat{t}, \hat{x})| < \eta/2$. For $\alpha \in \mathcal{A}_{\hat{t}}$, set

$$\theta^\alpha := \inf\left\{ s \geq \hat{t} \;\middle|\; (s, X_s^{\hat{t},\hat{x},\alpha}) \notin B_r(t_0, x_0) \right\} \in \mathcal{T}^{\hat{t}}_{\hat{t},T}.$$

Applying the product rule to $Y_s^{\hat{t},\hat{x},1,\alpha}\tilde{h}(s, X_s^{\hat{t},\hat{x},\alpha})$, we get

$$\tilde{h}(\hat{t},\hat{x}) = \mathbb{E}\Bigg[Y_{\theta^\alpha}^{\hat{t},\hat{x},1,\alpha}\tilde{h}(\theta^\alpha, X_{\theta^\alpha}^{\hat{t},\hat{x},\alpha})$$

$$+ \int_{\hat{t}}^{\theta^\alpha} Y_s^{\hat{t},\hat{x},1,\alpha}\left(c\tilde{h} - \frac{\partial \tilde{h}}{\partial t} + H^\alpha(\cdot, D_x\tilde{h}, D_x^2\tilde{h}) + f \right)(s, X_s^{\hat{t},\hat{x},\alpha})ds \Bigg]$$

$$> \mathbb{E}\left[Y_{\theta^\alpha}^{\hat{t},\hat{x},1,\alpha} h(\theta^\alpha, X_{\theta^\alpha}^{\hat{t},\hat{x},\alpha}) + \int_{\hat{t}}^{\theta^\alpha} Y_s^{\hat{t},\hat{x},1,\alpha} f(s, X_s^{\hat{t},\hat{x},\alpha}, \alpha_s)ds \right] + \eta$$

Proof (continued)

By our choice of $(\hat{t}, \hat{x})$, $U(\hat{t}, \hat{x}) + \eta/2 > \tilde{h}(\hat{t}, \hat{x})$. Thus,

$$U(\hat{t}, \hat{x}) > \mathbb{E}\left[Y_{\theta^\alpha}^{\hat{t},\hat{x},1,\alpha} h(\theta^\alpha, X_{\theta^\alpha}^{\hat{t},\hat{x},\alpha}) + \int_{\hat{t}}^{\theta^\alpha} Y_s^{\hat{t},\hat{x},1,\alpha} f(s, X_s^{\hat{t},\hat{x},\alpha}, \alpha_s) ds\right] + \frac{\eta}{2},$$

for any $\alpha \in \mathcal{A}_{\hat{t}}$.

How to get a contradiction to this??

By the definition of U,

$$U(\hat{t},\hat{x}) \leq \sup_{\alpha\in\mathcal{A}_{\hat{t}}} \mathbb{E}\left[F\left(\mathbf{X}^{\hat{t},\hat{x},1,0,\alpha}_{\pi^*[\alpha]}\right)\right]$$

$$\leq \mathbb{E}\left[F\left(\mathbf{X}^{\hat{t},\hat{x},1,0,\hat{\alpha}}_{\pi^*[\hat{\alpha}]}\right)\right] + \frac{\eta}{4}, \text{ for some } \hat{\alpha}\in\mathcal{A}_{\hat{t}}.$$

$$\leq \mathbb{E}\left[Y^{\hat{t},\hat{x},1,\hat{\alpha}}_{\theta^{\hat{\alpha}}} h(\theta, X^{\hat{t},\hat{x},\hat{\alpha}}_{\theta^{\hat{\alpha}}}) + Z^{\hat{t},\hat{x},1,0,\hat{\alpha}}_{\theta^{\hat{\alpha}}}\right] + \frac{\eta}{4} + \frac{\eta}{4}.$$

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$$\le \mathbb{E}\left[Y_{\theta^{\hat{\alpha}}}^{\hat{t},\hat{x},1,\hat{\alpha}} h(\theta, X_{\theta^{\hat{\alpha}}}^{\hat{t},\hat{x},\hat{\alpha}}) + Z_{\theta^{\hat{\alpha}}}^{\hat{t},\hat{x},1,0,\hat{\alpha}}\right] + \frac{\eta}{4} + \frac{\eta}{4}.$$

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Weak DPP for U

Proposition (Weak DPP for U)

Fix $(t, \mathbf{x}) \in [0, T] \times \mathcal{S}$ and $\varepsilon > 0$. For any $\pi \in \Pi^t_{t,T}$ and $\varphi \in LSC([0, T] \times \mathbb{R}^d)$ with $\varphi \geq U$, $\exists\ \pi^* \in \Pi^t_{t,T}$ s.t. $\forall \alpha \in \mathcal{A}_t$,

$$\mathbb{E}\left[F(\mathbf{X}^{t,\mathbf{x},\alpha}_{\pi^*[\alpha]})\right] \leq \mathbb{E}\left[Y^{t,x,y,\alpha}_{\pi[\alpha]}\varphi\left(\pi[\alpha], X^{t,x,\alpha}_{\pi[\alpha]}\right) + Z^{t,x,y,z,\alpha}_{\pi[\alpha]}\right] + 4\varepsilon.$$

An important Lemma

To prove this weak DPP, we need **the very technical lemma**

Lemma

Fix $t \in [0, T]$. For any $\pi \in \Pi_{t,T}^{t}$, $L^{\pi} : [0, t] \times \mathcal{S} \mapsto \mathbb{R}$ defined by $L^{\pi}(s, \mathbf{x}) := \sup_{\alpha \in \mathcal{A}_s} \mathbb{E}\left[F(\mathbf{X}_{\pi[\alpha]}^{s,\mathbf{x},\alpha})\right]$ is continuous.

Idea of Proof: Generalize the arguments in Krylov[1980] for control problems with fixed horizon to our case with random horizon.

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Fix $t \in [0, T]$. For any $\pi \in \Pi^{t}_{t,T}$, $L^{\pi} : [0, t] \times \mathcal{S} \mapsto \mathbb{R}$ defined by $L^{\pi}(s, \mathbf{x}) := \sup_{\alpha \in \mathcal{A}_s} \mathbb{E}\left[F(\mathbf{X}^{s,\mathbf{x},\alpha}_{\pi[\alpha]}) \right]$ is continuous.

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This lemma is why we need the stopper to use strategies.

Sketch of proof for "Weak DPP for U":

1. **Separate $[0, T] \times \mathcal{S}$ into small pieces.** Since $[0, T] \times \mathcal{S}$ is Lindelöf, take $\{(t_i, x_i)\}_{i\in\mathbb{N}}$ s.t. $\bigcup_{i\in\mathbb{N}} B(t_i, x_i; r^{(t_i,x_i)}) = (0, T] \times \mathcal{S}$,

$$\text{with } B(t_i, x_i; r^{(t_i,x_i)}) := (t_i - r^{(t_i,x_i)}, t_i] \times B_{r^{(t_i,x_i)}}(x_i).$$

Take a disjoint subcovering $\{A_i\}_{i\in\mathbb{N}}$ s.t. $(t_i, x_i) \in A_i$.

2. **Pick ε-optimal strategy $\pi^{(t_i,x_i)}$ in each A_i.** For each (t_i, x_i), by def. of $\bar{U}$, $\exists\ \pi^{(t_i,x_i)} \in \Pi^{t_i}_{t_i,T}$ s.t.

$$\sup_{\alpha\in\mathcal{A}_{t_i}} \mathbb{E}\left[F(\mathbf{X}^{t_i,x_i,\alpha}_{\pi^{(t_i,x_i)}[\alpha]})\right] \leq \bar{U}(t_i, x_i) + \varepsilon.$$

Set $\bar{\varphi}(t, x, y, z) := y\varphi(t, x) + z$. For any $(t', x') \in A_i$,

$$\begin{aligned} L^{\pi^{(t_i,x_i)}}(t', x') &\underset{\text{usc}}{\leq} L^{\pi^{(t_i,x_i)}}(t_i, x_i) + \varepsilon \leq \bar{U}(t_i, x_i) + 2\varepsilon \\ &\leq \bar{\varphi}(t_i, x_i) + 2\varepsilon \underset{\text{lsc}}{\leq} \bar{\varphi}(t', x') + 3\varepsilon. \end{aligned} \tag{7}$$

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3. **Paste $\pi^{(t_i,x_i)}$ together.** For any $n \in \mathbb{N}$, set $B^n := \cup_{1 \le i \le n} A_i$ and define $\pi^n \in \Pi^t_{t,T}$ by

$$\pi^n[\alpha] := T 1_{(B^n)^c}(\pi[\alpha], \mathbf{X}^{t,\mathbf{x},\alpha}_{\pi[\alpha]}) + \sum_{i=1}^{n} \pi^{(t_i,x_i)}[\alpha] 1_{A_i}(\pi[\alpha], \mathbf{X}^{t,\mathbf{x},\alpha}_{\pi[\alpha]}).$$

4. **Estimations.**

$$\begin{aligned}
&\mathbb{E}[F(\mathbf{X}^{t,\mathbf{x},\alpha}_{\pi^n[\alpha]})] \\
&= \mathbb{E}\left[F(\mathbf{X}^{t,\mathbf{x},\alpha}_{\pi^n[\alpha]}) 1_{B^n}(\pi[\alpha], \mathbf{X}^{t,\mathbf{x},\alpha}_{\pi[\alpha]})\right] + \mathbb{E}\left[F(\mathbf{X}^{t,\mathbf{x},\alpha}_{T}) 1_{(B^n)^c}(\pi[\alpha], \mathbf{X}^{t,\mathbf{x},\alpha}_{\pi[\alpha]})\right] \\
&\le \mathbb{E}[\bar{\varphi}(\pi[\alpha], \mathbf{X}^{t,\mathbf{x},\alpha}_{\pi[\alpha]})] + 3\varepsilon + \varepsilon,
\end{aligned}$$

where RED PART follows from (7) and BLUE PART holds for $n \ge n^*(\alpha)$.

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Then we get

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$$= \mathbb{E}[Y^{t,x,y,\alpha}_{\pi[\alpha]}\varphi(\theta, X^{t,x,\alpha}_{\pi[\alpha]}) + Z^{t,x,y,z,\alpha}_{\pi[\alpha]}] + 4\varepsilon.$$

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Supersolution Property of V_*

Proposition (Weak DPP for V)

Fix $(t, \mathbf{x}) \in [0, T] \times \mathcal{S}$ and $\varepsilon > 0$. For any $\alpha \in \mathcal{A}_t$, $\theta \in \mathcal{T}_{t,T}^t$ and $\varphi \in USC([0, T] \times \mathbb{R}^d)$ with $\varphi \leq V$,

(I) $\mathbb{E}[\bar{\varphi}^+(\theta, \mathbf{X}_\theta^{t,\mathbf{x},\alpha})] < \infty$;

(II) If, moreover, $\mathbb{E}[\bar{\varphi}^-(\theta, \mathbf{X}_\theta^{t,\mathbf{x},\alpha})] < \infty$, then there exists $\alpha^* \in \mathcal{A}_t$ with $\alpha_s^* = \alpha_s$ for $s \in [t, \theta]$ s.t. for any $\tau \in \mathcal{T}_{t,T}^t$,

$$\mathbb{E}[F(\mathbf{X}_\tau^{t,\mathbf{x},\alpha^*})] \geq \mathbb{E}[Y_{\tau\wedge\theta}^{t,x,y,\alpha}\varphi(\tau \wedge \theta, X_{\tau\wedge\theta}^{t,x,\alpha}) + Z_{\tau\wedge\theta}^{t,x,y,z,\alpha}] - 4\varepsilon.$$

Proposition

The function V_* is a viscosity supersolution on $[0, T) \times \mathbb{R}^d$ to the obstacle problem of an HJB equation

$$\max\left\{c(t,x)w - \frac{\partial w}{\partial t} + H(t, x, D_x w, D_x^2 w),\ w - g(x)\right\} \geq 0.$$

Comparison

To state an appropriate comparison result, we assume

A. for any $t, s \in [0, T]$, $x, y \in \mathbb{R}^d$, and $u \in M$,

$$|b(t,x,u)-b(s,y,u)|+|\sigma(t,x,u)-\sigma(s,y,u)| \leq K(|t-s|+|x-y|).$$

B. $f(t,x,u)$ is uniformly continuous in (t,x), uniformly in $u \in M$.

The conditions **A** and **B**, together with the linear growth condition on b and σ, imply that the function H is continuous, and thus $H = H_*$.

Proposition (Comparison)

Assume **A** and **B**. Let u (resp. v) be an USC viscosity subsolution (resp. a LSC viscosity supersolution) with polynomial growth condition to (30), such that $u(T,x) \leq v(T,x)$ for all $x \in \mathbb{R}^d$. Then $u \leq v$ on $[0,T) \times \mathbb{R}^d$.

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Main Result

Lemma

For all $x \in \mathbb{R}^d$, $V_*(T, x) \geq g(x)$.

Theorem

Assume **A** and **B**. Then $U^* = V_*$ on $[0, T] \times \mathbb{R}^d$. In particular, $U = V$ on $[0, T] \times \mathbb{R}^d$, i.e. the game has a value, which is the unique viscosity solution to (30) with terminal condition $w(T, x) = g(x)$ for $x \in \mathbb{R}^d$.

$$U^* \geq U \geq V \geq V_*$$

1. Weak DPP for U

$\Rightarrow$ **subsolution** property of U^*

2. Weak DPP for V

$\Rightarrow$ **supersolution** property of V_*

3. A comparison result $\Rightarrow V_* \geq U^*$ (supersol. $\geq$ subsol.)
$\Rightarrow U^* = V_* \Rightarrow U = V$, i.e. **the game has a value.**

No a priori regularity needed!
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Thank you very much for your attention!
Q & A