“Backward” martingale representation and endogenous completeness in finance

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Martingale Integral Representation

$(\Omega, \mathcal{F}_1, \mathbf{F} = (\mathcal{F}_t)_{t\in[0,1]}, \mathbb{P})$: a complete filtered probability space.

$\mathbb{Q}$: an equivalent probability measure.

$S = (S_t^j)$: J-dimensional martingale under $\mathbb{Q}$.

We want to know whether any local martingale $M = (M_t)$ under $\mathbb{Q}$ admits an integral representation with respect to S, that is,

$$M_t = M_0 + \int_0^t H_u dS_u, \quad t \in [0, 1],$$

for some predictable S-integrable process $H = (H_t^j)$.

- Completeness in Mathematical Finance.
- Jacod's Theorem (2nd FTAP): the integral representation holds iff $\mathbb{Q}$ is the only martingale measure for S.
- Easy to verify if S is given in terms of *local characteristics* ("forward" description).

Backward Martingale Representation

Inputs: random variables $\zeta > 0$ and $\psi = (\psi^j)$

- The density of the martingale measure $\mathbb{Q}$ is defined by

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \text{const}\,\zeta.$$

- ψ is the terminal value for S:

$$S_t \triangleq \mathbb{E}^{\mathbb{Q}}[\psi|\mathcal{F}_t], \quad t \in [0,1].$$

Problem

Determine (easily verifiable) conditions on ζ and ψ so that the martingale representation property holds under $\mathbb{Q}$ and S.

Radner equilibrium

Inputs:

- M agents with utility functions U_m and initial random endowments Λ^m,
- interest rate $r = 0$, J stocks with terminal values (dividends) $\psi = (\psi^j)$.

Output: stocks' price process $S = (S^j_t)$ such that

1. $S_1 = \psi$.
2. Given the S-market, the agents' optimal strategies (stock's quantities) $H^m = (H^{m,j}_t)$ satisfy the *clearing condition*:

$$\sum_{m=1}^{M} H^{m,j}_t = 0, \quad t \in [0,1], \; j = 1, \dots, J.$$

Construction of Radner equilibrium

Two steps:

1. Find static (Arrow-Debreu) equilibrium, that is, find a pricing measure $\mathbb{Q}$ such that if economic agents can trade *any* payoff ξ at the price
$$p = \mathbb{E}^{\mathbb{Q}}[\xi],$$
then the clearing condition holds: the total wealth does not change. Main tool: Brower's fixed point theorem.
2. Define $S_t = \mathbb{E}^{\mathbb{Q}}[\psi|\mathcal{F}_t]$, $t \in [0, 1]$, (ψ is the terminal dividend) and verify *endogenous* completeness of the S-market $\implies$ Radner equilibrium.

Other applications: equilibrium based price impact models (a project with Peter Bank; David German) and model's completion with options (Davis and Obloj).

Diffusion framework

The random variables $\psi = S_1$ and $\zeta = \text{const}\,\frac{d\mathbb{Q}}{d\mathbb{P}}$ are given by

$$\zeta \triangleq G(X_1)e^{\int_0^1 \beta(t,X_t)dt},$$

$$\psi^j \triangleq F^j(X_1)e^{\int_0^1 \alpha^j(t,X_t)dt} + \int_0^1 f^j(t,X_t)e^{\int_0^t \alpha^j(s,X_s)ds}dt$$

$$+ \int_0^1 \frac{g^j(t,X_t)}{Y_t}e^{\int_0^t (\alpha^j(s,X_s)+\beta(s,X_s))ds}dt, \quad j = 1,\dots,J,$$

where $Y_t \triangleq \mathbb{E}[\zeta|\mathcal{F}_t]$ and

- $F^j, G : \mathbb{R}^d \to \mathbb{R}$ and $f^j, g^j, \alpha^j, \beta : [0,1]\times\mathbb{R}^d \to \mathbb{R}$ are deterministic functions;
- $X = (X_t^i)$ is a d-dimensional diffusion:

$$X_t = X_0 + \int_0^t b(s,X_s)ds + \int_0^t \sigma(s,X_s)dW_s, \quad t \in [0,1],$$

  with drift and volatility functions $b^i, \sigma^{ij} : [0,1]\times\mathbb{R}^d \to \mathbb{R}$.

Assumptions on functions

1. The functions $F^j = F^j(x)$ and $G = G(x)$ are weakly differentiable and have exponential growth:

$$|\nabla F^j| + |\nabla G| \leq Ne^{N|x|}.$$

2. The Jacobian matrix $\left(\frac{\partial F^j}{\partial x^i}\right)$ has rank d almost surely under the Lebesgue measure on $\mathbb{R}^d$.
3. The maps $t \mapsto e^{-N|\cdot|} f^j(t, \cdot) \triangleq \left(e^{-N|x|} f^j(t, x)\right)_{x \in \mathbb{R}^d}$, $t \mapsto e^{-N|\cdot|} g^j(t, \cdot)$ and $t \mapsto \alpha^j(t, \cdot)$, $t \mapsto \beta(t, \cdot)$ of $[0, 1]$ to $\mathbf{L}_\infty$ are analytic on $(0, 1)$ and Hölder continuous on $[0, 1]$.

- Careful with item 3: stronger than pointwise analyticity! Seems to be overlooked in the cited literature.

Assumptions on the diffusion X

1. The map $t \mapsto b^i(t,\cdot)$ of $[0,1]$ to $\mathbf{L}_\infty$ is analytic on $(0,1)$ and Hölder continuous on $[0,1]$.
2. The map $t \mapsto \sigma^{ij}(t,\cdot)$ of $[0,1]$ to $\mathbf{C}$ is analytic on $(0,1)$ and Hölder continuous on $[0,1]$. Moreover, $\sigma = \sigma(t,x)$ is uniformly continuous with respect to x:

   $$|\sigma(t,x) - \sigma(t,y)| \leq \omega(|x-y|).$$

   for some strictly increasing function $\omega = (\omega(\epsilon))_{\epsilon>0}$ such that $\omega(\epsilon) \to 0$ as $\epsilon \downarrow 0$, and has a bounded inverse:

   $$|\sigma^{-1}(t,x)| \leq N \quad \text{(uniform ellipticity for } \sigma\sigma^*).$$

- Counter-example on t-analyticity condition in $\sigma = \sigma(t,x)$.

Main result

Theorem

Assume that $\mathbb{F} = \mathbb{F}^X$. Under the conditions above the martingale representation property holds for the probability measure $\mathbb{Q}$ with the density

$$\frac{d\mathbb{Q}}{d\mathbb{P}} \triangleq \frac{\zeta}{\mathbb{E}[\zeta]},$$

and the $\mathbb{Q}$-martingale

$$S_t \triangleq \mathbb{E}^{\mathbb{Q}}[\psi|\mathcal{F}_t], \quad t \in [0,1].$$

Comparison with the literature

Assumptions on functions: In Anderson and Raimondo (2008), Hugonnier et al. (2012), and Riedel and Herzberg (2012)

- The Jacobian matrix $\left(\frac{\partial F^j}{\partial x^i}\right)$ needs to have full rank only on some open set (counter-example in our setting).
- The "small letter" functions, that is, the functions of (t, x) should be (t, x)-analytic.

Assumptions on diffusion X:

- In Anderson and Raimondo (2008) X is a Brownian motion.
- In Hugonnier et al. (2012) the diffusion coefficients $b = b(t, x)$ and $\sigma = \sigma(t, x)$ are either *analytic* with respect to (t, x) or the transitional probability is $\mathbf{C}^7$.

Idea of the proof when $\mathbb{P} = \mathbb{Q}$

$$\psi^j \triangleq F^j(X_1)e^{\int_0^1 \alpha^j(t,X_t)dt} + \int_0^1 e^{\int_0^t \alpha^j(s,X_s)ds} f^j(t,X_t)dt, \quad j = 1,\ldots,J.$$

Ito's formula implies that

$$S_t^j \triangleq \mathbb{E}[\psi^j|\mathcal{F}_t] = e^{\int_0^t \alpha^j(s,X_s)ds} u^j(t,X_t) + \int_0^t e^{\int_0^r \alpha^j(s,X_s)ds} f^j(r,X_r)dr,$$

where $u^j = u^j(t,x)$ solves the parabolic equation:

$$u_t^j + (L(t) + \alpha^j)u^j + f^j = 0,$$
$$u^j(T,x) = F^j(x).$$

Here $L(t)$ the infinitesimal generator of X:

$$L(t) = \frac{1}{2}\sum_{i,j=1}^d (\sigma\sigma^*)^{ij}(t,x)\frac{\partial^2}{\partial x^i \partial x^j} + \sum_{i=1}^d b^i(t,x)\frac{\partial}{\partial x^i}.$$

Idea of the proof when $\mathbb{P} = \mathbb{Q}$

Our assumptions imply that

1. $u^j(t,\cdot) \to F^j$ as $t \to T$ in Sobolev's spaces $\mathbf{W}^1_p$ for all $p > 1$.
2. $t \to u^j(t,\cdot)$ is an analytic map of $(0,1)$ to $\mathbf{W}^2_p$.

It follows that $(t,x) \to \det u_x(t,x)$ is

1. continuous on $[0,1] \times \mathbb{R}^d$;
2. t-analytic on $(0,1)$;
3. $\det u_x(1,x) = \det F_x(x) \neq 0$, $x \in \mathbf{R}^d$ a.s.

Then $\det u_x(t,x) \neq 0$ on $[0,1] \times \mathbf{R}^d$ a.s.. As

$$S_t = u(0,X_0) + \int_0^t e^{\int_0^s \alpha(r,X_r)dr} u_x(s,X_s)\sigma(s,X_s)dW_s,$$

we deduce that every martingale is a stochastic integral under S.

Elements of the proof

Parabolic PDEs:

- Evolution equations in $\mathbf{L}_p$ spaces (maximal regularity, analyticity theorem by Kato and Tanabe).
- Elliptic equations in Sobolev spaces (sectoriality property).
- Interpolation theory ($\mathbf{W}_p^1$ is the midpoint of $\mathbf{L}_p$ and $\mathbf{W}_p^2$ in complex interpolation).

Stochastic Analysis:

- Krylov's variant of Ito's formula (instead of $\mathbf{C}^2$ we can have $\mathbf{W}_p^2$ with $p \geq d$ under the uniform ellipticity condition).

Counter-example on t-analyticity in $\sigma = \sigma(t, x)$

$$X_t = X_0 + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dW_s, \quad t \in [0, 1].$$

Recall that $b = b(t, x)$ and $\sigma = \sigma(t, x)$ have

- only minimal *classical* regularity conditions with respect to x;
- very strong *analyticity* assumptions with respect to t.

We gave an *explicit* (!) example which shows that the t-analyticity assumption on the volatility matrix can not be removed. In our construction,

- dimension $d = J = 2$ (will not work for $d = 1$)
- both σ and its inverse σ^{-1} are $\mathbf{C}^\infty$-matrices on $[0, 1] \times \mathbb{R}^2$ which are bounded with all their derivatives and have analytic restrictions to $[0, \frac{1}{2}) \times \mathbb{R}^2$ and $(\frac{1}{2}, 1] \times \mathbb{R}^2$.

Counter-example on t-analyticity in $\sigma = \sigma(t, x)$

- $g = g(t)$ is a $\mathbf{C}^\infty$-function on $[0, 1]$ which equals 0 on $[0, \frac{1}{2}]$, while it is analytic and strictly positive on $(\frac{1}{2}, 1]$.
- $h = h(t, y)$ is an analytic function on $[0, 1] \times \mathbb{R}$ such that $0 \leq h \leq 1$, $h(1, \cdot) \neq \text{const}$, and

  $$\frac{\partial h}{\partial t} + \frac{1}{2}\frac{\partial^2 h}{\partial y^2} = 0.$$

  For instance, we can take

  $$h(t, y) = \frac{1}{2}(1 + e^{\frac{t-1}{2}} \sin y).$$

- 2-dimensional diffusion (X, Y) on $[0, 1]$:

  $$X_t = \int_0^t \sqrt{1 + g(s)h(s, Y_s)}dB_s,$$
  $$Y_t = W_t,$$

  where B and W are independent Brownian motions.

Counter-example on t-analyticity in $\sigma = \sigma(t, x)$

Set

$$\mathbb{Q} = \mathbb{P}, \quad \psi = (F(X_1, Y_1), H(X_1, Y_1)),$$

where

$$F(x, y) = x,$$
$$H(x, y) = x^2 - 1 - h(1, y) \int_0^1 g(t)dt.$$

The determinant of the Jacobian matrix for (F, H):

$$\frac{\partial F}{\partial x}\frac{\partial H}{\partial y} - \frac{\partial F}{\partial y}\frac{\partial H}{\partial x} = -\frac{\partial h}{\partial y}(1, y) \int_0^1 g(t)dt \neq 0,$$

as $h(1, \cdot)$ is non-constant and analytic.

Counter-example on t-analyticity in $\sigma = \sigma(t, x)$

Ito's formula shows that

$$S_t \triangleq \mathbb{E}[F(X_1, Y_1)|\mathcal{F}_t] = X_t,$$
$$R_t \triangleq \mathbb{E}[H(X_1, Y_1)|\mathcal{F}_t] = X_t^2 - t - h(t, Y_t) \int_0^t g(s)ds,$$

As $g(t) = 0$ for $t \in [0, \frac{1}{2}]$, it follows that on $[0, \frac{1}{2}]$

$$S_t = B_t,$$
$$R_t = B_t^2 - t.$$

Clearly, the Brownian motion $Y = W$ can not be written as a stochastic integral with respect to (S, R).

Summary

- We gave conditions on diffusion X and functions F^j, G, α^j, $\beta = \beta(t,x)$, and f^j, g^j so that the integral representation holds for the measure $\mathbb{Q}$ and the $\mathbb{Q}$-martingale $S = (S^j)$ defined by $d\mathbb{Q}/d\mathbb{P} = \text{const}\,\zeta$ and $S^j_1 = \psi^j$, where

$$\zeta \triangleq G(X_1)e^{\int_0^1 \beta(t,X_t)dt},$$

$$\psi^j \triangleq F^j(X_1)e^{\int_0^1 \alpha^j(t,X_t)dt} + \int_0^1 f^j(t,X_t)e^{\int_0^t \alpha^j(s,X_s)ds}dt + \int_0^1 \frac{g^j(t,X_t)}{Y_t}e^{\int_0^t (\alpha^j(s,X_s)+\beta(s,X_s))ds}dt, \quad j = 1,\dots,J,$$

  and $Y_t \triangleq \mathbb{E}[\zeta|\mathcal{F}_t]$.
- The diffusion coefficients of X have only minimal x-regularity. However, they are *t-analytical* (a counter-example for σ).
- The study is motivated by the problem of *endogenous completeness* in financial economics.