Three-dimensional Brownian motion and its applications to trading

Alexander Lipton

Bank of America & University of Oxford

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de la Vega's Quote

"The Dutch call the option business "opsies," a term derived from the Latin word optio, which means choice, because the payer of the premium has the choice of delivering the shares to the acceptor of the premium or demanding them from him."
de la Vega, Joseph (1688) Confusion de Confusiones, Amsterdam.

What is Mathematical Finance?

Mathematical Finance is one of several competing scientific disciplines aimed at understanding the behavior of financial markets.
There are other disciplines aiming to achieve similar goals, such as

- Economics;
- Theory of Finance;
- Econometrics;
- Statistics;
- Econo-physics;
- Actuarial science;
- Etc.

Practitioners of each of these disciplines claim to possess a unique set of tools and a special angle to deal with financial markets. There is a lot of friendly (and not so friendly) competition among them.

Mathematical Finance covers a large variety of financial instruments

Important asset classes and topics of interest:

- Equities;
- Non-risky debt;
- Risky debt;
- Forex;
- Commodities;
- Derivatives written on the above assets;
- Structure of financial exchanges and market intermediation (including market making);
- Asset and liability management and theory of investing;
- Risk management;
- Capital calculations, etc.

Constituent parts

- Investors;
- Pension funds;
- Asset managers;
- Banks;
- Insurance companies;
- Hedge funds;
- Various supporting players, such as custodians, credit rating agencies, etc.

Purposes of investing

- Wealth preservation;
- Wealth creation;
- Life-time wealth transfer;
- Intergenerational wealth transfer.

"The importance of money essentially flows from it being the link between the present and the future." John Maynard Keynes

Mathematical models

- Stochastic nature of financial time series and their non-stationarity;
- Different time scales of modern finance: from milliseconds to decades;
- Different intellectual sources for model building: probability theory, numerical methods, statistics, economics (the weakest link);
- Main difficulty: "fallacy of the historical data series";
- Main purposes of the models: investing, trading, risk management, speculation;
- Important point: risk-neutral probability vs. real-world probability pricing of derivatives.

Brownian motion (BM)

- 1D BM W_t on a positive semi-axis;
- 2D BM $\left(W_t^1, W_t^2\right)$, $dW^1 dW^2 = \rho^{12} dt$, in a positive quadrant;
- 3D BM $\left(W_t^1, W_t^2, W_t^3\right)$, $dW^i dW^j = \rho^{ij} dt$, in a positive octant.
- Our strategy is to construct the corresponding Green's function and apply it to solve the relevant financial math problems.
- In 1D case it is very simple, in 2D case it is not too complicated, in 3D case it is very difficult.

One-dimensional case, Green's function on positive semi-axis

Green's function can be constructed via the method of images:

$$G(\tau, y_0, y') = \frac{1}{\sqrt{2\pi\tau}} \left(e^{-\frac{(y'-y_0)^2}{2\tau}} - e^{-\frac{(y'+y_0)^2}{2\tau}} \right).$$

Alternatively, Green's function can be constructed via the eigenfunction expansion method:

$$G(\tau, y_0, y') = \frac{1}{\sqrt{2\pi}} \int_0^\infty \sin(ky_0) \sin(ky) \, e^{-\frac{k^2\tau}{2}} dk.$$

Needless to say that these expressions are in agreement with each other.

One-dimensional case, Green's function on a final interval

Finite interval $0 \leq y \leq L$

Green's function can be constructed via the method of images:

$$G(\tau, y_0, y') = \frac{1}{\sqrt{2\pi\tau}} \sum_{n=-\infty}^{\infty} \left(e^{-\frac{(y'-y_0+2nL)^2}{2\tau}} - e^{-\frac{(y'+y_0+2nL)^2}{2\tau}} \right).$$

Alternatively, Green's function can be constructed via the eigenfunction expansion method:

$$G(\tau, y_0, y') = \frac{2}{L} \sum_{n=1}^{\infty} \sin\left(\frac{2\pi n}{L} y_0\right) \sin\left(\frac{2\pi n}{L} y'\right).$$

Needless to say that these expressions are in agreement with each other.

Close relation to the Poisson summation formula!

Two-dimensional case

General pricing equation for the value function $V(t,x,y)$ in the positive quadrant:

$$V_t + \frac{1}{2}V_{xx} + \frac{1}{2}V_{yy} + \rho_{xy}\ V_{xy} - \varrho V = 0.$$

It is supplied with appropriate boundary and terminal conditions at $x=0$, $y=0$, $t=T$.

Changes of independent and dependent variables

$$\begin{gathered} U(t,x,y) = e^{\varrho(T-t)}V(t,x,y), \\ \alpha = x, \quad \beta = (-\rho_{xy}x + y)/\bar{\rho}_{xy}, \\ \alpha = r\sin\varphi, \quad \beta = r\cos\varphi, \\ \bar{\rho}_{xy} = \sqrt{1-\rho_{xy}^2}. \end{gathered}$$

Final form of the pricing equation:

$$U_t + \frac{1}{2}\left(U_{rr} + \frac{1}{r}U_r + \frac{1}{r^2}U_{\varphi\varphi}\right) = 0.$$

The new domain is an angle $0 \le \varphi \le \varpi$, where $\varpi = \arccos\left(-\rho_{xy}\right)$.

Green's function via the eigenfunction expansion method

Green's function solves the forward equation:

$$G_\tau - \frac{1}{2}\left(G_{r'r'} + \frac{1}{r'}G_{r'} + \frac{1}{r'^2}G_{\varphi'\varphi'}\right) = 0.$$

Initial condition:

$$G(0, r', \varphi') = \frac{1}{r_0}\delta(r' - r_0)\delta(\varphi' - \varphi_0).$$

Boundary conditions:

$$G\left(\tau, r', 0\right) = 0, \quad G(\tau, r', \varpi) = 0, \quad G(\tau, 0, \varphi') = 0, \quad G(\tau, r', \varphi') \underset{r'\to\infty}{\longrightarrow} 0.$$

Solution obtained through the eigenfunction expansion method:

$$G\left(\tau, r_0, r', \varphi_0, \varphi'\right) = \frac{2e^{-\frac{r'^2+r_0^2}{2\tau}}}{\varpi\tau}\sum_{n=1}^{\infty} I_{\nu_n}\left(\frac{r'r_0}{\tau}\right)\sin\left(\nu_n\varphi'\right)\sin\left(\nu_n\varphi_0\right),$$

where $\nu_n = \frac{n\pi}{\varpi}$.

Green's function via the method of images

Define $\psi = \phi' - \phi_0$, $s_\pm = sign\left(\pi \pm \psi\right)$, and $f\left(p, q\right)$, $h\left(p, q\right)$, $p \geq 0$, $-\infty < q < \infty$,

$$f\left(p, q\right) = 1 - \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-p(\cosh(2q\zeta) - \cos(q))}}{\zeta^2 + \frac{1}{4}} d\zeta,$$

$$h\left(p, q\right) = \frac{1}{2}\left[s_+ f\left(p, \pi + q\right) + s_- f\left(p, \pi - q\right)\right].$$

We can define a *non-periodic* solution of the heat equation as follows

$$H\left(\tau, r_0, r', \varphi_0, \varphi'\right) = \frac{1}{2\pi\tau} e^{-\frac{r'^2 + r_0^2 - 2\cos(\psi) r' r_0}{2\tau}} h\left(\frac{r' r_0}{\tau}, \psi\right),$$

and obtain Green's function through the method of images:

$$G\left(\tau, r_0, r', \varphi_0, \varphi'\right) = \sum_{n=-\infty}^{\infty} \left[\begin{array}{c} H\left(\tau, r_0, r', \varphi_0 + 2n\varpi, \varphi'\right) \\ -H\left(\tau, r_0, r', -\varphi_0 + 2n\varpi, \varphi'\right) \end{array}\right].$$

Three-dimensional case

General pricing equation for the value function $V(t, x, y)$ in the positive quadrant:

$$V_t + \frac{1}{2}V_{xx} + \frac{1}{2}V_{yy} + \frac{1}{2}V_{zz} + \rho_{xy} \ V_{xy} + \rho_{xz} \ V_{xz} + \rho_{yz} \ V_{yz} - \varrho V = 0.$$

It is supplied with appropriate boundary and terminal conditions at $x = 0$, $y = 0$, $z = 0$, $t = T$.

Changes of independent and dependent variables (Cholesky decomposition)

$$\begin{gathered} U(t, x, y, z) = e^{\varrho(T-t)} V(t, x, y, z), \\ \alpha = x, \quad \beta = (-\rho_{xy} x + y)/\bar{\rho}_{xy}, \\ \gamma = \left(\left(\rho_{xy}\rho_{yz} - \rho_{xz} \right) x + \left(\rho_{xy}\rho_{xz} - \rho_{yz} \right) y + \bar{\rho}_{xy}^2 z \right) / \bar{\rho}_{xy}\chi, \\ \alpha = r \sin\varphi \sin\theta, \quad \beta = r\cos\varphi\sin\theta, \quad \gamma = r\cos\theta, \\ \bar{\rho}_{xy} = \sqrt{1 - \rho_{xy}^2}, \quad \chi = \sqrt{1 - \rho_{xy}^2 - \rho_{xz}^2 - \rho_{yz}^2 + 2\rho_{xy}\rho_{xz}\rho_{yz}}. \end{gathered}$$

Three-dimensional pricing problem

Pricing problem:

$$U_t + \frac{1}{2}\left[\frac{1}{r}\frac{\partial^2}{\partial r^2}(rU) + \frac{1}{r^2}\left(\frac{1}{\sin^2\theta}U_{\varphi\varphi} + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta U_\theta)\right)\right] = 0,$$

Computational domain (defined parametrically as function of ω):

$$\begin{gathered} r > 0, \\ 0 \leq \varphi \leq \varpi, \\ \varphi(\omega) = \arccos\left(\frac{1-\rho_{xy}\omega}{\sqrt{1-2\rho_{xy}\omega+\omega^2}}\right), \\ \theta(\omega) = \arccos\left(\frac{(\rho_{xy}\rho_{xz}-\rho_{yz})+(\rho_{xy}\rho_{yz}-\rho_{xz})\omega}{\sqrt{\bar{\rho}_{xy}\left(\bar{\rho}_{xz}^2+2\omega(\rho_{xz}\rho_{yz}-\rho_{xy})\omega+\bar{\rho}_{yz}^2\omega^2\right)}}\right), \\ \theta = \Theta(\varphi). \end{gathered}$$

Spherical domain, source Lipton & Savescu

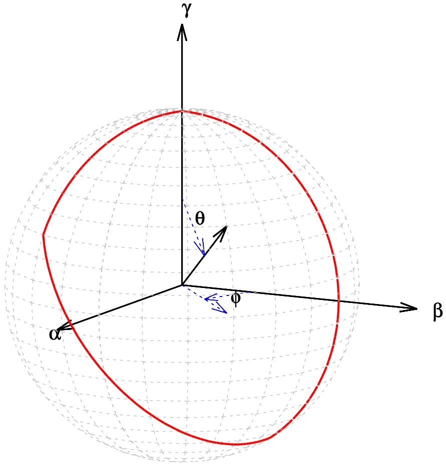


Figure:

Three-dimensional Green's function

Green's function satisfies the forward equation:

$$G_\tau - \frac{1}{2}\left[\frac{1}{r'}\frac{\partial^2}{\partial r'^2}\left(r'G\right) + \frac{1}{r'^2}\left(\frac{1}{\sin^2\theta'}G_{\varphi\varphi'} + \frac{1}{\sin\theta'}\frac{\partial}{\partial\theta'}\left(\sin\theta' G_{\theta'}\right)\right)\right] = 0,$$

with initial condition:

$$G(0, r', \varphi', \theta') = \frac{1}{r_0^2 \sin\theta_0}\delta\left(r' - r_0\right)\delta\left(\varphi' - \varphi_0\right)\delta\left(\theta' - \theta_0\right),$$

and boundary conditions:

$$G\left(\tau, r', 0, \theta'\right) = G(\tau, r', \varpi, \theta') = G\left(\tau, r', \varphi', 0\right) = 0,$$
$$G(\tau, r', \varphi', \Theta(\varphi')) = G(\tau, 0, \varphi', \theta') = 0, \quad G(\tau, r', \varphi', \theta') \underset{r'\to\infty}{\longrightarrow} 0.$$

Eigenfunction expansion for Green's function

- Construct Green's function by using eigenfunction expansion method.
- Separation of variables yields $G(\tau, r', \varphi', \theta') = g(\tau, r') \Psi(\varphi', \theta')$.
- Similar to 2D case, the radial part has the form

$$g(\tau, r') = \frac{e^{-\frac{r'^2 + r_0^2}{2\tau}}}{\tau\sqrt{r' r_0}} I_{\sqrt{\Lambda^2 + 1/4}}\left(\frac{r' r_0}{\tau}\right).$$

- The angular part solves the following 2D eigenvalue problem:

$$\frac{1}{\sin^2\theta'}\Psi_{\varphi'\varphi'} + \frac{1}{\sin\theta'}\frac{\partial}{\partial\theta'}\left(\sin\theta'\Psi_{\theta'}\right) = -\Lambda^2\Psi,$$
$$\Psi(0, \theta') = 0, \quad \Psi(\varpi, \theta') = 0, \quad \Psi(\varphi', 0) = 0, \quad \Psi(\varphi', \Theta(\varphi')) = 0.$$

Solution of 2D eigenvalue problem

- The corresponding eigenvalue problem is solved via the finite element method (FEM);
- The spherical domain is mapped onto the domain Ω in the (φ, θ) plane;
- The variational (weak) formulation of the problem is used:

$$\int_{\Omega} \frac{1}{\sin\theta'} \Psi_{\varphi'} \bar{\Psi}_{\varphi'} d\Omega + \int_{\Omega} \sin\theta' \Psi_{\theta'} \bar{\Psi}_{\theta'} d\Omega = \Lambda^2 \int_{\Omega} \sin\theta' \Psi \bar{\Psi} d\Omega.$$

- The domain Ω is triangulated by using iterative algorithm to construct adaptive mesh;
- After each iteration the Delaunay triangulation method is used.

Planar domain, source Lipton & Savescu

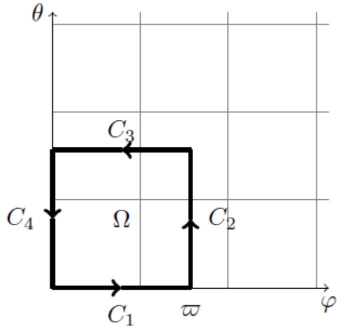


Figure 8: $\rho_{xy} = 0, \quad \rho_{xz} = 0, \quad \rho_{yz} = 0$

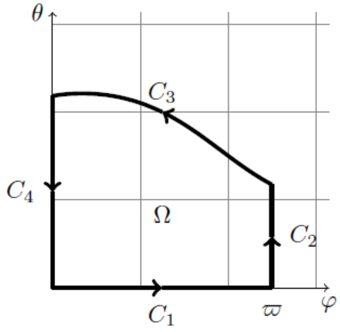
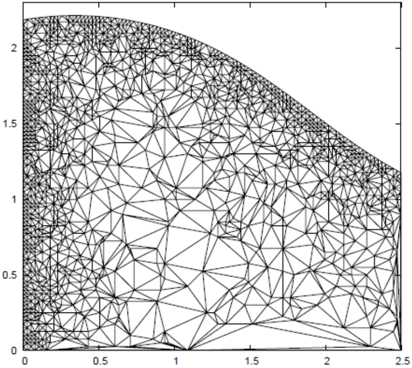


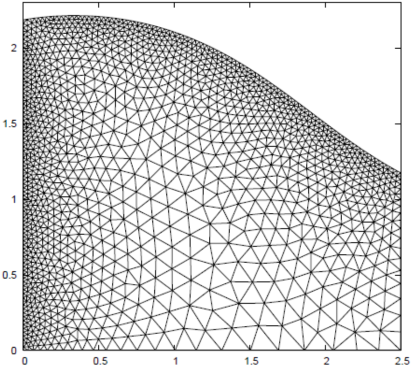
Figure 9: $\rho_{xy} = 0.8, \quad \rho_{xz} = 0.2, \quad \rho_{yz} = 0.5$

Figure:

Mesh, source Lipton & Savescu



(a) First iteration mesh



(b) Mesh after 100 iterations

Figure:

Details of FEM

- Dimension of the space in which eigenvectors are searched for is equal to the number of free points in the mesh;
- Basis in this space is represented by basis functions Φ_i, $1 \leq i \leq n$, linear in each triangle;
- The variational problem is approximated by a linear system

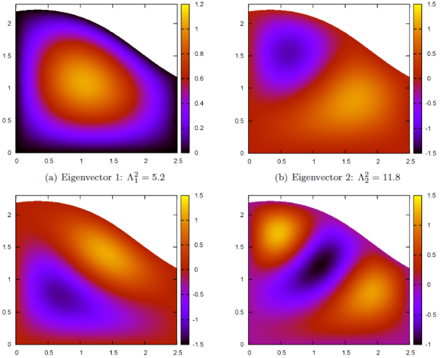
$$K\Psi = \Lambda^2 M\Psi.$$

- Here $K = (K_{ij})$ is the stiffness matrix, $M = (M_{ij})$ is the mass matrix:

$$\begin{gathered} K_{ij} = \int_\Omega \nabla\Phi_i \left(A\nabla\Phi_j\right) d\Omega, \\ M_{ij} = \int_\Omega \Phi_i\Phi_j \sin\theta' d\Omega, \\ A = \begin{pmatrix} \frac{1}{\sin\theta'} & 0 \\ 0 & \sin\theta' \end{pmatrix}. \end{gathered}$$

Eigenvectors, source Lipton & Savescu,

$\rho_{xy} = 80\%, \rho_{xz} = 20\%, \rho_{yz} = 50\%$



(a) Eigenvector 1: $\Lambda_1^2 = 5.2$
(b) Eigenvector 2: $\Lambda_2^2 = 11.8$
(c) Eigenvector 3: $\Lambda_3^2 = 16.3$
(d) Eigenvector 4: $\Lambda_4^2 = 21.3$

Figure:

Green's function

Eigenfunction expansion for Green's function:

$$G(\tau, r', \varphi', \theta') = \sum_{n=1}^{\infty} C_n g_n(\tau, r') \Psi_n(\varphi', \theta').$$

Coefficients C_n - can be computed by imposing the initial condition:

$$G(0, r', \varphi', \theta') = \frac{1}{r_0^2 \sin\theta_0} \delta(r' - r_0)\delta(\varphi' - \varphi_0)\delta(\theta' - \theta_0),$$

and we obtain $C_n = \Psi_n(\varphi_0, \theta_0)$.
Final formula for Green's function:

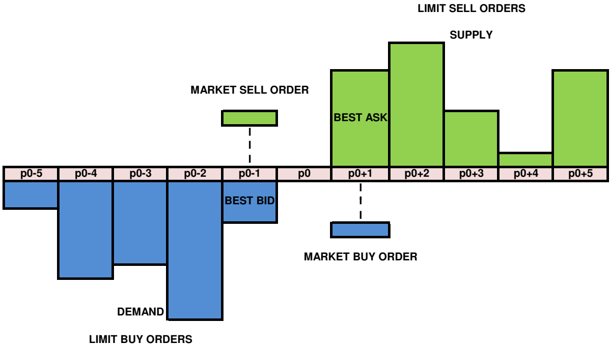
$$G\left(\tau, r_0, r', \varphi_0, \varphi', \theta_0, \theta'\right) = \frac{e^{-\frac{r'^2 + r_0^2}{2\tau}}}{\tau\sqrt{r' r_0}} \sum_{n=1}^{\infty} I_{\sqrt{\Lambda_n^2 + \frac{1}{4}}} \left(\frac{r' r_0}{\tau}\right) \Psi_n(\varphi_0, \theta_0) \Psi_n(\varphi', \theta'$$

HFT in retrospective

Historically, "HFT" technologies were opposed to: (A) Racing pigeons; (B) Telegraph; (C) Telephone; (D) Radio; (E) Screen trading; etc.

- Yet, various HFT strategies persisted in spite of these objections.
  Here is an interesting example.
  In 15th century Florence state built galleys to send goods to London and Bruges.
  These were leased to the highest bidder, who, in turn, subleased them to others.
  An auction lasted until a candle burned out. Since everyone tried to put their bid as late as possible, rules were changed.
  Now an auction would end with the clock on the tower of Palazzo della Signoria, which was audible but not visible.

LOB


LIMIT SELL ORDERS
SUPPLY
MARKET SELL ORDER
BEST ASK
p0-5
p0-4
p0-3
p0-2
p0-1
p0
p0+1
p0+2
p0+3
p0+4
p0+5
BEST BID
MARKET BUY ORDER
DEMAND
LIMIT BUY ORDERS

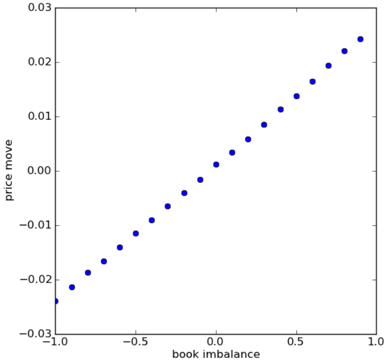
HF market participants

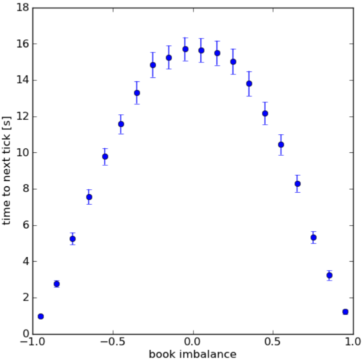
Algorithmic traders in high frequency (HF) electronic markets can be roughly divided into the following categories:

- Market makers, who provide liquidity and try to capture bid-ask spread;
- Systematic traders and arbitrageurs, who try to profit from price dislocations and statistical relationships among different prices;
- Agency brokers, who execute large trades for clients and earn fees.

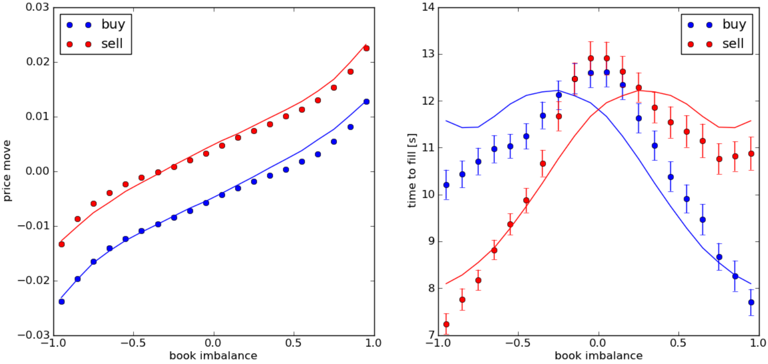
We are interested in the agency broker point of view.

LOB Imbalance - I, source Lipton, Pesavento, Sotiropoulos





LOB Imbalance - II, source Lipton, Pesavento, Sotiropoulos



A simple model

We model the number of shares q^b and q^a posted at a simple stochastic process in a positive quadrant:

$$\left(dq^b, dq^a\right) = \left(dW^b, dW^a\right),$$

where W^b, W^a are correlated Brownian motions.
To capture the joint dynamics of the bid and ask queues and trade arrival, we introduce another stochastic process to model the arrival of trades on the near side of the book:

$$\left(dq^b, dq^a, d\phi\right) = \left(dW^b, dW^a, dW^\phi\right).$$

In what follows, we follow a recent paper "Trade arrival dynamics and quote imbalance in a limit order book" by A. Lipton, U. Pesavento, M. Sotiropoulos. References to prior art are mentioned in the end.

2D-Problem formulation and solution

- By using Cholesky-style transformations, we can write the exit probability problem on the computational interval $[0, \varpi]$ in the form

$$P_{\phi\phi}(\phi) = 0,$$

$$P(0) = 0, \qquad P(\varpi) = 1.$$

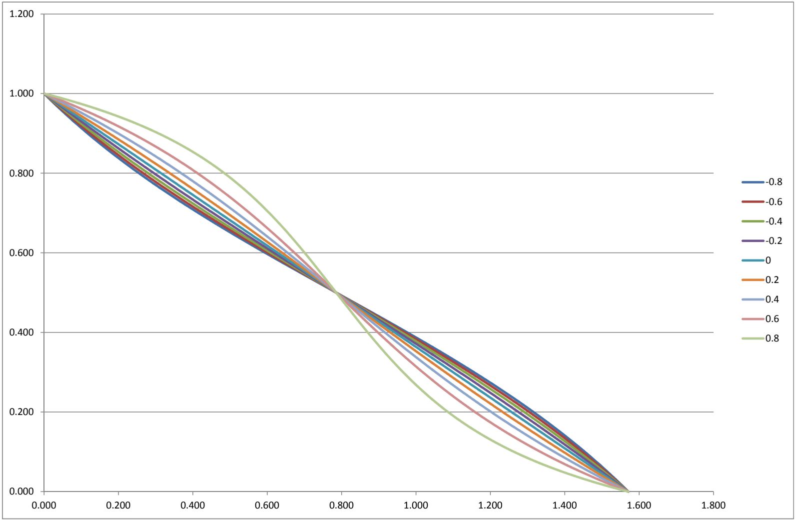
- Its solution is straightforward

$$P(\phi) = \frac{\phi}{\varpi}.$$

- When expressed in the original (x, y) coordinates, this probability has the form

$$P(x, y) = \frac{1}{2}\left(1 - \frac{\arctan\left(\sqrt{\frac{1+\rho}{1-\rho}}\frac{y-x}{y+x}\right)}{\arctan\left(\sqrt{\frac{1+\rho}{1-\rho}}\right)}\right).$$

2D-Hitting Probability



3D-Problem formulation

- We wish to solve the following classical problem - compute the exit probability for 3D Wiener process in a positive octant with absorbing boundaries. To the best of our knowledge our solution is new.
- By using Cholesky-style transformations, we can write the exit probability problem in the computational domain Ω in the form

$$\frac{1}{\sin^2\theta} P_{\phi\phi}(\phi,\theta) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta P_\theta(\phi,\theta)\right) = 0,$$

$$P(0,\theta) = 0, \qquad P(\varpi,\theta) = 0, \qquad P(\phi,\Theta(\phi)) = 1.$$

- Introduce new variable $\zeta = \ln\left(\tan\left(\frac{\theta}{2}\right)\right)$ and rewrite the exit problem as follows

$$P_{\phi\phi}(\phi,\zeta) + P_{\zeta\zeta}(\phi,\zeta) = 0,$$

$$P(0,\zeta) = 0, \qquad P(\varpi,\zeta) = 0, \qquad P(\phi, Z(\phi)) = 1.$$

- Computational domain is now a semi-infinite strip with curvilinear boundary

$$\zeta = Z(\phi) = \ln\left(\tan\left(\frac{\Theta(\phi)}{2}\right)\right).$$

3D-Problem solution - I

- We look for the solution of the Dirichlet problem for the Laplace equation in the form

$$P(\phi, \zeta) = \sum_{n=1}^{\infty} c_n \sin(k_n \phi) e^{k_n \zeta}, \qquad k_n = \frac{\pi n}{\varpi}.$$

- It is clear that each individual term is a harmonic function. We choose coefficients c_n in such a way that $P(\phi, Z(\phi)) = 1$. Specifically,

$$\sum_{n=1}^{\infty} c_n \sin(k_n \phi) e^{k_n Z(\phi)} = 1.$$

- Thus, we need to build a theory of Fourier series expansion with respect to the following set of (non-orthonormal) basis functions

$$E_n(\phi) = \sin(k_n \phi) e^{k_n Z(\phi)},$$

$$\sum_{n=1}^{\infty} c_n E_n(\phi) = 1.$$

3D-Problem solution - II

- Introduce the following integrals

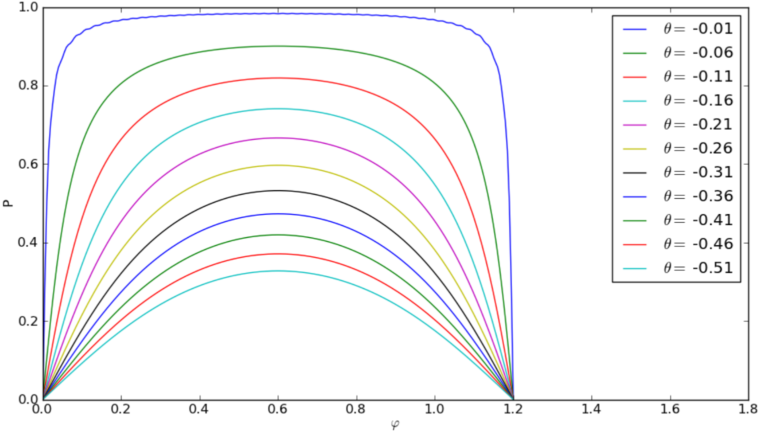
$$\begin{aligned} J_{mn} &= \int_0^{\varpi} \sin(k_m\phi)\sin(k_n\phi)\, e^{(k_m+k_n)Z(\phi)} d\phi, \\ I_m &= \int_0^{\varpi} \sin(k_m\phi)\, e^{k_m Z(\phi)} d\phi. \end{aligned}$$

- It is clear that

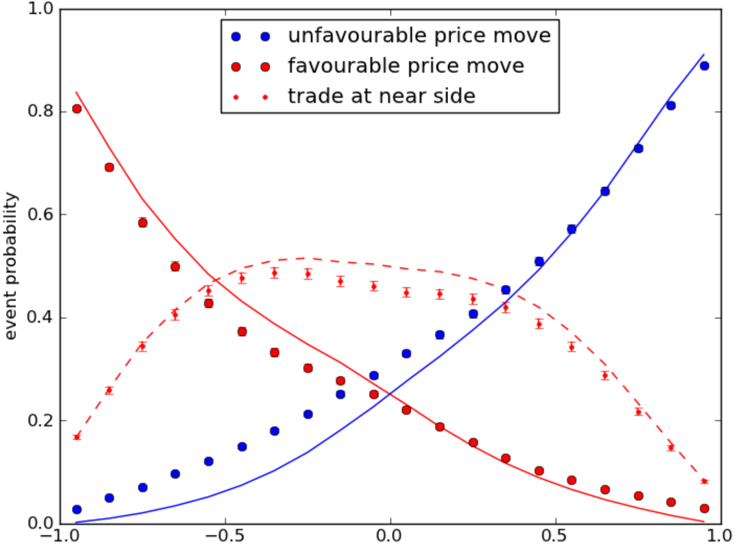
$$\begin{gathered} \sum_n J_{mn} c_n = I_m, \\ \overrightarrow{c} = \widehat{J}^{-1} \overrightarrow{I}. \end{gathered}$$

- In general, this matrix problem is difficult to invert on the boundary $\zeta = Z(\phi)$, however, away from the boundary automatic regularization kicks in, and everything works very well.
- When the boundary is (approximately) linear, the corresponding integrals can be found analytically.
- As a by-product, we managed to find a new(?) solution of a long-standing problem of exit probabilities for three correlated Brownian motions in a positive octant.

3D-Hitting probability, source Lipton, Pesavento, Sotiropoulos



Calibration, source Lipton, Pesavento, Sotiropoulos



Conclusions

- Mathematical finance is a thriving discipline which poses extremely intricate and important questions.
- It requires a diverse skill set and ability to apply both sophisticated and simple tools in an appropriate fashion.
- Its future belongs to people who are willing to analyzeh very big data, and are able to extract some sense out of a seemingly never ending time-series of disjoint data points.
- Successful practitioners and academic should be prepared to deal with extraordinary large data sets, industrial strength problems of mind-boggling complexity, and heavy computational burdens.
- The time of closed-form solutions is (sadly) passed once and for all, or is it?
- Come join the party!!

Bibliography - I

M. Avellaneda and J. Zhu. Distance to default. Risk, 14(12):125 129, 2001.

F. Black and J. Cox. Valuing corporate securities: some effects of bond indenture provisions. Journal of Finance, 31:351 367, 1976.

Ch. Blanchet-Scaillet and F. Patras. Structural Counterparty Risk Valuation for Credit Default Swaps. In T. Bielecki, D. Brigo, and F. Patras, editors, Credit Risk Frontiers: Subprime Crisis, Pricing and Hedging, CVA, MBS, Ratings, and Liquidity. Wiley, 2011.

J. Cariboni and W. Schoutens. Pricing credit default swaps under Lévy models. Journal of Computational Finance, 10(4):71, 2007.

H. He, W.P. Keirstead, and J. Rebholz. Double lookbacks. Mathematical Finance, 8(3): 201 228, 1998.

B. Hilberink and L.C.G. Rogers. Optimal capital structure and endogenous default. Finance and Stochastics, 6(2):237 263, 2002.

J. Hull and A. White. Valuing credit default swaps II: modeling default correlations. Journal of Derivatives, 8(3):12 22, 2001.

T. Hyer, A. Lipton, D. Pugachevsky, and S. Qui. A hidden-variable model for risky bonds. Technical report, working paper, 1999.

Bibliography - II

J. Kim, K. Ramaswamy, and S. Sundaresan. Does default risk in coupons affect the valuation of corporate bonds? A contingent claim model. Financial Management, (22):117 131, 1993.

H. Leland and K. Toft. Optimal capital structure, endogenous bankruptcy, and the term structure of credit spreads. Journal of Finance, (51):987 1019, 1996.

A. Lipton. Mathematical Methods for Foreign Exchange: a Financial Engineer's Approach. World Scientific, 2001.

A. Lipton. Assets with jumps. Risk, 15(9):149 153, 2002.

A. Lipton and A. Sepp. Credit value adjustment for credit default swaps via the structural default model. The Journal of Credit Risk, 5(2):123 146, 2009.

F. Longsta and E. Schwartz. A simple approach to valuing risky fixed and floating rate debt. Journal of Finance, (50):789 819, 1995.

R.C. Merton. On the pricing of corporate debt: The risk structure of interest rates. The Journal of Finance, 29(2):449 470, 1974.

F. Patras. A refection principle for correlated defaults. Stochastic processes and their applications, 116:690 698, 2006.

Bibliography - III

A. Sepp. Analytical pricing of double-barrier options under a double-exponential jump diffusion process: applications of Laplace transform. International Journal of Theoretical and Applied Finance, 7(2):151 175, 2004.

M. Valuzis. On the probabilities of correlated defaults: a first passage time approach. Nonlinear Analysis: Modelling and Control, 13(1):117 133, 2008.

C. Zhou. An Analysis of Default Correlations and Multiple Defaults. The Review of Financial Studies, 14(2):555 576, 2001a.

C. Zhou. The term structure of credit spreads with jump risk. Journal of Banking & Finance, 25(11):2015 2040, 2001b.

Bibliography - IV

T. Parks. Medici Money. Norton, 2005.

R. Almgren, C. Thum, H. L. Hauptmann, and H. Li. Equity market impact. Risk, 18:57, 2005.

M. Avellaneda and S. Stoikov. High-frequency trading in a limit order book. Quantitative Finance, 8:217–224, 2008.

E. Bacry and J.F. Muzy. Hawkes model for price and trades high-frequency dynamics. SIAM Journal of Financial Mathematics, submitted, 2013.

J.-P. Bouchaud, J. D. Farmer, and F. Lillo. How markets slowly digest changes in supply and demand. In T. Hens and K Schenk-Hoppe, editors, Handbook of Financial Markets: Dynamics and Evolution.

J.-P. Bouchaud, D. Mezard, and M. Potters. Statistical properties of stock order books: empirical results and models. Quantitative Finance, 2:251–256, 2002.

R. Cont and A. de Larrard. Order book dynamics in liquid markets: limit theorems and diffusion approximations. Working paper, 2012.

R. F. Engle. The econometrics of ultra-high frequency data. Econometrica, 68:1–22, 2000.

Bibliography - V

J. Hasbrouck. Measuring the information content of stock trades. Journal of Finance, 46:179–207, 1991.

A. Lipton. A structural approach to pricing credit derivatives with counterparty adjustments: additional thoughts. Global Derivatives, Trading and Risk Management, Amsterdam, 2013.

A. Lipton and I. Savescu. CDSs, CVA and DVA - a structural approach. Risk, 26(4), 2013.

S. Stoikov R. Cont and R. Talreja. A stochastic model for order book dynamics. Operations research, 58(3):549–563, 2010.

E. Smith, J.D. Farmer, L. Gillemot, and S. Krishnamurthy. Statistical theory of the continuous double auction. Quantitative Finance, 3:481–514, 2003.

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