Bellman integro-PDE in Hilbert spaces and optimal control of stochastic PDE driven by Lévy type noise

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HJB integro-PDE and control problems

- Does there exist a theory of Hamilton-Jacobi-Bellman integro-PDE in Hilbert spaces?
- Can it be applied to optimal control problems for stochastic PDE driven by Lévy type noise?
- We need a notion of viscosity solution which can give us existence of unique solution with good properties.
  Uniqueness - comparison principle.
  Existence - proof that value functions are viscosity solutions using Dynamic Programming Principle.

HJB integro-PDE

H, U - real, separable Hilbert spaces.
A - linear, maximal monotone operator in H.
We consider parabolic integro-PDE of the form

$$\begin{cases} u_t - \langle Ax, Du\rangle + F(x, Du, D^2u, u(t,\cdot)) = 0, & \text{in } (0,T)\times H, \\ u(T,x) = g(x), \end{cases}$$

where for $(x,p,X,v) \in H \times H \times \mathcal{S}(H) \times UC_{\mathrm{b}}^2(H)$,

$$F(x,p,X,v) = \inf_{a\in\Lambda} \Big\{ \frac{1}{2}\mathrm{Tr}\left(\Sigma(x,a)X\right) + \langle b(x,a), p\rangle + f(x,a) + \int_U \left(v(x+\gamma(x,a,y)) - v(x) - \langle \gamma(x,a,y), Dv(x)\rangle \mathbf{1}_{\{\|y\|<1\}} \right) \nu(dy) \Big\}$$

$\Sigma(x,a) = (\sigma(x,a)Q^{\frac{1}{2}})(\sigma(x,a)Q^{\frac{1}{2}})^*$, $\mathcal{S}(H)$ - bounded, self-adjoint operators on H. $\langle\cdot,\cdot\rangle$ - inner product in H.

Integro-PDE

Theory in finite dimensional spaces: Mikulyavichyus, Pragarauskas, Soner, Sayah, Barles, Garroni, Menaldi, Robin, Buckdahn, Pardoux, Bensoussan, P. L. Lions, Gimbert, Cardaliaguet, Ley, Imbert, Ciomaga, Alibaud, Monteillet, Amadori, Alvarez, Tourin, Karlsen, Jakobsen, La Chioma, Arisawa, Monteillet, Chasseigne, Bensaoud, Ishikawa, Pham, Caffarelli, Silvestre, Kassmann, Bass, Abels, Priola, Schwab, Guillen....
Classical and viscosity approaches, connection with optimal control of jump diffusions, regularity theory, ...

Theory in infinite dimensional spaces: Viscosity solutions theory for equations without non-local terms: Crandall-P. L. Lions, Ishii, Swiech, Kelome, Gozzi, Rouy, Kocan, Sritharan, Nisio, ...
Other approaches (linear and semi-linear equations): Regular and mild solutions, L^2-approach, approach through backward-forward SPDE.

NO RESULTS FOR INTEGRO-PDE IN INFINITE DIMENSIONAL SPACES.

Viscosity solutions of integro-PDE

Need solutions continuous in weaker topology which allows to produce maxima and minima (by perturbed optimization) with appropriate test functions. Based on notion of so called B-continuous viscosity solutions introduced by Crandall-Lions for 1st order equations and later generalized to 2nd order equations by Swiech.

B - bounded, positive, self-adjoint operator in H such that A^*B is bounded and

$$A^*B + cB \geq 0 \quad \text{for some } c \geq 0.$$

$H_{-1} :=$ completion of H with respect to the norm

$$\|x\|_{-1} = \|B^{\frac{1}{2}}x\|.$$

A function $u : (0,T) \times H \to \mathbb{R}$ is called B-continuous if u is continuous in $|\cdot| \times \|\cdot\|_{-1}$ norm on bounded subsets of $(0,T) \times H$.

B-continuous viscosity solutions

- Notion of solution based on the typical idea of “differentiation by parts”.
- Solutions are B-continuous.
- Test functions:

$$\psi(t,x)=\varphi(t,x)+\delta(t,x)h(\|x\|)$$

- $\varphi,\delta\in UC_{\rm b}^{1,2}$, φ is B-lsc, δ is B-continuous, $A^*D\varphi, A^*D\delta$ are uniformly continuous, $\delta\geq 0$.
- $h\in UC_{\rm b}^2(\mathbb{R})$, is even, $h'(r)\geq 0$ for $r\geq 0$.
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$$\langle Ax, D\psi\rangle \quad \text{is interpreted as} \quad \langle x, A^*D\varphi\rangle+\langle x, A^*D\delta\rangle h(\|x\|).$$

The term $\delta(t,x)\langle Ax, Dh(\|x\|)\rangle$ is dropped since it is nonnegative.

B-continuous viscosity solutions

Definition

A bounded B-upper semicontinuous function $u : (0, T) \times H \to \mathbb{R}$ is a viscosity subsolution if whenever $u - \psi$ has a global maximum at a point (t, x) for a test function $\psi(s, y) = \varphi(s, y) + \delta(s, y)h(\|y\|)$ then

$$\psi_t(t,x) - \langle x, A^* D\varphi(t,x) + h(\|x\|)A^* D\delta(t,x)\rangle \\ +F(x, D\psi(t,x), D^2\psi(t,x), \psi(t,\cdot)) \geq 0.$$

A bounded B-lower semicontinuous function $u : (0, T) \times H \to \mathbb{R}$ is a viscosity supersolution if whenever $u + \psi$ has a global minimum at a point (t, x) for a test functions ψ then

$$-\psi_t(t,x) + \langle x, A^* D\varphi(t,x) + h(\|x\|)A^* D\delta(t,x)\rangle \\ +F(x, -D\psi(t,x), -D^2\psi(t,x), -\psi(t,\cdot)) \leq 0.$$

v. solution = v. subsolution & v. supersolution.

Assumptions

- ρ is such that $\inf_{\{\|y\|>r\}} \rho(y) > 0$ for every $r > 0$ and

$$\int_U ((\rho(z))^2 \mathbf{1}_{\{\|z\|<1\}} + \mathbf{1}_{\{\|z\|\geq 1\}})\nu(dz) < +\infty.$$

- $b(\cdot, a), \sigma(\cdot, a)$ are uniformly Lipschitz continuous in $\|\cdot\|_{-1}$ norm. $f(\cdot, a), g$ are uniformly continuous in $\|\cdot\|_{-1}$ norm.
-

$$\|\gamma(x, a, z) - \gamma(y, a, z)\| \leq C\rho(z)\|x - y\|_{-1}$$

$$\|\gamma(0, a, z)\| \leq C\rho(z)$$

- The control set Λ is compact.

Comparison principle

Theorem

Let $u \in BUC([0,T] \times H_{-1})$ be a viscosity subsolution and $v \in BUC([0,T] \times H_{-1})$ be a viscosity supersolution. Suppose that $u(T,x) \leq v(T,x)$ for all $x \in H$. Then $u \leq v$.

Comparison principle can also be proved for HJB equations with zero order term (discounting) and for unbounded sub- and supersolutions. One then has to impose more conditions on the integrability of ν. For instance $\int_U (\rho(z))^2 \nu(dz) < +\infty$.

Existence of viscosity solutions: Value function for the associated optimal control problem is a viscosity solution.

Comparison principle

How to prove comparison principle?

u - viscosity subsolution, v - viscosity supersolution,

Typical idea: Cut-off, doubling and penalization.

$$u(t,x) - v(s,y) - \frac{\|x-y\|_{-1}^2}{\epsilon} - \delta(\|x\|^2 + \|y\|^2) - \frac{(t-s)^2}{\beta}$$

- The functions used are not valid test functions. Need localization.
- Need maximum principle for B-upper semicontinuous functions applicable to integro-PDE. Combine finite dimensional non-local maximum principle with reduction technique to finite dimensions of P. L. Lions. If there is no 2nd order PDE terms this problem does not exist. Proof is much easier.

Existence of solutions: Control problem

$T > 0$. For $t \in [0, T]$, the set of admissible controls $\mathcal{U}_t$ is the collection of all 6-tuples $\left(\Omega, \mathcal{F}, \mathcal{F}_s^t, \mathbb{P}, L, a(\cdot)\right)$ such that:

- $(\Omega, \mathcal{F}, \mathbb{P})$ is a complete probability space;
- L is a U-valued ν Lévy process, i.e.

$$\mathbb{E}\left[e^{i\langle u, L(t_2)-L(t_1)\rangle_U}\right] = e^{-(t_2-t_1)\psi(u)},$$

$$\psi(u) = \int_U \left(1 - e^{i\langle u,z\rangle_U} + \mathbf{1}_{\{\|z\|_U<1\}} i\langle u, z\rangle_U\right)\nu(dz),$$

  $L(t) = 0$, $\mathbb{P}$ a.s., and $\mathcal{F}_s^t$ is the filtration generated by L, augmented by the $\mathbb{P}$-null sets in $\mathcal{F}$;
- $a : [t, T] \times \Omega \to \Lambda$, is an $\mathcal{F}_s^t$-predictable processes in a complete separable metric space Λ.

$\left(\Omega, \mathcal{F}, \mathcal{F}_s^t, \mathbb{P}, L\right)$ is called a reference probability space.

Existence of solutions: Control problem

Poisson random measure of jumps of L:

$$\pi([0,t],B) = \sum_{0<s\le t} \mathbf{1}_B(L(s)-L(s-)), \quad B \in \mathcal{B}(U \setminus \{0\}),$$

$$L(s-) = \lim_{t\uparrow s} L(t),$$

Compensated Poisson random measure of jumps:

$$\widehat{\pi}(dt,dz) = \pi(dt,dz) - dt\,\nu(dz).$$

We assume that ν is a Borel measure on $U \setminus \{0\}$ such that

$$\int_U ((\rho(z))^2 \nu(dz) < +\infty.$$

Control problem

State equation:

$$\begin{cases} dX(s) = (-AX(s) + b(X(s), a(s)))ds + \int_{U\setminus\{0\}} \gamma(X(s-), a(s), z)\widehat{\pi}(dt, dz \\ X(t) = x \in H. \end{cases}$$

Minimize cost functional:

$$J(t, x; a(\cdot)) = \mathbb{E}\left\{\int_t^T f(X(s), a(s))ds + g(X(T))\right\}$$

over all admissible controls in $\mathcal{U}_t$. The dynamic programming equation: Integro-PDE Hamilton-Jacobi-Bellman (HJB) equation.

$$\begin{cases} u_t - \langle Ax, Du\rangle + \inf_{a\in\Lambda}\{\langle b(x,a), Du\rangle + f(x,a) \\ + \int_U (u(t, x+\gamma(x,a,z)) - u(t,x) - \langle\gamma(x,a,z), Du(t,x)\rangle)\,\nu(dz)\} \\ u(T, x) = g(x). \end{cases}$$

State equation

Theorem

Let $0 \leq t \leq t_1 < T$, $\left(\Omega, \mathcal{F}, \left\{\mathcal{F}_s^t\right\}_{s \in [t,T]}, \mathbb{P}, L\right)$ *be a reference probability space. Let* $a(\cdot) \in \mathcal{U}_t$*, and let* ξ *be* $\mathcal{F}_{t_1}^t$*-measurable and such that* $\mathbb{E}\|\xi\|^2 < +\infty$*. Then there exists a unique mild solution* $X(\cdot)$ *of the state equation with* $X(t_1) = \xi$*. The solution has cádlág trajectories. We have*

$$\sup_{t_1 \leq s \leq T} \mathbb{E}\left[\|X_1(s) - X_2(s)\|_{-1}^2\right] \leq C_T \|x_1 - x_2\|_{-1}^2,$$

$$\mathbb{E}\left[\sup_{t_1 \leq \tau \leq s} \|X(\tau) - x\|^2\right] \leq \omega_{T,x}(s - t_1)$$

for some modulus $\omega_{T,x}$.

Uniqueness in law

Let $(\Omega, \mathcal{F})$ be a measurable space, $(\Omega_i, \mathcal{F}_i, \mathbb{P}_i), i = 1, 2$ be two probability spaces, and $X_i : [t, T] \times \Omega_i \to \Omega$ be two stochastic processes, and let $D \subset [t, T]$. We say that $X_1(\cdot)$ and $X_2(\cdot)$ have the same laws on D if they have the same finite dimensional distributions on D, i.e. if for any $t \le t_1 < t_2 < ... < t_n \le T, t_i \in D$, and $A \in \mathcal{F} \times \mathcal{F} \times ... \times \mathcal{F}$ (n-fold product),

$$\mathbb{P}_1 \{(X_1(t_1), ..X_1(t_n))(\omega_1) \in A\} = \mathbb{P}_2 \{(X_2(t_1), ..X_2(t_n))(\omega_2) \in A\} .$$

Theorem

Let $\mu_i = \left(\Omega_i, \mathcal{F}_i, \mathcal{F}_s^{i,t}, \mathbb{P}_i, L_i\right), i = 1, 2,$ *be two reference probability spaces,* π_i *be the Poisson random measures for* L_i*,* $a_i \in \mathcal{U}_t^{\mu_i}$*, and* $\zeta_i \in L^2(\Omega_i, \mathcal{F}_t^{i,t}, \mathbb{P}_i)$*. Let* $\mathcal{L}_{\mathbb{P}_1}(a_1(\cdot), L_1(\cdot), \zeta_1) = \mathcal{L}_{\mathbb{P}_2}(a_2(\cdot), L_2(\cdot), \zeta_2)$ *on some subset* $D \subset [0, T]$ *of full measure. Denote by* $X_i(\cdot)$ *the unique mild solution in the reference probability space* μ_i *and with* $X_i(t) = \xi_i, i = 1, 2.$ *Then* $\mathcal{L}_{\mathbb{P}_1}(X_1(\cdot), a_1(\cdot)) = \mathcal{L}_{\mathbb{P}_2}(X_2(\cdot), a_2(\cdot))$ *on* D*.*

Uniqueness in law

The proof is based on the lemma below and the fact that mild solutions are obtained as fixed points of a contraction map so they are limits of iterations of this map that preserve uniqueness in law.

Lemma

Let $\mu_i = \left(\Omega_i, \mathcal{F}_i, \mathcal{F}_s^{i,t}, \mathbb{P}_i, L_i\right), i = 1, 2$*, be two reference probability spaces. Let* π_i*, be the Poisson random measures for* L_i*. Let* $\Phi_i \colon [t, T] \times \Omega_i \times U \to H, i = 1, 2$ *be two* $\mathcal{F}_s^{i,t}$*-predictable fields such that* $\Phi_i \in L^2_{\mu_i, T}$*. Let* $(\tilde{\Omega}, \tilde{\mathcal{F}})$ *be a measurable space and* $\xi_i \colon \Omega_i \to \tilde{\Omega}, i = 1, 2$ *be two random variables. Assume that, for some subset* $D \subset [t, T]$ *of full measure,*

$$(\Phi_i(\cdot), L_i(\cdot), \xi_i), \quad i = 1, 2$$

have the same finite dimensional distributions on D*. Then on* $[t, T]$

$$\mathcal{L}_{\mathbb{P}_1}\left(\int_t^{\cdot} \int_U \Phi_1(s, z)\widehat{\pi}_1(ds, dz), \xi_1\right) = \mathcal{L}_{\mathbb{P}_2}\left(\int_t^{\cdot} \int_U \Phi_2(s, z)\widehat{\pi}_2(ds, dz), \xi_2\right)$$

Value function

Value function:

$$V(t,x) = \inf_{a(\cdot)\in\mathcal{U}_t} J(t,x;a(\cdot)).$$

- $J(t,\cdot;a(\cdot))$ is uniformly continuous in the $\|\cdot\|_{-1}$ norm, uniformly for $t\in[0,T]$ and $a(\cdot)\in\mathcal{U}_t$.
- V is continuous and $V(t,\cdot)$ is uniformly continuous in the $\|\cdot\|_{-1}$ norm, uniformly for $t\in[0,T]$.
- V is independent of the choice of the reference probability space, i.e. for every reference probability space μ,

$$V(t,x) = V^{\mu}(t,x) := \inf_{a(\cdot)\in\mathcal{U}_t^{\mu}} J(t,x;a(\cdot)).$$

The last property is a consequence of the following lemma.

Value function

Canonical reference probability space on $[t, T]$:
$\mu_L := (D_U[t,T], \mathcal{F}_*, \mathbb{P}_*, \mathcal{B}_s^t, \mathcal{L})$, where $D_U[t,T]$ is the space of U valued cádlág functions, $\mathbb{P}_*$ is the measure on $(D_U[t,T], \mathcal{B}(D_U[t,T]))$ such that the mapping $\mathcal{L}(s)(\omega) = \omega(s)$ is a ν $\mathcal{B}_s^t$-Lévy process in U, $\mathcal{F}_*$ is the completion of $\mathcal{B}(D_U[t,T])$, and for $s \in [t,T]$, $\mathcal{B}_s^{t,0} = \sigma(\mathcal{L}(\tau) : t \le \tau \le s)$, $\mathcal{B}_s^t = \sigma\left(\mathcal{B}_s^{t,0}, \mathcal{N}^*\right)$, where $\mathcal{N}^*$ are the $\mathbb{P}_*$-null sets.

Lemma

Let $a(\cdot) = \big(\Omega, \mathcal{F}, \mathcal{F}_s^t, \mathbb{P}, L, a(\cdot)\big) \in \mathcal{U}_t$ *be* $\mathcal{F}_s^{t,0}$*-predictable. Then there exists a* $\mathcal{P}_{[t,T]}^{D_U[t,T]}/\mathcal{B}(\Lambda)$*-measurable function* $F : [t,T] \times D_U[t,T] \to \Lambda$ *such that*

$$a(s,\omega) = F(s, L(\cdot,\omega)), \quad \text{for } \omega \in \Omega, \ s \in [t,T].$$

$\mathcal{P}_{[t,T]}^{D_U[t,T]}$ is the sigma field of $\mathcal{B}_s^{t,0}$-predictable sets.

DPP

Uniqueness in law and the previous lemma are key results in proving the Dynamic Programming Principle.

Theorem (Dynamic Programming Principle)

Let $0 \leq t < \eta \leq T, x \in H$*, and denote, for* $a(\cdot) \in \mathcal{U}_t$*,*
$X(s) := X(s; t, x, a(\cdot)),\ s \in [t, T]$*. Then*

$$V(t, x) = \inf_{a(\cdot) \in \mathcal{U}_t} \mathbb{E} \left[\int_t^\eta f(X(s), a(s))\, ds + V(\eta, X(\eta)) \right].$$

DPP is also true if η is replaced by a stopping time for every $a(\cdot) \in \mathcal{U}_t$.

Theorem

The value function V *is a viscosity solution of the HJB equation.*

Example

Stochastic wave equation:

$$\begin{cases} \frac{\partial^2 u}{\partial t^2}(t,\xi) = \Delta u(t,\xi) + f(u(t,\xi)) + \frac{\partial}{\partial t}\tilde{L}(t,\xi), & t>0,\ \xi\in\mathcal{O}, \\ u(t,\xi) = 0, & t>0,\ \xi\in\partial\mathcal{O}, \\ u(0,\xi) = u_0(\xi), & \xi\in\mathcal{O}, \\ \frac{\partial u}{\partial t}(0,\xi) = v_0(\xi), & \xi\in\mathcal{O}, \end{cases}$$

with $\tilde{L}$, $L^2(\mathcal{O})$ valued square integrable ν Lévy process in $L^2(\mathcal{O})$, $\mathcal{O}$ a bounded regular domain in $\mathbb{R}^d$, $f:\mathbb{R}\to\mathbb{R}$ a Lipschitz function and $u_0\in H_0^1(\mathcal{O})$, $v_0\in L^2(\mathcal{O})$.
Setting

$$X(t) = \begin{pmatrix} u(t) \\ v(t) \end{pmatrix},\ t\geq 0,$$

we can rewrite this equation in an abstract way:

Example

$$dX(t) = \left(\begin{pmatrix} 0 & I \\ \Delta & 0 \end{pmatrix} X(t) + F(X(t)) \right) dt + dL(t),$$

where

$$F\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ F_1(u) \end{pmatrix}, \qquad L(t) = \begin{pmatrix} 0 \\ \tilde{L}(t) \end{pmatrix}$$

with $D(-\Delta) = H^2(\mathcal{O}) \cap H_0^1(\mathcal{O})$. Then

$$A = \begin{pmatrix} 0 & -I \\ -\Delta & 0 \end{pmatrix}, \qquad D(A) = \begin{pmatrix} H^2(\mathcal{O}) \cap H_0^1(\mathcal{O}) \\ \times \\ H_0^1(\mathcal{O}) \end{pmatrix}$$

is maximal monotone in Hilbert space $H = \begin{pmatrix} H_0^1(\mathcal{O}) \\ \times \\ L^2(\mathcal{O}) \end{pmatrix}$, equipped with

Example

inner product

$$\left\langle \begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} \bar{u} \\ \bar{v} \end{pmatrix} \right\rangle_H = \left\langle (-\Delta)^{1/2} u, (-\Delta)^{1/2} \bar{u} \right\rangle_{L^2(\mathcal{O})} + \langle v, \bar{v} \rangle_{L^2(\mathcal{O})}.$$

Moreover, $A^* = -A$. The operator

$$B = \begin{pmatrix} (-\Delta)^{-1/2} & 0 \\ 0 & (-\Delta)^{-1/2} \end{pmatrix}$$

is bounded, positive, self-adjoint on H, $A^* B$ is bounded,

$$\left\langle (A^* + I) B \begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} u \\ v \end{pmatrix} \right\rangle_H = \|(-\Delta)^{1/4} u\|^2 + \|(-\Delta)^{-1/4} v\|^2 \geq 0,$$

$$\left\| \begin{pmatrix} u \\ v \end{pmatrix} \right\|_{-1} = \left(\|(-\Delta)^{1/4} u\|^2 + \|(-\Delta)^{-1/4} v\|^2 \right)^{1/2}.$$

Example

Thus $F = \binom{0}{F_1}$ is Lipschitz from H_{-1} into H (required condition) if and only if

$$\|F_1(u) - F_1(\bar{u})\|_H \leq c\|(-\Delta)^{1/4}(u - \bar{u})\|, \qquad u, \bar{u} \in H_0^1(\mathcal{O}).$$

Here

$$F_1(u)(\xi) = f(u(\xi)), \qquad \xi \in \mathcal{O}.$$

The condition is satisfied if f is a Lipschitz function.

HJMM model

Heath-Jarrow-Morton (HJM) model in Musiela parametrization (HJMM):

• The bond price at moment t with maturity $t+\xi$, denoted by $P(t,\xi)$, is determined by the so called forward rates $r(t,\xi)$, $\xi \geq 0$, by the formula

$$P(t,\xi) = e^{-\int_0^\xi r(t,\eta)\,d\eta}, \qquad t \geq 0, \quad \xi \geq 0.$$

• The rates evolve according to stochastic evolution

$$dr(s,\xi) = \left(\frac{\partial r}{\partial \xi}(s,\xi) + F(s,\xi)\right) ds + \sigma(s,\xi)\,dL(s),$$

L is a Lévy process with Laplace exponent $-J$ (i.e. $\mathbb{E}\left[e^{-zL(t)}\right] = e^{tJ(z)}$), where $J(z) = \int_{\mathbb{R}}(e^{-zu} - 1 + zu)\,\nu(du), \qquad z \in \mathbb{R}$.

• Arbitrage free condition implies that discounted price process should be a local martingale which implies that

$$F(s,\xi) = \frac{\partial}{\partial \xi} J\left(\int_0^\xi \sigma(s,\eta)d\eta\right) = J'\left(\int_0^\xi \sigma(s,\eta)d\eta\right)\sigma(s,\xi).$$

HJMM model

• If a bond derivative pays at a fixed time T the amount $g(r(T))$ then its price at $t < T$, given by the so called non-arbitrage pricing, should be

$$v(t,x) = \mathbb{E}\left[e^{-\int_t^T r^+(s,0)ds} g(r(T))\right].$$

The price is a functional of the rate $x(\xi)$, $\xi \geq 0$, observed at the moment t of signing the contract. Thus $r(\cdot,\cdot)$ is a solution of state equation with the initial condition $r(t,\xi) = x(\xi)$, Since solutions may not be nonnegative, we take r^+ above.

• Suppose that one expects that volatilities belong to a set Λ of functions of $\xi \geq 0$. Therefore, the so called *super-price*, favorable for the seller of the derivative, could be defined as the largest of the above expressions over all processes $\sigma(s)$, $s \in [t,T]$ taking values in Λ. If we think of the processes $\sigma(\cdot)$ as controls, this is the value function $V(t,x)$ of a control problem.

Black-Scholes-Barenblatt equation

HJB (Black-Scholes-Barenblatt) equation for the problem:

$$\begin{cases} V_t(t,x) + \langle Ax, DV(t,x)\rangle - V(t,x)x^+(0) + \sup_{\sigma\in\Lambda}\Big[\langle b(\sigma), DV(t,x)\rangle \\ \quad + \int_{\mathbb{R}} [V(t,x+z\sigma) - V(t,x) - z\langle\sigma, DV(t,x)\rangle]\nu(dz)\Big] = 0, \\ V(T,x) = g(x), \qquad\qquad (t,x)\in(0,T)\times H. \end{cases}$$

- $Ax = -\frac{\partial}{\partial\xi}x$.
- $b(\sigma)$ is given by the formula

$$b(\sigma)(\xi) = J'\left(\int_0^\xi \sigma(\eta)\,d\eta\right)\sigma(\xi), \qquad \xi \geq 0,$$

where

$$J'(z) = \int_{R^1} u(1-e^{-zu})\,\nu(du).$$

Black-Scholes-Barenblatt equation

• Take $H = H^{1,\gamma}[0, +\infty), \gamma > 0$, the space of absolutely continuous functions x for which

$$\|x\|^2_{H^{1,\gamma}} = \int_0^{+\infty} e^{\gamma\xi}[(x(\xi))^2 + (x'(\xi))^2]\,d\xi < +\infty.$$

• Λ is a compact subset of $H^{1,\gamma}_+[0, +\infty)$.
• ν is concentrated on $[0, +\infty)$ and $\int_0^{+\infty} z^2\,\nu(dz) < +\infty$.
• Take $B = ((\lambda I + A)(\lambda I + A^*))^{-1/2}$, where $\lambda > 0$.
• g is uniformly continuous in the $\|\cdot\|_{-1}$ norm on bounded sets of H.
For instance for a European swaption on a swap with cash-flows $C_i, i = 1, ..., n$, at times $T < T_1 < ... < T_n$,

$$g(x) = \big(K - \sum_{i=1}^{n} C_i e^{\int_0^{T_i - T} x(\xi)d\xi}\big)^+$$

for some $K > 0$. This g is weakly sequentially continuous and thus satisfies the above condition.

Black-Scholes-Barenblatt equation

Theorem

The drift b and the diffusion coefficient $\gamma(x, \sigma, z) := z\sigma$, satisfy the assumptions for existence and uniqueness of viscosity solutions with $\rho(z) = z$. Thus the value function V is the unique viscosity solution of the non-local Black-Scholes-Barenblatt (BSB) equation.

• One can also consider an obstacle problem for the non-local BSB equation and the associated optimal stopping problem. It would correspond to pricing of American options.

OTHER APPLICATIONS OF V. SOLUTIONS OF INTEGRO-PDE: Large deviations for solutions of stochastic PDE with small Lévy noise. Use integro-PDE to prove existence of the so called Laplace limits at single times by showing that viscosity solutions of singularly perturbed equations converge to the viscosity solution of the limiting HJB equation. Then follow the program developed by Feng and Kurtz.