Stochastic Target Games with Controlled Loss

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Joint work with L. Moreau (ETH-Zürich) and M. Nutz (Columbia)

Plan of the talk

- Problem formulation

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- Examples

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- Assumptions for this talk

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- Geometric dynamic programming

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- Geometric dynamic programming
- Application to the monotone case

Problem formulation and Motivations

Problem formulation

Provide a PDE characterization of the *viability* sets

$$\Lambda(t) := \{(z,p) : \exists\, \mathfrak{u} \in \mathfrak{U} \text{ s. t. } \mathbb{E}\left[\ell(Z_{t,z}^{\mathfrak{u}[\vartheta],\vartheta}(T))\right] \geq m \;\forall\, \vartheta \in \mathcal{V}\}$$

In which :

- $\mathcal{V}$ is a set of admissible adverse controls
- $\mathfrak{U}$ is a set of admissible strategies
- $Z_{t,z}^{\mathfrak{u}[\vartheta],\vartheta}$ is an adapted $\mathbb{R}^d$-valued process s.t. $Z_{t,z}^{\mathfrak{u}[\vartheta],\vartheta}(t) = z$
- ℓ is a given loss/utility function
- m a threshold.

Application in finance

□ $Z_{t,z}^{\mathfrak{u}[\vartheta],\vartheta} = (X_{t,x}^{\mathfrak{u}[\vartheta],\vartheta}, Y_{t,x,y}^{\mathfrak{u}[\vartheta],\vartheta})$ where

- $X_{t,x}^{\mathfrak{u}[\vartheta],\vartheta}$ models financial assets or factors with dynamics depending on ϑ
- $Y_{t,x,y}^{\mathfrak{u}[\vartheta],\vartheta}$ models a wealth process
- ϑ is the control of the market : parameter uncertainty (e.g. volatility), adverse players, etc...
- $\mathfrak{u}[\vartheta]$ is the financial strategy given the past observations of ϑ.

□ Flexible enough to embed constraints, transaction costs, market impact, etc...

Examples

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- Expected loss control for $\ell(z) = -[y - g(x)]^-$

- Give sense to problems that would be degenerate under $\mathbb{P}-$a.s. constraints : B. and Dang (guaranteed VWAP pricing).

Examples

□ Constraint in probability :

$$\Lambda(t) := \{(z,p) : \exists\, \mathfrak{u} \in \mathfrak{U} \text{ s.t. } \mathbb{P}\left[Z_{t,z}^{\mathfrak{u}[\vartheta],\vartheta}(T) \in \mathcal{O}\right] \geq m \;\forall\, \vartheta \in \mathcal{V}\}$$

for $\ell(z) = 1_{z\in\mathcal{O}}$, $m \in (0,1)$.
$\Rightarrow$ Quantile-hedging in finance for $\mathcal{O} := \{y \geq g(x)\}$.

Examples

□ Matching a P&L distribution = Multiple constraints in probability :

$$\{\exists\, \mathfrak{u} \in \mathfrak{U} \text{ s.t. } \mathbb{P}\left[\text{dist}\left(Z_{t,z}^{\mathfrak{u}[\vartheta],\vartheta}(T), \mathcal{O}\right) \leq \gamma_i\right] \geq m_i \ \forall\, i \leq I,\ \forall\, \vartheta \in \mathcal{V}\}$$

(see B. and Thanh Nam)

Examples

☐ Almost sure constraint :

$$\Lambda(t) := \{z : \exists\, \mathfrak{u} \in \mathfrak{U} \text{ s.t. } Z_{t,z}^{\mathfrak{u}[\vartheta],\vartheta}(T) \in \mathcal{O}\ \mathbb{P} - \text{a.s. } \forall\, \vartheta \in \mathcal{V}\}$$

for $\ell(z) = 1_{z \in \mathcal{O}}$, $m = 1$.
$\Rightarrow$ Super-hedging in finance for $\mathcal{O} := \{y \geq g(x)\}$.

Unfortunately not covered by our assumptions...

Setting for this talk

(see the paper for an abstract version)

Brownian diffusion setting

Brownian diffusion setting

☐ **State process :** $Z^{\mathfrak{u}[\vartheta],\vartheta}$ solves (μ and σ continuous, uniformly Lipschitz in space)

$$Z(s) = z + \int_t^s \mu(Z(r), \mathfrak{u}[\vartheta]_r, \vartheta_r)\, dr + \int_t^s \sigma(Z(r), \mathfrak{u}[\vartheta]_r, \vartheta_r)\, dW_r.$$

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Brownian diffusion setting

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- $\mathcal{V}$ is the set of predictable processes with values in $V \subset \mathbb{R}^d$.
- $\mathfrak{U}$ is set of non-anticipating maps $\mathfrak{u} : \vartheta \in \mathcal{V} \mapsto \mathcal{U}$, i.e.

$$\{\vartheta_1 =_{(0,s]} \vartheta_2\} \subset \{\mathfrak{u}[\vartheta_1] =_{(0,s]} \mathfrak{u}[\vartheta_2]\} \quad \forall\, \vartheta_1, \vartheta_2 \in \mathcal{V},\, s \le T,$$

where $\mathcal{U}$ is the set of predictable processes with values in $U \subset \mathbb{R}^d$.

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where $\mathcal{U}$ is the set of predictable processes with values in $U \subset \mathbb{R}^d$.

□ The loss function ℓ has polynomial growth and is continuous.

The game problem

- **Worst expected loss** for a given strategy :

$$J(t,z,\mathfrak{u}) := \operatorname*{ess\,inf}_{\vartheta\in\mathcal{V}} \mathbb{E}\left[\ell\left(Z_{t,z}^{\mathfrak{u}[\vartheta],\vartheta}(T)\right)|\mathcal{F}_t\right]$$

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- **The *viability* sets** are given by

$$\Lambda(t) := \{(z,p) : \exists\, \mathfrak{u} \in \mathfrak{U} \text{ s.t. } J(t,z,\mathfrak{u}) \geq p \ \mathbb{P}-\text{a.s.}\}.$$

Compare with the formulation of games in Buckdahn and Li (2008).

Geometric dynamic programming principle

How are the properties

$(z, m) \in \Lambda(t)$ and $(Z_{t,z}^{\mathfrak{u}[\vartheta],\vartheta}(\theta), ?) \in \Lambda(\theta)$

related ?

□ **First direction :** Take $\underline{(z, m) \in \Lambda(t)}$ and $\mathfrak{u} \in \mathfrak{U}$ such that

$$\operatorname*{ess\,inf}_{\vartheta \in \mathcal{V}} \mathbb{E}\left[\ell\left(Z_{t,z}^{\mathfrak{u}[\vartheta],\vartheta}(T)\right) | \mathcal{F}_t\right] \geq m$$

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To take care of the evolution of the worst case scenario conditional expectation, we introduce :

$$S_r^{\vartheta} := \operatorname*{ess\,inf}_{\bar{\vartheta} \in \mathcal{V}} \mathbb{E}\left[\ell\left(Z_{t,z}^{\mathfrak{u}[\vartheta \oplus_r \bar{\vartheta}],\vartheta \oplus_r \bar{\vartheta}}(T)\right)|\mathcal{F}_r\right].$$

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$$S^{\vartheta} \text{ is a submartingale and } S_t^{\vartheta} \geq m \text{ for all } \vartheta \in \mathcal{V},$$

and we can find a martingale M^{ϑ} such that

$$S^{\vartheta} \geq M^{\vartheta} \text{ and } M_t^{\vartheta} = S_t^{\vartheta} \geq m.$$

□ We have for all stopping times θ (may depend on $\mathfrak{u}$ and ϑ)

$$\operatorname*{ess\,inf}_{\bar{\vartheta}\in\mathcal{V}} \mathbb{E}\left[\ell\left(Z_{t,z}^{\mathfrak{u}[\vartheta\oplus_\theta\bar{\vartheta}],\vartheta\oplus_\theta\bar{\vartheta}}(T)\right)|\mathcal{F}_\theta\right] = S_\theta^\vartheta \geq M_\theta^\vartheta.$$

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If the above is usc in space uniformly in the strategies, θ takes finitely many values, we can find a covering formed of $B_i \ni (t_i, z_i)$, $i \geq 1$, such that

$$J(t_i, z_i; \mathfrak{u}) \geq M_\theta^\vartheta - \varepsilon \text{ on } A_i := \{(\theta, Z_{t,z}^{\mathfrak{u}[\vartheta],\vartheta}(\theta)) \in B_i\}.$$

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Hence

$$K(t_i, z_i) \geq M_\theta^\vartheta - \varepsilon \text{ on } A_i.$$

where

$$K(t_i, z_i) := \operatorname*{ess\,sup}_{\mathfrak{u}\in\mathfrak{U}} \operatorname*{ess\,inf}_{\vartheta\in\mathcal{V}} \mathbb{E}\left[\ell\left(Z_{t_i,z_i}^{\mathfrak{u}[\vartheta],\vartheta}(T)\right)|\mathcal{F}_{t_i}\right] \text{ is deterministic.}$$

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□ If K is lsc

$$K(\theta(\omega), Z_{t,z}^{\mathfrak{u}[\vartheta],\vartheta}(\theta)(\omega)) \geq K(t_i, z_i) - \varepsilon \geq M_\theta^\vartheta(\omega) - 2\varepsilon \text{ on } A_i.$$

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□ If K is lsc

$$\underline{(Z_{t,z}^{\mathfrak{u}[\vartheta],\vartheta}(\theta(\omega)), M_\theta^\vartheta(\omega) - 3\varepsilon) \in \Lambda(\theta(\omega))}$$

□ To get rid of ε, and for non-regular cases (in terms of K and J), we work by approximation : One needs to start from $(z, m-\iota) \in \Lambda(t)$ and obtain

$$\left(Z_{t,z}^{\mathfrak{u}[\vartheta],\vartheta}(\theta), M_\theta^\vartheta\right) \in \bar{\Lambda}(\theta) \ \mathbb{P}-\text{a.s.} \ \forall \ \vartheta \in \mathcal{V}$$

where

$$\bar{\Lambda}(t) := \left\{ \begin{array}{l} (z,m): \text{ there exist } (t_n, z_n, m_n) \to (t,z,m) \\ \text{such that } (z_n, m_n) \in \Lambda(t_n) \text{ and } t_n \geq t \text{ for all } n \geq 1 \end{array} \right\}$$

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Remark : $M^\vartheta = M_{t,p}^{\alpha^\vartheta} := p + \int_t^\cdot \alpha_s^\vartheta dW_s$, with $\alpha^\vartheta \in \mathcal{A}$, the set of predictable processes such that the above is a martingale.

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Remark : The bounded variation part of S is useless : optimal adverse player control should turn S in a martingale.

□ **Reverse direction :** Assume that $\{\theta^{\vartheta}, \vartheta \in \mathcal{V}\}$ takes finitely many values and that

$$\underline{(Z_{t,z}^{\mathfrak{u}[\vartheta],\vartheta}(\theta^{\vartheta}), M_{t,m}^{\alpha^{\vartheta}}(\theta^{\vartheta})) \in \Lambda(\theta^{\vartheta})\ \mathbb{P}-\text{a.s. } \forall\ \vartheta \in \mathcal{V}.}$$

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Then

$$K(\theta^\vartheta, Z_{t,z}^{\mathfrak{u}[\vartheta],\vartheta}(\theta^\vartheta)) \geq M_{t,m}^{\alpha^\vartheta}(\theta^\vartheta).$$

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Then

$$K(\theta^\vartheta, Z^{\mathfrak{u}[\vartheta],\vartheta}_{t,z}(\theta^\vartheta)) \geq M^{\alpha^\vartheta}_{t,m}(\theta^\vartheta).$$

Play again with balls + concatenation of strategies (assuming smoothness) to obtain $\bar{\mathfrak{u}} \in \mathfrak{U}$ such that

$$\mathbb{E}\left[\ell\left(Z^{(\mathfrak{u}\oplus_{\theta^\vartheta}\bar{\mathfrak{u}})[\vartheta],\vartheta}_{t,z}(T)\right)|\mathcal{F}_\theta\right] \geq M^{\alpha^\vartheta}_{t,m}(\theta^\vartheta) - \varepsilon \; \mathbb{P}-\text{a.s.} \; \forall \; \vartheta \in \mathcal{V}.$$

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By taking expectation

$$\mathbb{E}\left[\ell\left(Z_{t,z}^{(\mathfrak{u}\oplus_{\theta^\vartheta}\bar{\mathfrak{u}})[\vartheta],\vartheta}(T)\right)|\mathcal{F}_t\right] \geq m - \varepsilon\ \mathbb{P}-\text{a.s. } \forall\ \vartheta \in \mathcal{V}$$

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so that

$$(z, m-\varepsilon) \in \Lambda(t).$$

□ To cover the general case by approximations, we need to start with

$$(Z_{t,z}^{\mathfrak{u}[\vartheta],\vartheta}(\theta^\vartheta), M_{t,m}^{\alpha^\vartheta}(\theta^\vartheta)) \in \mathring{\Lambda}_\iota(\theta^\vartheta)\ \mathbb{P}-\text{a.s.}\ \forall\ \vartheta \in \mathcal{V},$$

where

$$\mathring{\Lambda}_\iota(t) := \left\{(z,p) :\ (t',z',p') \in B_\iota(t,z,p) \text{ implies } (z',p') \in \Lambda(t')\right\}.$$

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□ To concatenate while keeping the non-anticipative feature :

- Martingale strategies : α^ϑ replaced by $\mathfrak{a} : \vartheta \in \mathcal{V} \mapsto \mathfrak{a}[\vartheta] \in \mathcal{A}$ in a non-anticipating way (corresponding set $\mathfrak{A}$)
- Non-anticipating stopping times : $\theta[\vartheta]$ (typically first exit time of $(Z^{\mathfrak{u}[\vartheta],\vartheta}), M^{\mathfrak{a}[\vartheta]})$ from a ball.

The geometric dynamic programming principle

(GDP1) : If $(z, m - \iota) \in \Lambda(t)$ for some $\iota > 0$, then $\exists\ \mathfrak{u} \in \mathfrak{U}$ and $\{\alpha^{\vartheta}, \vartheta \in \mathcal{V}\} \subset \mathcal{A}$ such that

$$(Z_{t,z}^{\mathfrak{u}[\vartheta],\vartheta}(\theta), M_{t,m}^{\alpha^{\vartheta}}(\theta)) \in \bar{\Lambda}(\theta)\ \mathbb{P} - \text{a.s. } \forall\ \vartheta \in \mathcal{V}.$$

(GDP2) : If $(\mathfrak{u}, \mathfrak{a}) \in \mathfrak{U} \times \mathfrak{A}$ and $\iota > 0$ are such that

$$(Z_{t,z}^{\mathfrak{u}[\vartheta],\vartheta}(\theta[\vartheta]), M_{t,m}^{\mathfrak{a}[\vartheta]}(\theta[\vartheta])) \in \mathring{\Lambda}_{\iota}(\theta[\vartheta])\ \mathbb{P} - \text{a.s. } \forall\ \vartheta \in \mathcal{V}$$

for some family $(\theta[\vartheta], \vartheta \in \mathcal{V})$ of non-anticipating stopping times, then

$$(z, m - \varepsilon) \in \Lambda(t)\ ,\ \forall\ \varepsilon > 0.$$

Rem : Relaxed version of Soner and Touzi (2002) and B., Elie and Touzi (2009).

Application to the monotone case

□ **Monotone case :** $Z^{\mathfrak{u}[\vartheta],\vartheta}_{t,x,y} = (X^{\mathfrak{u}[\vartheta],\vartheta}_{t,x}, Y^{\mathfrak{u}[\vartheta],\vartheta}_{t,x,y})$ with values in $\mathbb{R}^d \times \mathbb{R}$ with $X^{\mathfrak{u}[\vartheta],\vartheta}_{t,x}$ independent of y and $Y^{\mathfrak{u}[\vartheta],\vartheta}_{t,x,y} \uparrow y$.

□ **Monotone case :** $Z_{t,x,y}^{\mathfrak{u}[\vartheta],\vartheta} = (X_{t,x}^{\mathfrak{u}[\vartheta],\vartheta}, Y_{t,x,y}^{\mathfrak{u}[\vartheta],\vartheta})$ with values in $\mathbb{R}^d \times \mathbb{R}$ with $X_{t,x}^{\mathfrak{u}[\vartheta],\vartheta}$ independent of y and $Y_{t,x,y}^{\mathfrak{u}[\vartheta],\vartheta} \uparrow y$.

□ **The value function is :**

$$\varpi(t,x,m) := \inf\{y : (x,y,m) \in \Lambda(t)\}.$$

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- If φ is usc and $\varphi \geq \varpi$ then

$$\mathring{\Lambda}_\iota(t) \supset \{(x, y, m) : y \geq \varphi(t, x, m) + \eta_\iota\} \text{ for some } \eta_\iota > 0.$$

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(GDP2) : Let $\varphi \in C^0$ be such that $\arg\max(\varpi^* - \varphi) = (t, x, m)$. Assume that $(\mathfrak{u}, \mathfrak{a}) \in \mathfrak{U} \times \mathfrak{A}$ and $\eta > 0$ are such that

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Remark : In the spirit of the weak dynamic programming principle (B. and Touzi, and, B. and Nutz).

PDE characterization - "waving hands" version

- Assuming smoothness, existence of optimal strategies...

- $y = \varpi(t, x, m)$ implies
$Y^{\mathfrak{u}[\vartheta],\vartheta}(t+) \geq \varpi(t+, X^{\mathfrak{u}[\vartheta],\vartheta}(t+), M^{\mathfrak{a}[\vartheta]}(t+))$ for all ϑ.

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Hence, for all ϑ,

$$
\begin{array}{rcl}
\mu_Y(x, y, \mathfrak{u}[\vartheta]_t, \vartheta_t) & \geq & \mathcal{L}_{X,M}^{\mathfrak{u}[\vartheta]_t, \vartheta_t, \mathfrak{a}[\vartheta]_t} \varpi(t, x, m) \\
\sigma_Y(x, y, \mathfrak{u}[\vartheta]_t, \vartheta_t) & = & \sigma_X(x, \mathfrak{u}[\vartheta]_t, \vartheta_t) D_x \varpi(t, x, p) \\
& & + \mathfrak{a}[\vartheta]_t D_m \varpi(t, x, m)
\end{array}
$$

with $y = \varpi(t, x, m)$

PDE characterization - "waving hands" version

- **Supersolution property**

$$\inf_{v \in V} \sup_{(u,a) \in \mathcal{N}^v \varpi} \left(\mu_Y(\cdot, \varpi, u, v) - \mathcal{L}_{X,M}^{u,v,a} \varpi \right) \geq 0$$

where

$$\mathcal{N}^v \varpi := \{(u,a) \in U \times \mathbb{R}^d : \sigma_Y(\cdot, \varpi, u, v) = \sigma_X(\cdot, u, v) D_x \varpi + a D_m \varpi\}.$$

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- Subsolution property

$$\sup_{(u[\cdot],a[\cdot])\in\mathcal{N}^{[\cdot]}\varpi}\ \inf_{v\in V}\Big(\mu_Y(\cdot,\varpi,u[v],v)-\mathcal{L}^{u[v],v,a[v]}_{X,M}\varpi\Big)\leq 0$$

where

$$\mathcal{N}^{[\cdot]}\varpi := \{\text{loc. Lip. }(u[\cdot],a[\cdot])\text{ s.t. }(u[\cdot],a[\cdot])\in\mathcal{N}^{\cdot}\varpi(\cdot)\}.$$

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