Branching Diffusions Representation for Nonlinear PDEs

Nizar Touzi
Ecole Polytechnique, Paris

Joint work with Nadia Oudjane, Pierre Henry-Labordère, Xiaolu Tan and Xavier Warin

University of Southern California
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Objective

Design numerical approximation for the equation :

$$\partial_t v + \mu \cdot Dv + \frac{1}{2}\sigma^2 : D^2 v + F(t, x, v, Dv, D^2 v) = 0, \quad v(T, .) = g$$

• Finite differences, finite elements : very efficient in $1-2$ dim, curse of dimensionality, path dependency increases dimension

• **Probabilistic representation** $\Longrightarrow$ Monte Carlo / Probabilistic numerical methods

• **An important issue :** extension to the path-dependent case ? ?

Intuition from the linear case

The heat equation :

$$\partial_t v + \frac{1}{2}\Delta v = 0, \quad v(T,.) = g$$

has the following two possible probabilistic representations :

(i) $v(0,x) = \mathbb{E}\big[g(B_T)|B_0 = x\big]$; with B a BM

(ii) $v(t,x) = e^{\beta T}\mathbb{E}\big[g(B_T)\mathbb{1}_{\{T<\tau\}}|B_0 = x\big]$; $\tau \sim \text{Exp}(\beta) \perp B$

Both representations are valid in the path-dependent case

From linear representation (i) to nonlinear

Representation (i) extended by

• BSDEs (Pardoux & Peng, Bouchard & NT, Zhang, ...)

$$dv_t = -F_t(v_t, \zeta_t)dt + \zeta_t dB_t, \quad v_T = g(B_T)$$

• 2BSDEs (Cheridito, Soner, NT & Victoire, Fahim, NT & Warin, Zhang & Zhuo, Possamaï & Tan)

$$dv_t = -F_t(v_t, \zeta_t, \gamma_t)dt + \zeta_t dB_t, \quad d\zeta_t = \ldots dt + \gamma_t dB_t, \quad v_T = g(B_T)$$

New formulation : Soner, NT & Zhang, and Possamaï, Tan & Zhou

A probabilistic numerical scheme for fully nonlinear PDEs

X^n : discrete-time approximation of diffusion with drift μ and diffusion $\sigma = 1$ (also $d = 1$ for simplicity)

$$
\begin{aligned}
Y^n_{t_n} &= g\left(X^n_{t_n}\right), \\
Y^n_{t_{i-1}} &= \mathbb{E}^n_{i-1}\left[Y^n_{t_i}\right] + f\left(X^n_{t_{i-1}}, Y^n_{t_{i-1}}, Z^n_{t_{i-1}}, \Gamma^n_{t_{i-1}}\right)\Delta t_i, \quad 1 \le i \le n, \\
Z^n_{t_{i-1}} &= \mathbb{E}^n_{i-1}\left[Y^n_{t_i}\frac{\Delta W_{t_i}}{\Delta t_i}\right] \\
\Gamma^n_{t_{i-1}} &= \mathbb{E}^n_{i-1}\left[Y^n_{t_i}\frac{|\Delta W_{t_i}|^2 - \Delta t_i}{|\Delta t_i|^2}\right]
\end{aligned}
$$

Then $Y^n_0 \longrightarrow v(0,x)$ as $n \to \infty$ + Error estimate...

Automatic differentiation

$\Longrightarrow$ Integration by parts

$$\partial_x \mathbb{E}[\phi(X_t)] \;=\; \mathbb{E}\left[\phi(X_t)\frac{W_h}{h}\right]$$

For simplicity, consider the one-dimensional case $X_t = x + W_t$:

$$\begin{aligned}
\mathbb{E}[\phi_x(x+W_h)] &= \int \phi_x(x+y)\frac{e^{-y^2/(2h)}}{\sqrt{2\pi}}dy \\
&= \int \phi_x(x+y)\frac{y}{h}\frac{e^{-y^2/(2h)}}{\sqrt{2\pi}}dy \\
&= \mathbb{E}\left[\phi(x+W_h)\frac{W_h}{h}\right]
\end{aligned}$$

From linear representation (ii) to nonlinear

- Consider KPP equations

$$\text{(KPP)} \quad \partial_t v + \mu \cdot Dv + \frac{1}{2}\sigma^2 : D^2 v + \beta\Big(\sum_{i=1}^n p_i v^i - v\Big) = 0$$

with $p_i > 0$ and $\sum_{k=1}^n p_i = 1$

- Branching diffusions representation :

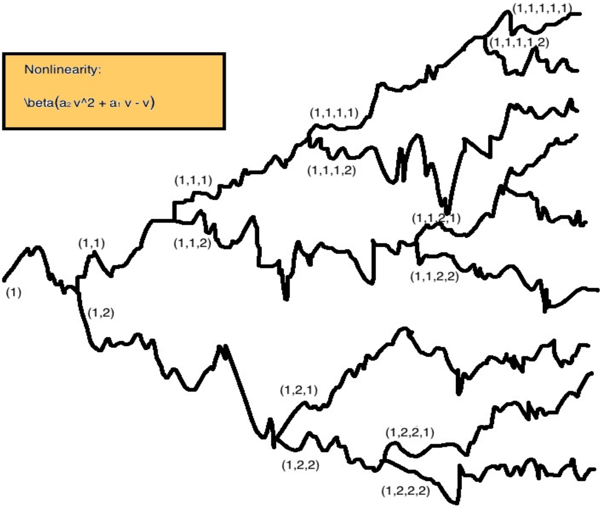
$$v(0,x) \quad = \quad \mathbb{E}\Big[\prod_{k\in\mathcal{K}_T} g(Z_T^k)\Big], \quad \text{where} \quad Z^k : \; k-\text{th particle}$$

and

$$\mathcal{K}_t \quad := \quad \big\{\text{All particles alive at time } t\big\}$$

[Skorokhod, Watanabe, McKean]

Branching diffusion ($n = 2$)



Generalized KPP equation

Let $a_i(t,x)$ be bounded functions, and consider the PDE

$$\partial_t v + \mu(t,x)\cdot Dv + \frac{1}{2}\sigma^2(t,x) : D^2 v + \beta\Big(\sum_{i=1}^{n} p_i a_i(t,x) v^i - v\Big) = 0$$

$$v(T,.) = g$$

Introduce the branching diffusion :

- $(\tau_k)_k$ iid Expo(β) : branching times
- $(I_k)_k$ iid Multinomial($p_1,\ldots,p_n$) : number of decendents
- Particle k dies out at the branching event T_k, and I_k independent particles follow the diffusion with drift and diffusion (μ,σ)

The branching diffusion representation

Recall

- $\mathcal{K}_T :=$ {particles present at T}
- $\overline{\mathcal{K}}_T := \cup_{t \le T} \mathcal{K}_t$: all particles

Theorem (Henry-Labordère, Tan & NT SPA '14)

$v(0,x) = \mathbb{E}\big[\psi_{0,x}\big]$ *where*

$$\psi_{0,x} \quad := \quad \prod_{k \in \mathcal{K}_T} g(Z_T^k) \prod_{k \in \overline{\mathcal{K}}_T \setminus \mathcal{K}_T} a_{I_k}(T_k, Z_{T_k}^k)$$

Moreover, this representation extends to the path-dependent case

- Numerical implications
- In the rest of he talk : extension to more general nonlinearities

Regression versus branching diffusions methods

BSDE representation : (backward) regression-based methods $\Longrightarrow$

- $\oplus$ no explosion restrictions
- $\ominus$ High complexity, curse of dimension is back !
- $\ominus$ Markovian feature is crucial

Branching diffusions $\Longrightarrow$

- $\oplus$ Purely forward Monte Carlo
- $\oplus$ Suitable for path-dependency
- $\oplus$ Very easy to implement, complexity linear in d^2
- $\ominus$ Need to control from explosion of solution
- $\ominus$ and of the variance (in the subsequent extensions)...

Main objective

• Branching diffusion representation for a larger class of PDEs (beyond KPP)

Including nonlinearity in the gradient

• Unbiased simulation / Monte Carlo approximation

Treat both Gradient and Hessian as nonlinearities...

Outline

1. Unbiased simulation of SDEs
   - The constant diffusion case
   - Regime switching and automatic differentiation

2. Age-dependent branching diffusions and semilinear PDEs
   - Complexity of the Monte Carlo approximation

Weak approximation of SDEs

Objective is to approximate **without discretization error** :

$$V_0 := \mathbb{E}\left[g(X_T)\right]$$

where X is solution of the SDE

$$dX_t \quad = \quad \mu(t, X_t)dt + \sigma(t, X_t)dW_t$$

- W is a Brownian motion
- μ and σ satisfy the Lipschitz bounded, σ^{-1} bounded
- more conditions on μ and σ will pop up

Our algorithm in the case of constant diffusion $\sigma = I_d$ (I)

- (N_t) : Poisson process with intensity β, arrival times $(\tau_i)_{i\geq 1}$
- Set $\tau_0 := 0$, $T_i := \tau_i \wedge T$, and

$$\Delta T_i := T_i - T_{i-1}, \qquad \Delta W_{T_i} := W_{T_i} - W_{T_{i-1}}$$

- Consider the "Euler discretization along the arrival times τ_i"

$$\hat{X}_{T_i} = \hat{X}_{T_{i-1}} + \mu(T_{i-1}, \hat{X}_{T_{i-1}})\Delta T_i + \Delta W_{T_i},$$

$$\text{for } i = 1, \ldots, N_T + 1$$

$\Longrightarrow$ branching diffusion with one descendent at each default

Unbiased simulation for constant diffusion

Define the **exactly simulatable** r.v.

$$\hat{\xi} := \beta^{-N_T} e^{\beta T} \left[g(\hat{X}_T) - g(\hat{X}_{T_{N_T}}) \mathbb{1}_{\{N_T>0\}} \right] \prod_{k=1}^{N_T} \hat{\mathcal{W}}_k^1$$

where

$$\hat{\mathcal{W}}_k^1 \quad := \quad \left(\mu(T_k, \hat{X}_{T_k}) - \mu(T_{k-1}, \hat{X}_{T_{k-1}})\right) \cdot \frac{\Delta W_{T_{k+1}}}{\Delta T_{k+1}}$$

Theorem (Henry-Labordère, Tan & NT '15)

Assume μ $\frac{1}{2}-$Hölder in t, Lip in x, and g Lipschitz. Then

$$\hat{\xi} \in \mathbb{L}^2 \quad \text{and} \quad \mathbb{E}\left[g(X_T)\right] = \mathbb{E}\left[\,\hat{\xi}\,\right]$$

Main ideas

Define

$$\hat{X}_0 := X_0, \;\; d\hat{X}_t = \mu(\Theta_t)dt + \sigma dW_t$$

with $\Theta_t := (T_{N_t}, \hat{X}_{T_{N_t}})$. In other words,

$$\hat{X}_{T_{k+1}} = \hat{X}_{T_k} + \int_{T_k}^{T_{k+1}} \mu(T_k, \hat{X}_{T_k}) ds + \int_{T_k}^{T_{k+1}} \sigma dW_s$$

i.e. the drift coefficient changes at each arrival time T_k

First main idea

Define $u(t,x) := \mathbb{E}_{t,x}\big[g(X_T)\big]$, $t \leq T$, $x \in \mathbb{R}$

Proposition

Let $\beta > 0$, $\theta \in [0,T) \times \mathbb{R}^d$, $(t,x) \in [0,T) \times \mathbb{R}^d$. *Then*

$$u(t,x) = e^{\beta(T-t)}\mathbb{E}_{t,x,\theta}\Big[\mathbb{1}_{\{N_T=0\}}\, g\big(\hat{X}_T\big) + \mathbb{1}_{\{N_T>0\}}\frac{1}{\beta}\Delta\mu \cdot Du\big(T_1, \hat{X}_{T_1}\big)\Big]$$

where $\Delta\mu := \mu - \mu(\theta)$

Sketch of proof of the lemma

The function $\tilde{u} := e^{-\beta(T-t)}\mathbb{E}_{t,x}\big[g(X_T)\big]$ solves

$$-\partial_t \tilde{u} - \mu \cdot D\tilde{u} - \frac{1}{2}\sigma^2 : D^2\tilde{u} + \beta\tilde{u} = 0 \quad \text{and} \quad \tilde{u}(T,.) = g$$

Equivalently, with $\phi := (\mu - \mu(\theta)) \cdot D\tilde{u}$,

$$-\partial_t \tilde{u} - \mu(\theta) \cdot D\tilde{u} - \frac{1}{2}\sigma^2 : D^2\tilde{u} + \beta\tilde{u} = \phi \quad \text{and} \quad \tilde{u}(T,.) = g$$

By the Feynman-Kac representation :

$$\begin{aligned} u(0, X_0) &= e^{\beta T}\mathbb{E}\Big[e^{-\beta T} g(\hat{X}_T) + \int_0^T e^{-\beta t}\phi(t, \hat{X}_t)dt\Big] \\ &= e^{\beta T}\mathbb{E}\Big[g(\hat{X}_T)\mathbb{1}_{\{\tau \geq T\}} + \frac{1}{\beta}\phi(\tau, \hat{X}_\tau)\mathbb{1}_{\{\tau < T\}}\Big] \end{aligned}$$

where τ is an independent Expo(β)

Second main idea : use Monte Carlo automatic differentiation

By the last proposition,

$$\begin{aligned}
u(t,x) &= \mathbb{E}_{t,x,\theta}\Big[e^{\beta(T_1-t)}\Big(\mathbb{1}_{\{N_T=0\}}\, g(\hat{X}_T) \\
&\qquad + \mathbb{1}_{\{N_T>0\}}\frac{\Delta\mu_{T_1}}{\beta}\cdot Du(T_1,\hat{X}_{T_1})\Big)\Big] \\
&= \mathbb{E}_{t,x,\theta}\Big[e^{\beta(T_1-t)}\Big(\mathbb{1}_{\{N_T=0\}}\, g(\hat{X}_T) \\
&\qquad + \mathbb{1}_{\{N_T=1\}}\frac{\Delta\mu_{T_1}}{\beta}\cdot\frac{\Delta W_{T_2}}{\Delta T_2}g(\hat{X}_T) \\
&\qquad + \mathbb{1}_{\{N_T>1\}}\frac{\Delta\mu_{T_2}}{\beta^2}\cdot Du(T_2,\hat{X}_{T_2})\Big)\Big]
\end{aligned}$$

by the assumption. And so on...

Back to unit diffusion : square integrability lost... in general

Iterating as above, and passing to limits, we would arrive at

$$\mathbb{E}[\xi] \quad \text{where} \quad \xi := \beta^{-N_T} e^{\beta T} g(\hat{X}_T) \prod_{k=1}^{N_T} \hat{\mathcal{W}}_k^1$$

where, in the case of unit diffusion :

$$\hat{\mathcal{W}}_k^1 \quad := \quad \left[\mu(T_k, \hat{X}_{T_k}) - \mu(T_{k-1}, \hat{X}_{T_{k-1}})\right] \cdot \frac{\Delta W_{T_{k+1}}}{\Delta T_{k+1}}$$

Notice that $\frac{\Delta W_{T_1}}{\Delta T_1} \sim (\Delta T_1)^{-1/2}$, then :

$$\mu \text{ Lip in } x, \ \frac{1}{2} - \text{Hölder in } t \quad \Longrightarrow \quad \hat{\xi} \in \mathbb{L}^2$$

Gaining more integrability

Oudjane & Warin '15 give up on the Expo(β) distribution $\Longrightarrow$
Age-dependent branching...

$$-\partial_t u - \mu(\theta) \cdot Du - \frac{1}{2}\sigma^2 : D^2 u = \phi \quad \text{and} \quad u(T,.) = g$$

with $\phi := (\mu - \mu(\theta)) \cdot Du$. By the Feynman-Kac representation :

$$\begin{aligned} u(0, X_0) &= \mathbb{E}\Big[g(\hat{X}_T) + \int_0^T \phi(t, \hat{X}_t)dt\Big] \\ &= \mathbb{E}\Big[\bar{\rho}_T^{-1} g(\hat{X}_T)\mathbb{1}_{\{\tau \geq T\}} + \rho(\tau)^{-1}\phi(\tau, \hat{X}_\tau)\mathbb{1}_{\{\tau < T\}}\Big] \end{aligned}$$

where τ is an independent r.v. with density ρ, and $\bar{\rho}_T := \int_T^\infty \rho(t)dt$

Choose ρ so as to guarantee square integrability...
Gamma distribution does the job
$\rho(t) = \Gamma(\kappa)^{-1}\beta^\kappa t^{\kappa-1} e^{-\beta t}, \;\; \kappa \leq \frac{1}{2}$

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1. Unbiased simulation of SDEs
   - The constant diffusion case
   - Regime switching and automatic differentiation
2. Age-dependent branching diffusions and semilinear PDEs
   - Complexity of the Monte Carlo approximation

A class of semilinear PDEs with polynomial nonlinearity

Consider the PDE (unit diffusion for simplicity)

$$\partial_t u + \frac{1}{2}\Delta u + f(t,x,u,Du) = 0, \quad u_T = g$$

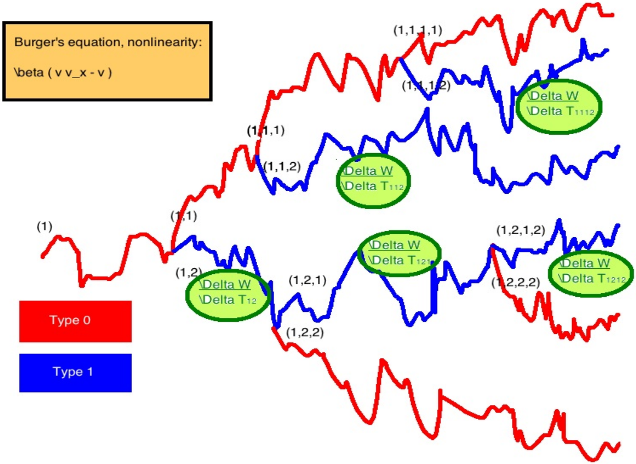
with nonlinearity

$$f(t,x,y,z) = \sum_{(\ell_i)_{0\le i\le n}\in L} p_\ell c_\ell(t,x) y^{\ell_0} \prod_{i=1}^{n} \big(b_i(t,x)\cdot z\big)^{\ell_i}$$

- L finite subset of $\mathbb{N}^{n+1}$
- $p_\ell > 0$ with $\sum_{\ell\in L} p_\ell = 1$
- $b_i(t,x)$ bounded functions

Example : Burgers equation $d = 1$ and $f(t,x,u,u_x) = u\, u_x$

Branching diffusion for the Burger equation



Marked branching diffusion representation

- $(\tau_k)_k$ iid arrival times, $T_k := \tau_k \wedge T$
- If $T_1 < T$: particle dies out, and is replaced with probability p_ℓ by

$$\ell_i \text{ particles of type } i,\ i = 0, \ldots, n$$

- For a particle $k \in \overline{\mathcal{K}}_T$, denote by
  - $D(k)$ its type
  - $k-$ its parent particle
    $\Longrightarrow$ Particle k lives between T_{k-} and T_k

Using automatic differentiation

- Automatic differentiation :

$$\mathcal{W}_k := \mathbb{1}_{\{D(k)=0\}} + \mathbb{1}_{\{D(k)\neq 0\}} b_{D(k)}(T_{k-}, X^k_{T_{k-}}) \cdot \frac{\Delta W_{T_k}}{\Delta T_k}$$

The limiting random variable is :

$$\psi := \prod_{k\in\mathcal{K}_T} \bar{F}_\rho(\Delta T_k)^{-1}\left[g\left(X^k_T\right) - \mathbb{1}_{\{D_k\neq 0\}} g\left(X^k_{T_{k-}}\right)\right]\mathcal{W}_k$$
$$\times \prod_{k'\in\overline{\mathcal{K}}_T\setminus\mathcal{K}_T} \left[\rho(\Delta T_{k'})\right]^{-1} b_{I_{k'}}(T_{k'}, X^{k'}_{T_{k'}})\mathcal{W}_{k'}$$

Sufficient condition for square integrability

For independent BM W, $\tau \sim \rho$, and $T_1 := \tau \wedge T$, define :

$$A_p := \max_{\ell} \frac{|g|_\infty^p \vee \|W_{T_1}\|^p \| b_\ell \cdot \frac{W_{T_1}}{T_1} \|^p}{\bar{F}_\rho(T)^{p-1}}$$

$$B_p := \max_{\ell} \|\Delta \tau\|^{p/2} \left\| b_\ell \cdot \frac{W_{T_1}}{T_1} \right\|^p \left[|b_\ell|_\infty \sup_{t \le T} \frac{t^{-\frac{p}{2(p-1)}}}{\rho(t)} \right]^{p-1}$$

Theorem (Henry-Labordère, Oudjane, Tan, NT, Warin '16)

Assume that g Lipschitz and, for some $p > 1$,

$$\int_{A_p}^{\infty} \left[B_p \sum |b_\ell|_\infty |x|^{|\ell|} \right]^{-1} dx \quad > \quad T$$

Then $v(0,x) = \mathbb{E}_{0,x}[\psi]$, and $\psi \in \mathbb{L}^2$

The complexity of the algorithm

- Average number of particles in one simulation $m(t) := \mathbb{E}[\#\overline{\mathcal{K}}_t]$ satisfies, for $n_0 := \sum_\ell |\ell| p_\ell$,

$$m(t) = 1 + n_0 \int_0^t m(t-s)\rho(s)ds.$$

When $\rho \sim \Gamma(\kappa, \theta)$, one has

$$m(T) \;=\; \gamma(\kappa, T/\theta) \sum_{k=0}^{\infty} \frac{n_0^k}{\Gamma(k\kappa)}.$$

- Number of v.a. simulated for one particle

$$d + 1 + 1$$

- Number of computation for one particle

$$Cd$$

Numerical example

- Define

$$u(t,x) = \cos(x_1 + \cdots + x_d)\exp(\alpha(T-t))$$

is solution of semilinear PDE

$$\partial_t u + \frac{1}{2}\Delta u + c\ u(b_1 \cdot Du) + b_0 \;=\; 0.$$

- For numerical implementation, we choose

$$\alpha = 0.2, \quad c = 0.15, \quad b_1 = (1+\frac{1}{d}, 1+\frac{2}{d}, \cdots, 2).$$

A numerical example of dimension $d = 20$

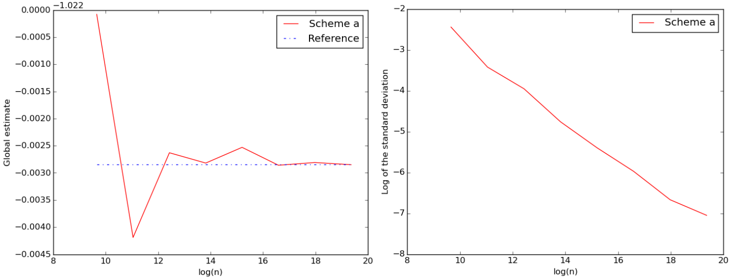


Figure: Estimation and standard deviation observed in dimension $d = 20$ depending on the log of the number of simulation used.

Comments on the Monte-Carlo method

- Choice of ρ, $(p_\ell)_{\ell\in L}$, the expression of nonlinearity ...
- possible to use importance sampling, particles method,...
- open to parallel computing

Monte Carlo approximation of nonlinear PDEs

Fully nonlinear PDEs... (e.g. HJB equations)

If $T_1 < T$: particle dies out, and is replaced with probability p_ℓ by

- i_ℓ particles of type 0
- j_ℓ particles of type 1 $\Longrightarrow$ first order differentiation weight
- h_ℓ particles of type 2 $\Longrightarrow$ second order differentiation weight

Automatic differentiation for particles k of type $D(k) = 2$:

$$\frac{\Delta W_T^2 - \Delta T}{(\Delta T)^2} \quad !!$$

THANK YOU FOR YOUR ATTENTION