Changes of the filtration and the default event risk premium

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April 22 2013
– *Math Finance Colloquium*–
USC

Change of the probability measure

• The change of the probability measure is the cornerstone of the Arbitrage Pricing Theory.

• It permits to use the martingale theory for pricing financial instruments.

• $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ filtered probability space.

In a financial market model:

- $\mathbb{P}$ represents the subjective probability measure.
- $(\mathcal{F}_t)$ represents the information flow available to market investors to evaluate contingent claims.
- Typically, this is the natural filtration of a vector of price processes (locally bounded semi-martingales) $S = (S_t)_{t\geq 0}$, with $S := (S^0, \ldots, S^n)$.
- S^0 stands for the locally risk-free asset (i.e., a safe bank account); the remaining assets are risky.

- **Absence of arbitrage opportunities**: by betting on the process S, it may not be possible to obtain a gain out of nothing and without bearing any risk.
- **Fundamental Theorem of Asset Pricing**: The existence of an equivalent martingale measure $\mathbb{Q}$ is *essentially* equivalent to absence of arbitrage opportunities. (*Delbaen-Schachermayer-1994*).

$$\frac{S}{S^0} = \underbrace{A}_{\text{predictable,finite variation}} + \underbrace{M}_{\text{local martingale under } \mathbb{P}}$$
$$= \underbrace{\tilde{M}}_{\text{local martingale under } \mathbb{Q}}$$

- A: term reflecting the risk aversion of investors (risk premium)

Aim of the talk:

We want to show that a different technique, the change of the underlying filtration, provides a new characterization of risk premiums attached to particular events (such as the default event of a firm).

We shall present a selection of results from:

- *From the decompositions of a stopping time to risk premium decompositions*, NCCR Working Paper 615.
- *Hazard processes and martingale hazard processes* (with A. Nikeghbali), Mathematical Finance.

• Pricing derivative products (i.e., any $\mathcal{F}_\infty$-measurable random variable, X) consists in computing:

$$X_t := \mathbb{E}^{\mathbb{Q}}[X|\mathcal{F}_t] \quad \mathbb{Q}: \text{ equivalent martingale measure.}$$

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• Hence, it is implicitly assumed that among all public information available to investors, $(\mathcal{F}_t)$ contains all pertinent information to use, that is, if $(\mathcal{F}_t) \subset (\mathcal{G}_t)$:

$$\boxed{\mathbb{E}^{\mathbb{Q}}[X|\mathcal{G}_t] = \mathbb{E}^{\mathbb{Q}}[X|\mathcal{F}_t] \quad \forall\, X\ \mathcal{F}_\infty\text{-measurable random variable.}} \tag{1}$$

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• In financial economics, this property is known as *semi-strong market efficiency* (see Fama, 1970).

• In probability theory, this property is known as the *immersion property*:

Theorem (Dellacherie-Meyer, 1978)

Let $(\mathcal{F}_t) \subset (\mathcal{G}_t)$ be two filtrations. Then, condition (1) is equivalent to:

(H) *Every $(\mathcal{F}_t)$-martingale is a $(\mathcal{G}_t)$-martingale.*

Change of the filtration

- The change of the filtration:
  - useful when $(\mathcal{F}_t)$ does not contain all the relevant information.
  - in this case the filtration $(\mathcal{F}_t)$ needs to be enlarged to incorporate more events.
- The **theory of the enlargements of a filtration** can be effectively used for pricing defaultable claims.
  - This theory was developed mainly by T. Jeulin and M. Yor in the 70s.

Outline of the talk:

The remaining of the talk is structured as follows:

1. We give a short overview of the needed tools from the theory of enlargements of filtrations.
2. We show a useful application to the decompositions of a default time.
3. We provide an application to pricing of default able claims and compute default event risk premiums.

Previous research:

- Direct approach (without enlargement of filtrations): Duffie, Schroder, Skiadas (1996)
- Using enlargement of filtrations: Elliott, Jeanblanc, Yor (2000)

Random times and enlargements of filtrations

$(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ will always denote a filtered probability space satisfying the usual conditions.
From now on: $\mathbb{P}$ is supposed to be a martingale measure.

Definition

A ***random time*** τ *is a nonnegative random variable* $\tau : (\Omega, \mathcal{F}) \to [0, \infty]$.

The progressively enlarged filtration:

$$\mathcal{G}_t :\equiv \mathcal{F}_t \vee \sigma\{\tau \wedge s, s \leq t\}.$$

Key processes

- the $(\mathcal{F}_t)$-supermartingale

$$\boxed{Z_t^\tau = \mathbb{P}[\tau > t \mid \mathcal{F}_t]}$$

- the $(\mathcal{F}_t)$-dual optional (resp. predictable) projection of the process $1_{\{\tau \leq t\}}$, denoted by A_t^τ (resp. a_t^τ);
- the càdlàg martingale

$$\mu_t^\tau = \mathbb{E}[A_\infty^\tau \mid \mathcal{F}_t] = A_t^\tau + Z_t^\tau$$

- The compensator of the process $1_{\tau \leq t}$, given by:

$$\Lambda_{t\wedge\tau} = \int_0^{t\wedge\tau} \frac{da_s^\tau}{Z_{s-}^\tau}$$

A general question:

How are the $(\mathcal{F}_t)$-semimartingales modified when considered as stochastic processes in the larger filtration $(\mathcal{G}_t)$?

Theorem (Jeulin-Yor, 1978)

Every $(\mathcal{F}_t)$ local martingale (M_t), stopped at τ, is a $(\mathcal{G}_t)$-semimartingale, with canonical decomposition:

$$\boxed{M_{t\wedge\tau} = \widetilde{M}_t + \int_0^{t\wedge\tau} \frac{d\langle M, \mu^\tau\rangle_s}{Z^\tau_{s-}}}$$

where $(\widetilde{M}_t)$ is a $(\mathcal{G}_t)$-local martingale.

2 classes of random times will play an important role:

Definition

(1.) τ *is a* $(\mathcal{F}_t)$ ***pseudo-stopping time*** *if for every bounded* $(\mathcal{F}_t)$*-martingale* (M_t) *we have*

$$\mathbf{E}M_\tau = \mathbf{E}M_0. \tag{2}$$

(2.) τ *is a* ***honest time*** *if it is the end of an* $(\mathcal{F}_t)$ *optional set O, i.e:*

$$L = \sup\{t : (t, \omega) \in O\}.$$

General properties of stopping times

$(\Omega, \mathcal{F}, (\mathcal{G}_t)_{t\geq 0}, \mathbb{P})$ be a filtered probability space, usual assumptions.

Classification of stopping times

Let τ be a stopping time.

(i) τ is a predictable stopping time if there exists a sequence of stopping times $(\tau_n)_{n\geq 1}$ such that τ_n is increasing, $\tau_n < \tau$ on $\{\tau > 0\}$ for all n, and $\lim_{n\to\infty} \tau_n = \tau$ *a.s.*.

(ii) τ is an accessible stopping time if there exists a sequence of predictable stopping times $(\tau_n)_{n\geq 1}$, such that:

$$\mathbb{P}(\cup_n \{\omega : \tau(\omega) = \tau_n(\omega) < \infty\}) = 1.$$

(iii) τ is totally inaccessible if, for every predictable stopping time T,

$$\mathbb{P}(\{\omega : \tau(\omega) = T(\omega) < \infty\}) = 0.$$

A motivating example

Example

Let $(B_t)_{t\geq 0}, (\beta_t)_{t\geq 0}$ *be 2 correlated Brownian motions.* ***Market price information*** *strong solution of:*

$$dY_t := \sigma(t, Y_t)d\beta_t + \mu(t, Y_t)dt, \quad Y_0 = y_0.$$

Assets, value of the firm: $X_t := F(t, B_t)$, *F increasing in the second argument.* ***Default time****:*

$$\tau := \inf\{t \geq 0 | X_t \leq b(t)\}.$$

where b is a continuous function of time. The reference filtration is $\mathcal{F}_t := \sigma(\beta_s, s \leq t)$ *and the market filtration is constructed as* $\mathcal{G}_t := \mathcal{F}_t \vee \sigma(\tau \wedge s, s \leq t)$.

Are default time totally inaccessible?

Assume:

(NA) There exist $\mathbb{Q} \sim \mathbb{P}$ such that all traded price processes are local martingales.

(L) Suppose that (X_t^d) is the price of a defaultable claim. Assume that $\Delta X_\tau^d < 0$ *a.s.*, i.e., there is a loss in case of default with probability one.

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Proposition

Under (NA) and (L), the default time τ does not have a predictable part.

Default times, more properties

We now come back to the market model for default able claims with 2 filtrations:

- $(\mathcal{F}_t)$ filtration of the hedging instruments and
- $\mathcal{G}_t = \mathcal{F}_t \vee (\tau \wedge t)$.

We shall assume form now on the immersion property ($(\mathcal{F}_t)$ -martingales remain $(\mathcal{G}_t)$-martingales).

Proposition

Let τ be a finite $(\mathcal{G}_t)$ stopping time. Then, there exists a sequence $(T^i)_{i\geq 1}$, of $(\mathcal{F}_t)$ stopping times, such that

$$\mathbb{P}(T^i = T^j < \infty) = 0 \quad i \neq j$$

and $\mathbb{P}(\tau = T^i) > 0$ whenever $\mathbb{P}(T^i < \infty) > 0$ and a t.i. $(\mathcal{G}_t)$ stopping time T^0 such that T^0 avoids all finite $(\mathcal{F}_t)$ stopping times and such that:

$$\boxed{\tau = \sum_{i\geq 0} T^i 1_{\{T^i=\tau\}}.} \tag{3}$$

The $(\mathcal{G}_t)$ stopping time τ is totally inaccessible if and only if the $(\mathcal{F}_t)$ stopping times $(T^i)_{i\geq 1}$ are totally inaccessible.

Financial interpretation

Definition

Let τ be totally inaccessible. Then, we shall call:

- $T^i, i \in \mathbb{N}^*$ *times of economic shocks,*
- T^0 *is the idiosyncratic default time (i.e. when default is due to the unique and specific circumstances of the company, as opposed to the overall market circumstances).*

Proposition

The Azéma supermartingale of τ *is given by:*

$$\boxed{Z_t^\tau := 1 - \left(\sum_{i\geq 1} p^i_{T^i} 1_{\{T^i\leq t\}} + a^0_t\right),} \qquad (4)$$

and its Doob-Meyer decomposition is: $Z_t^\tau = \left(1 - \hat{N}_t\right) - a_t^\tau$,

$$a_t^\tau := \sum_{i\geq 1}\int_0^t (p^i_{s-} + \upsilon^i_s) d\Lambda^i_{T^i\wedge s} + a^0_s; \quad \hat{N}_t := \mathbb{E}[N_t|\mathcal{F}_t]$$

and for $i \geq 1$,

- υ^i *is a predictable processes that satisfies* $\langle N^i, p^i\rangle = \int \upsilon^i_s d\Lambda^i_{s\wedge T^i}$.
- $p^i_t = \mathbb{P}(\tau = T^i|\mathcal{F}_t)$

τ has an intensity λ if and only if intensities exist for the times T^i, $i \geq 0$. Then, denoting by λ^i the $(\mathcal{G}_t)$ intensity of T^i, the following relation holds:

$$\lambda_t = \sum_{i \geq 1} \left(\frac{p^i_{t-} + \upsilon^i_t}{Z^\tau_t} \right) \lambda^i_t 1_{\{T^i \geq t\}} + \lambda^0_t.$$

Applications to pricing of defaultable claims

Defaultable claims: $\mathcal{G}_T$ measurable random variables ($T > 0$ constant) that have the specific form:

$$X = P1_{\{\tau > T\}} + C_\tau 1_{\{\tau \leq T\}},$$

where:

- P is a positive square integrable, $\mathcal{F}_T$-measurable random variable
- (C_t) is a positive bounded, $(\mathcal{F}_t)$-predictable process.

Denote $R_t = \int_0^t r_u du$, where r_u is the locally risk-fee interest rate (so that $S_t^0 = e^{R_t}$ is the safe bank account).

The arbitrage-free price of a defaultable claim is given by the following conditional expectation:

$$S(X)_t := e^{R_t}\mathbb{E}[Pe^{-R_T}1_{\{\tau>T\}} + C_\tau e^{-R_\tau}1_{\{\tau\leq T\}}|\mathcal{G}_t]1_{\{t\leq T\}}.$$

Pre-default prices can always be expressed in terms of an $(\mathcal{F}_t)$-adapted process, via projections on the smaller filtration $(\mathcal{F}_t)$ as:

$$1_{\{\tau>t\}}S(X)_t = 1_{\{\tau>t\}}\tilde{S}(X)_t$$

where $\tilde{S}(X)$ is $(\mathcal{F}_t)$-adapted, given by:

$$\tilde{S}(X)_t := \frac{e^{R_t}}{Z_t^\tau}\mathbb{E}[Pe^{-R_T}1_{\{\tau>T\}} + C_\tau e^{-R_\tau}1_{\{\tau\in(t,T]\}}|\mathcal{F}_t].$$

We recall below a well known expression of the pre-default price process which holds in a particular case of our framework:

Proposition (Elliott-Jeanblanc-Yor, 2000)

Suppose (Z_t^τ) *continuous, decreasing. Then*

$$\tilde{S}(X)_t = e^{-(R_t+\Lambda_t)}\mathbb{E}\left[\int_t^T C_u e^{-(R_u+\Lambda_u)} d\Lambda_u + P e^{-(R_T+\Lambda_T)} | \mathcal{F}_t\right], \quad t \geq 0. \tag{5}$$

Theorem (C.-Nikeghbali, 2010)

Z^τ *continuous decreasing if and only if* τ *is a pseudo-stopping time that avoids all* $(\mathcal{F}_t)$ *stopping times.*

Definition

Suppose that the pre-default price process $\tilde{S}(X)$ has a Doob Meyer decomposition under the risk neutral measure $\mathbb{Q}$:

$$\tilde{S}(X)_t = \tilde{S}(X)_0 + \int_0^t \tilde{S}(X)_u d\nu(X)_u + M_t$$

where $(\nu(X)_t)$ is a finite variation, predictable process and (M_t) a martingale $M_0 = 0$. We call ***default event risk premium*** *the process*

$$\pi(X)_t = \nu(X)_t - R_t, \quad 0 \leq t \leq T.$$

Notice that $\pi(X)$ represents a compensation for:

- the jump (loss) that will occur in the price of the claim X at the default time τ;
- the change of the martingale property in the enlarged filtration $(\mathcal{G}_t)$.

Example 1: Z^τ continuous decreasing

If Z^τ is continuous decreasing, then, it can be easily checked that:

$$\boxed{\pi(X)_t = \Lambda_t - \int_0^t \tilde{C}_u d\Lambda_u}$$

where $\tilde{C} = C/\tilde{S}(X)$.

Example

Let $(B_t)_{t\geq 0}, (\beta_t)_{t\geq 0}$ be 2 correlated Brownian motions. ***Market price information*** *strong solution of:*

$$dY_t := \sigma(t, Y_t)d\beta_t + \mu(t, Y_t)dt, \quad Y_0 = y_0.$$

Assets, value of the firm: $X_t := F(t, B_t)$, F increasing in the second argument. ***Default time****:*

$$\tau := \inf\{t \geq 0 | X_t \leq b(t)\}.$$

Question: What changes when Z^τ is not continuous decreasing?

We introduce the exponential martingale:

$$D_t := \mathcal{E}\left(\int_0^{\cdot} \frac{dm_s^\tau}{Z_{s-}^\tau}\right)_t$$

Proposition

Suppose that $(D_t)_{0\leq t\leq T}$ is a square integrable martingale, and define the default-adjusted measure as:

$$d\mathbb{Q}^\tau := D_T \cdot d\mathbb{P} \quad on\ \mathcal{F}_T.$$

Then, it follows that the pre-default price of the defaultable claims is:

$$\tilde{S}_t(X) = e^{\tilde{R}_t}\mathbb{E}^{\mathbb{Q}^\tau}\left[\int_t^T C_u e^{-\tilde{R}_u} d\Lambda_u + X e^{-\tilde{R}_T} | \mathcal{F}_t\right] \quad t < T;$$

where $\tilde{R}_t = R_t + \Lambda_t$.

Corollary

Under the above assumptions, there exists a martingale (M_t) under the measure $\mathbb{P}$ such that the pre-default price process satisfies the following SDE:

$$\frac{d\tilde{S}_t(X)_t}{\tilde{S}_t(X)_t} = dR_t + \underbrace{\left(d\Lambda_t - \tilde{C}_t d\Lambda_t - \frac{1}{Z_t} d\langle M, m^\tau\rangle_t\right)}_{\text{default event risk premium}} + dM_t$$

where $\tilde{C}_t = C_t/\tilde{S}_t(X)_t$.

Example 2: Z^τ not predictable

Suppose that:

$$\tau = \begin{cases} T^0, & \mathbb{P}(\tau = T^0 | \mathcal{F}_t) = p \\ T^1, & \mathbb{P}(\tau = T^1 | \mathcal{F}_t) = 1 - p \end{cases}$$

- $T^0 \sim exp(\gamma)$ firm-specific factor;
- $T^1 \sim exp(\alpha)$ macro factor.

Then, the Azéma supermartingale is given by:

$$Z_t^\tau = 1 - (1-p)1_{\{T^1 \le t\}} - p(1 - e^{-\gamma t})$$

and the $(\mathcal{G}_t)$-intensity of τ is stochastic and given by:

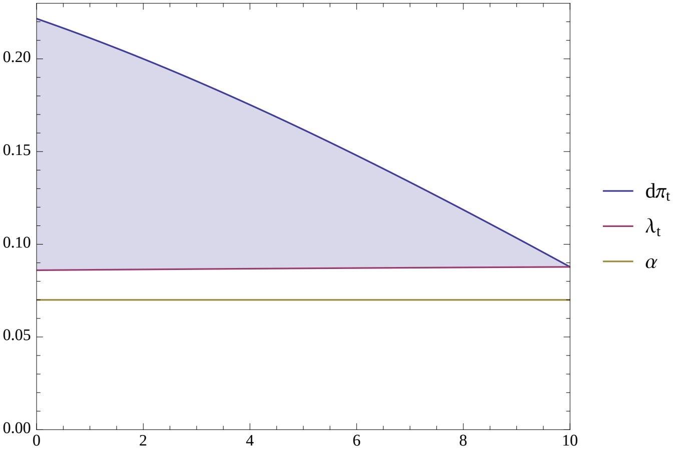
$$\lambda_t = 1_{\{T^1 \ge t\}} \frac{(1-p)\alpha + p\gamma e^{-\gamma t}}{1 - p + pe^{-\gamma t}} + 1_{\{T^1 < t\}} \gamma$$

For a ZCB with zero recovery in case of default (i.e., $X = 1_{\{\tau > T\}}$, we obtain that the default event risk premium is given by:

$$\pi(X)_t = \int_0^t (\lambda_u + h(u) 1_{\{T^1 \geq u\}}) du$$

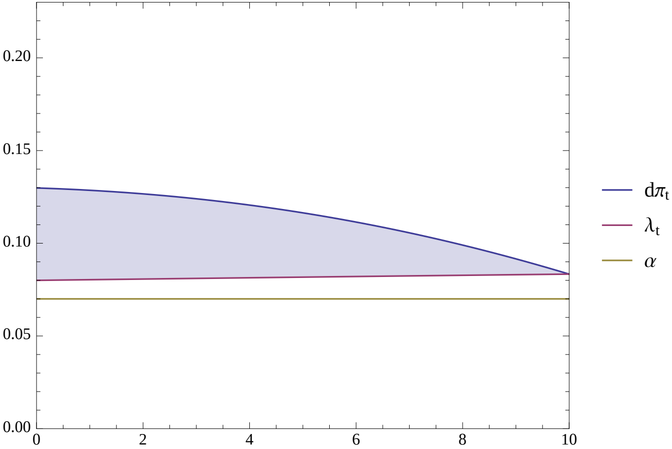
where:

$$h(t) := \frac{1 - e^{-(T-t)(\alpha-\gamma)}}{(1-p)e^{-(T-t)(\alpha-\gamma)} + pe^{-t\gamma}} \frac{1-p}{(1-p) + pe^{-\gamma}}.$$

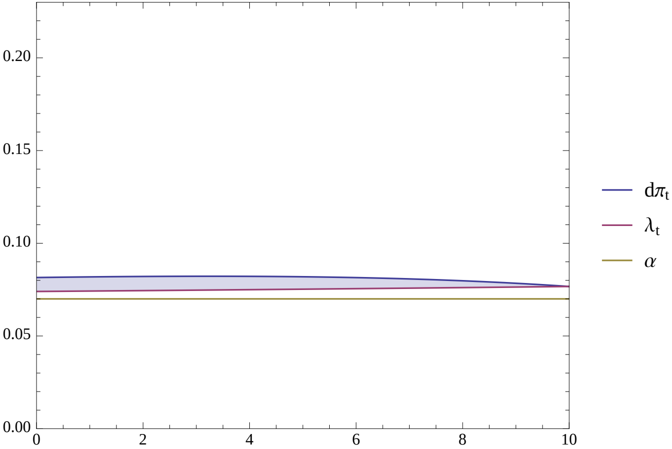


Instantaneous default event risk premium.

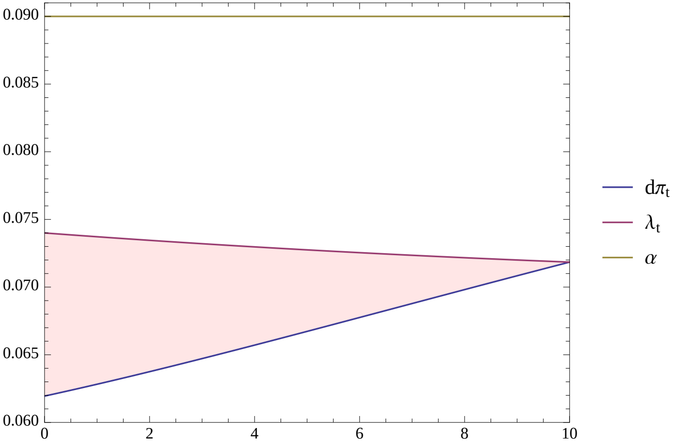
Parameters: $\alpha = 0.09$, $\gamma = 0.07$ and $p = 0.2$.



Instantaneous default event risk premium.
Parameters: $\alpha = 0.09$, $\gamma = 0.07$ and $p = 0.5$.



Instantaneous default event risk premium.
Parameters: $\alpha = 0.09$, $\gamma = 0.07$ and $p = 0.8$.



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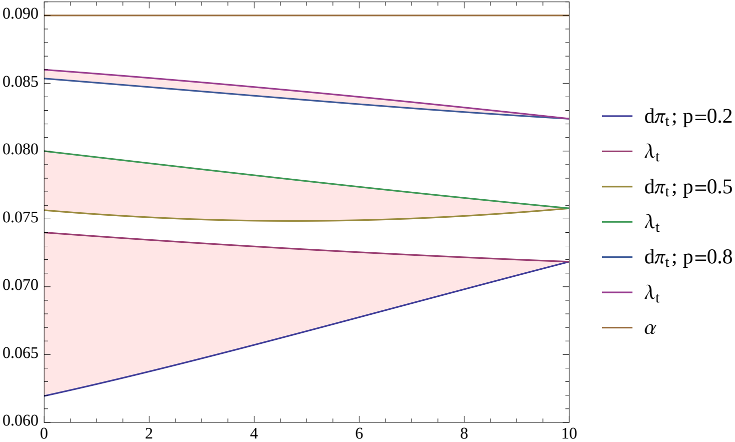

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Conclusions

We aimed at drawing the attention upon the following facts:

- Constructing the default model in two steps using the technique of enlarging the initial filtration is extremely useful if one has to consider such imperfections as:
  - economic shocks or
  - jumps in the collateral values at the default time.
- In this context it is possible to quantify the default event risk premium.
- The assumptions which usually appear in the literature, namely :
  - τ avoids the $(\mathcal{F}_t)$-stopping times;
  - the recovery process (C_t) is predictable.

  have tended to underestimate the default risk.

Thank you for your attention!