Endogenous Formation of Limit Order Books

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Flash Crash

- On **May 6, 2010, at 2:42pm** all major stock indices (S&P 500, Dow Jones Industrial Average and Nasdaq Composite) suffered a huge and rapid loss (about 10%) in 5 minutes, and recovered by 3:07 pm.
- On **Aug. 24** the Dow Jones index dropped roughly 7% in the first five minutes of trading.

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Market Microstructure

- **Classical** models of **Financial Mathematics**: **exogenous prices**, **trading mechanism hidden**.
- **Financial Economics**: **endogenous prices**, **trading mechanism hidden**.
- **Market Microstructure**: study **trading mechanism**.
- Typically, two types of mechanisms are considered:
  - **central market-maker** (“**quote-driven**”) exchanges;
  - and **auction-style** (“**order-driven**”) ones.
- We focus on the **auction-style exchanges**.
- The main object of our study is the **Limit Order Book (LOB)**.

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Example of LOB

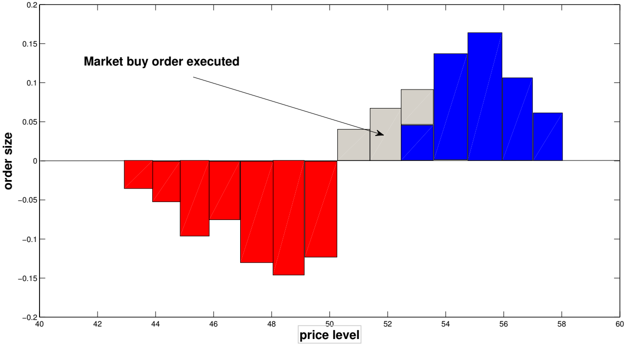

Figure: Limit buy (in red) and sell (in blue) orders.

Types of investigation

- Optimize agents' behavior, **given a model for LOB**.
  - Key empirical features of the market (such as **market resilience** and **price impact**) are modeled **exogenously**.
  - Then, the problem of **optimal execution** is solved.
  - **Literature**: *Almgren, Chriss, Bouchaud, Obizhaeva, Wang, Schied, Zhang, Gatheral, Alfonsi, Stoikov, Avellaneda, Cont, Talreja, Jaimungal, Cartea, Cvitanic, Shreve, Gueant, Lehalle, Pham, Bayraktar, Ludkovski, Moallemi, Carmona, Lacker, Cheridito, Guo, Pham, Ma*.
- Model LOB **endogenously** – as an **outcome of an equilibrium**.
  - **Fundamental price** or **demand** is modeled exogenously, but **LOB** arises **endogenously** from the agents' behavior in equilibrium.
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Our goals

1. Develop a **rigorous**, **precise** and **tractable** **modeling framework** for auction-style exchanges.
   - **Input**: **rules** (mechanics) of the exchange + agents' **beliefs** about future **demand** for (or **fundamental value** of) the asset.
   - **Output**: agents **actions** in **equilibrium**.
2. Study the **internal liquidity effects** (due to the agent's interaction). In particular,
   - how do changes in the **rules of the exchange** affect the **liquidity**?
   - how do changes in a **relevant factor** affect the agents **beliefs** and, in turn, the **liquidity**?

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The New York Times Magazine, by Jacob Goldstein, Oct. 8, 2013.

- "*In the old days, the stock market worked because there were people – so-called* **market makers**... *In the past decade, their jobs have been largely* **replaced by high-frequency traders** *who provide this middleman service.*"

- "*A trader using a high-speed connection to* **jump in front** *of a deal...* **isn't really improving** *the market.*"

- "*In practice, it can be difficult to* **distinguish** *between high-frequency traders who are simply* **adding liquidity** *and the ones who are* **profiting from unfair advantages**."

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Effects of trading frequency: main results

- If the **liquidity does not disappear** in equilibrium, then the **market efficiency increases** with trading **frequency**.
- The **liquidity does not disappear** in equilibrium **only if** the agents are **market-neutral**.
- In addition, we show **why** exactly the **liquidity disappears** when the agents are not market-neutral and connect it to the **adverse selection** effect.

External demand and beliefs

- **Time** is discrete: $n = 0, 1, \dots, N$.
- **External demand** in the time interval $(n-1, n]$ is given by the random function $D_n(p), \quad p \in \mathbb{R}$.
- $D_n(p)$ denotes the **total demand (both external and internal)** for the asset at price p and at all more favorable price levels, in the nth time period.
  - $D_n^+(p) = \max(D_n(p), 0)$ is the maximum quantity that will be **purchased** at or below price p (via market orders),
  - $D_n^-(p) = -\min(D_n(p), 0)$ is the maximum quantity that will be **sold** at or above price p.
- The **fundamental price** (or, "tipping point" of the demand) p_n^0 is the unique solution to: $D_n(p) = 0$.
- Every agent models future demand $(D_n(p))$ using the same information $\mathbb{F}$ but **different probability measures** $\mathbb{P}^\alpha << \mathbb{P}$, $\alpha \in \mathbb{A}$, which we call **beliefs**.

State space and controls

- **State space** $\mathbb{S} = \mathbb{R} \times \mathbb{A}$ represents the **inventory** of an agent and her **beliefs**.
- As the beliefs do not change, the **state process** of an agent, (S_n), represents her inventory.
- The **control** of an agent is given by adapted processes $(p_n, q_n, r_n)_{n=0}^{N-1}$, with values in $\mathbb{R}^2 \times \{0, 1\}$.
    - p_n is the **location** of a limit order placed at time n,
    - q_n is the **size** of the order (with **negative** values corresponding to **buy** orders).
    - r_n indicates whether the agent submits a **market order** (if $r_n = 1$) or a **limit order** (if $r_n = 0$).
- $(\mu_n)_{n=0}^{N}$ is the **empirical distribution** of the agents: $\mu_n(ds, d\alpha)$ denotes the **number of agents** at states $(ds, d\alpha)$ at time n.

LOB, State Dynamics and Revenue

- The **Limit Order Book (LOB)** is a pair of processes $(\nu_n^-, \nu_n^+)_{n=0}^{N-1}$, with values in the space of finite sigma-additive measures on $\mathbb{R}$.

- The **bid and ask prices** at time n are given by

$$p_n^b = \sup \text{supp}(\nu_n^-), \quad p_n^a = \inf \text{supp}(\nu_n^+) \quad p_N^{a,b} = p_{N-1}^{a,b} + \Delta p_N^0,$$

- **State process** S evolves as follows

$$S_{n+1} = \begin{cases} S_n - q_n, & r_n = 1, \\ S_n - q_n, & r_n = 0, \ q_n > 0, \ D_{n+1}^+(p_n) > \nu_n^+((-\infty, p_n)), \\ S_n - q_n, & r_n = 0, \ q_n < 0, \ D_{n+1}^-(p_n) > \nu_n^-((p_n, \infty)), \\ S_n, & \textit{otherwise} \end{cases}$$

- At every time step, the agent collects **revenue**: $-\Delta S_{n+1} p_n$, $-\Delta S_{n+1} p_n^a$ or $-\Delta S_{n+1} p_n^b$.

- At time N, the inventory is **marked to market**, adding $S_N^+ p_N^b - S_N^- p_N^a$.

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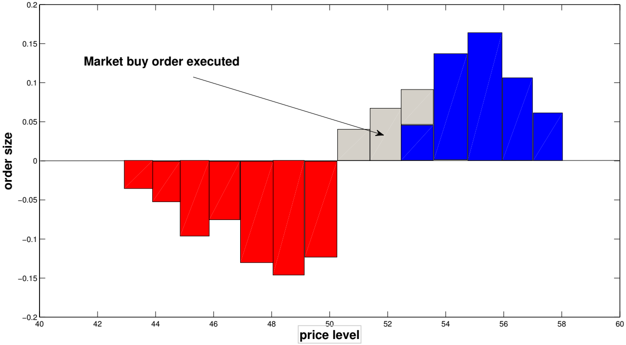

Figure: Limit buy (in red) and sell (in blue) orders.

Objective function and Equilibrium

- An agent aims to **maximize the total expected revenue**

$$\mathbb{E}^{\alpha}\left[S_N^+ p_N^b - S_N^- p_N^a - \sum_{n=0}^{N-1} \Delta S_{n+1}\left(p_n \mathbf{1}_{\{r_n=0\}} + p_n^a \mathbf{1}_{\{r_n=1, q_n<0\}} + p_n^b \mathbf{1}_{\{r_n=1, q_n>0\}}\right)\right.$$

- Fix an empirical distribution (μ_n). The LOB (ν_n^+, ν_n^-) and controls $(p_n(s,\alpha), q_n(s,\alpha), r_n(s,\alpha))$ form an **equilibrium**, if

  1. the controls $(p_n(s,\alpha), q_n(s,\alpha), r_n(s,\alpha))$ are **optimal** for an agent in state (s,α);
  2. and the collection of all limit orders $(p_n(s,\alpha), q_n(s,\alpha))$, over all (s,α) s.t. $r_n(s,\alpha) = 0$, should **reproduce the LOB** (ν_n^+, ν_n^-):

$$\nu_n^+((-\infty, x]) = \int_{\mathbb{S}} \mathbf{1}_{\{p_n(s,\alpha)\leq x,\, r_n(s,\alpha)=0\}}\, q_n^+(s,\alpha)\mu_n(ds, d\alpha), \quad \forall x \in \mathbb{R},$$

  and similarly for ν^-.

Degeneracy

- In equilibrium, it may happen that **no agents post limit orders** on a particular side of the book.
- Instead, they may choose to
  - submit market orders: $r_n = 1$ (**impatience**);
  - or wait: $q_n = 0$ (**adverse selection**).
- This constitutes a **liquidity crisis**.
- An equilibrium with LOB ν is **non-degenerate** if $\nu_n^+(\mathbb{R}) > 0$ and $\nu_n^-(\mathbb{R}) > 0$, $\mathbb{P}$-a.s., for all n.

Fundamental price in continuous time

- Every agent uses a **continuous time model** for the **demand**, on $[0, T]$.
- An agent with beliefs α models the continuous-time **fundamental price** as

$$\tilde{p}^0_t = p^0_0 + \int_0^t \mu^\alpha_s ds + \int_0^t \sigma_s dW^\alpha_s, \qquad p^0_0 \in \mathbb{R},$$

where W^α is a BM, μ^α and σ are prog. mbl. stochastic processes, s.t. $|\mu^\alpha| \leq C$, $1/C \leq \sigma \leq C$ and

$$\mathbb{P}^\alpha_t \left(\mathbb{E}^\alpha \left((\sigma_s - \sigma_\tau)^2 \mid \mathcal{F}_\tau \right) \leq \varepsilon(\Delta t) \right) = 1,$$

for $t \leq \tau \leq s \leq t + \Delta t$ and some determ. $\varepsilon(\Delta t) \to 0$.

- Given $\Delta t > 0$, the **discrete time model** is defined by discretizing the continuous time model. In particular, $p^0_n = \tilde{p}^0_{n\Delta t}$.
- The **demand size** $D_n \left(p + p^0_n \right)$ is **arbitrary**, but "not too flat".
- The **empirical distribution** process (μ_n) is **arbitrary**, but dominated by a deterministic measure.

Asymptotic efficiency and market-neutrality

- **Proposition 1**. For a sequence $\{\Delta t \to 0\}$, assume that every discrete model admits a **non-degenerate equilibrium**. Denote the **value function** of an agent by $V_n(s,\alpha)$. Then, as $\Delta t \to 0$,

$$\left|p_N^a - p_N^0\right|, \; \left|p_N^b - p_N^0\right|, \; \sup_{n=0,\dots,N,\; s\in\mathbb{R},\; \alpha\in\mathbb{A}} \left|V_n(s,\alpha)/s - p_n^0\right| \to 0$$

- **Theorem 1**. Under the assumptions of Proposition 1, and with an additional assumption of "uniform continuity in probability" of the process

$$\mathbb{E}^{\alpha}_{\cdot} \int_t^T \mu_s^{\alpha} ds,$$

we must have: $\tilde{p}^0$ is a **martingale** under all $\mathbb{P}^{\alpha}$ (i.e. if this condition **fails**, any equilibrium is **degenerate**).

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Brownian motion with drift

- $\mathbb{A} = \{\alpha\}\,, \;\; \alpha \in \mathbb{R}$.
- $\tilde{p}^0_t = p^0_0 + \alpha t + \sigma W_t$, for $t \in [0, T]$, where $\sigma, p^0_0 \in \mathbb{R}$ and W is a Brownian motion.
- As all agents have the same beliefs, the equilibrium can be constructed so that the **LOB is a combination of two delta-functions**:

$$\nu^+_n = h^a_n \, \delta_{p^a_n}, \qquad \nu^-_n = h^b_n \, \delta_{p^b_n}$$

- If $N = 1$, such an **equilibrium can be constructed** for any $\alpha \in \mathbb{R}$, provided T is small enough.
- If $\alpha = 0$, such an **equilibrium can be constructed** for any N and any T. Moreover, as $\Delta t = T/N \to 0$,
  - the **bid and ask prices**, p^b and p^a, **converge** to the **fundamental price** p^0,
  - along with the **expected execution prices** $V(s)/s$.

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Zero-drift case

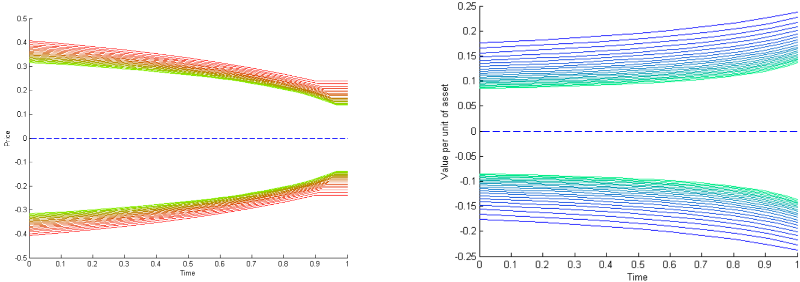

Figure: Bid and ask prices (left) and the associated expected execution prices (right), as functions of time horizon. Different curves correspond to different trading frequencies. Zero drift case.

Value function and DPP

$$V_n(s) = \text{esssup}_{p,q,r} \mathbb{E}^{\alpha}_n J^{\nu,p,q,r}_n (s)\,,$$

- there always exists an optimal control $(\hat{p}, \hat{q}, \hat{r})$, s. t. $\hat{q}_n(s) \in \{0, s\}$;
- $V_n(s) = s^+\lambda^a_n - s^-\lambda^b_n$, with $\lambda^{a,b}_N = p^{a,b}_N \approx p^0_N$;
- for $s > 0$ $(s = 1)$,
  - if $\hat{q}_n(s) = s$ and $\hat{r}_n(s) = 0$, then $\lambda^a = V(s)/s = V(1)$ follows:

$$\lambda^a_n = \mathbb{E}^{\alpha}_n \lambda^a_{n+1} + \sup_{p\in\mathbb{R}} \mathbb{E}^{\alpha}_n \left[\left(p - \lambda^a_{n+1}\right) \mathbf{1}_{\{p^0_{n+1} > p\}} \right],$$

    and $p = \hat{p}_n(s)$ attains the above supremum,
  - if $\hat{q}_n(s) = 0$ and $\hat{r}_n(s) = 0$, then $\lambda^a_n = \mathbb{E}^{\alpha}_n \lambda^a_{n+1}$,
  - if $\hat{r}_n(s) = 1$, then $\lambda^a_n = p^b_n$.

Positive drift

$$\tilde{p}_t^0 = p_0^0 + \alpha t + \sigma W_t, \text{ for } t \in [0, T].$$

- If $\alpha > 0$ and N is large enough, the agents become overly optimistic: at some step n, $\lambda_n^a = V_n(s)/s \geq p_n^0$ for $s > 0$.
- Then, the expected gain from executing a limit sell order at the ask price becomes negative:

$$\mathbb{E}_{n-1}\left[(ps - V_n(s))\,\mathbf{1}_{\{p_n^0 > p\}}\right] \leq \mathbb{E}_{n-1}\left[(ps - p_n^0 s)\,\mathbf{1}_{\{p_n^0 > p\}}\right] < 0, \qquad \forall p \in \mathbb{R}.$$

- Thus, it is **suboptimal** for the agents to post a **limit sell** order at any level at time $n-1$.
- This is precisely the **adverse selection** effect: if her limit sell order is executed, the agent will immediately **regret** it, because, in any such outcome, she will **expect** a **higher execution price**.
- It causes the agents with positive inventory to wait – and **stop providing liquidity** – so that the LOB **degenerates**.

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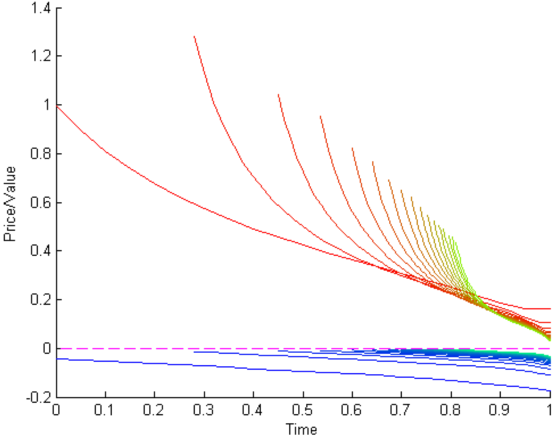

Figure: Ask prices p_n^a (in red) and the associated $\lambda_n^a = V_n(s)/s$ (in blue), as functions of time n. Different curves correspond to different trading frequencies. Positive drift: $\alpha > 0$.

Technical result

$$X_t = \int_0^t \mu_u du + \int_0^t \sigma_u dB_u, \quad t \geq 0$$

- Assume that $|\mu| \leq C$, $1/C \leq \sigma \leq C$ and

$$\mathbb{E}\left((\sigma_s - \sigma_\tau)^2 \mid \mathcal{F}_\tau \right) \leq \varepsilon(\Delta t), \quad a.s.,$$

for $0 \leq \tau \leq s \leq \Delta t$ and some determ. $\varepsilon(\Delta t) \to 0$.

- Then, $\exists C_1 > 0$ s.t., for all small enough $t > 0$ and all $x, z \geq 0$,

$$\mathbb{P}\left(X_t > x + z \middle| X_t > x \right) \leq C_1 e^{-z/\sqrt{t}}$$

Effects of Trading Frequency: summary

- I have presented a **modeling framework** for market microstructure, in which the mechanics of the exchange are reproduced very closely and the LOB arises **endogenously**, as an outcome of the game between market participants.
- Using this framework, we have verified that, even in the absence of any significant fundamental shocks, the agents may choose **not to provide liquidity** in equilibrium.
- We have analyze the **liquidity effects** of changing the **trading frequency**. We find that **trading frequency** has dual effect on liquidity:
  - if the agents are **market-neutral**, higher frequency makes market **more efficient**,
  - but higher frequency increases the risk of **degeneracy**, if the agents' **beliefs** deviate from **market-neutrality**.
- Typically, the **adverse selection** causes LOB to **degenerate**.

Introduction

- **Goals**:
  - model how the agents **form their beliefs** (e.g. depending on a relevant market factor);
  - develop a **quantitative model**, which can be **calibrated** to market data.
- Most of the **changes in LOB** occur **between Market Orders**.
- Hence, we formulate a **continuous time control-stopping game**, which **terminates** at the time when the **first market order** is submitted.

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- Hence, we formulate a **continuous time control-stopping game**, which **terminates** at the time when the **first market order** is submitted.

Model Setup

- The time changes on $[0, T]$.
- The **fundamental price** (p^0_t) changes by jumps
  - **jump times** are determined by a **Poisson random measure** N,
  - **jump sizes** are given by a random function adapted to $\mathbb{F}^W$, where W is an **independent BM**,
  - the above holds under **every** $\mathbb{P}^\alpha$, with the same BM W.
- The **demand size** $D_t(p + p^0_t)$ is an **arbitrary** $\mathbb{F}^W$-adapted random field (strictly decreasing in p and taking value zero at $p = 0$).
- The **empirical distribution** does not change: $\mu_n = \mu_0$.
- Agents always submit orders of the **size** s equal to their inventory.
- The **control** of each agent is given by (p_t, v_t), where
  - p_t is the **location** of a **limit order** at time t,
  - v_t is the threshold for executing a market order: e.g. an agent with $s > 0$ submits a market order at $\tau^v = \inf\{t \in [0, T] : p^b_t \geq v_t\}$.

Objective and Equilibrium

- The **game ends** when "a non-zero mass of" **market orders is executed**.
- If an agent's **limit order** is executed at time t, before the end of the game, she receives sp_t.
- If an agent executes a **market order** at time t, before the end of the game, she receives sp_t^a or sp_t^b.
- If an agent **has not executed** any order by the end of the game, and
  - the game ends at time t, due to **external market order**, then her payoff is sp_t^0,
  - the game ends at time t, due to **internal market order**, then her payoff is sp_t^a or sp_t^b.
- A combination of measure valued processes $(\nu_t^-, \nu_t^+, \theta_t^-, \theta_t^+)$ and controls $(p_t(s,\alpha), v_t(s,\alpha))$ is an **equilibrium**, if
  1. $(p_t(s,\alpha), v_t(s,\alpha))$ is optimal, given $(\nu_t^-, \nu_t^+, \theta_t^-, \theta_t^+)$;
  2. ν_t and θ_t are the **empirical distributions** of $\{p_t(s,\alpha)\}_{s,\alpha}$ and $\{v_t(s,\alpha)\}_{s,\alpha}$ w.r.t. μ.

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Two-player game

Under additional assumptions on the set of beliefs $\{\mathbb{P}^\alpha\}$, the **bid and ask prices**, as well as the **time of the first internal market order**, can be characterized by the solution to a **two-player controlled Dynkin game**:

$$V_t^a = \text{esssup}_{t\leq\tau^a\leq T,\, p^a}\mathbb{E}_t^{\alpha_0}\left(\int_0^{\tau^a\wedge\tau^b}\exp\left(-\int_t^s c_u^a(p_u^a,p_u^b)du\right)g_s^a(p_s^a,p_s^b)ds\right.$$

$$\left.+\exp\left(-\int_t^{\tau^a\wedge\tau^b}c_u^a(p_u^a,p_u^b)du\right)\left(p^b_{\tau^a}\mathbf{1}_{\{\tau^a\leq\tau^b\}}+p^a_{\tau^b}\mathbf{1}_{\{\tau^b<\tau^a\}}\right)\right),$$

$$V_t^b = \text{essinf}_{t\leq\tau^b\leq T,\, p^b}\mathbb{E}_t^{\beta_0}\left(\int_0^{\tau^a\wedge\tau^b}\exp\left(-\int_t^s c_u^b(p_u^a,p_u^b)du\right)g_s^b(p_s^a,p_s^b)ds\right.$$

$$\left.+\exp\left(-\int_t^{\tau^a\wedge\tau^b}c_u^b(p_u^a,p_u^b)du\right)\left(p^a_{\tau^b}\mathbf{1}_{\{\tau^b\leq\tau^a\}}+p^b_{\tau^a}\mathbf{1}_{\{\tau^a<\tau^b\}}\right)\right).$$

System of RBSDEs

The **value functions** and the associated **optimal controls** p^a, p^b and $\tau = \tau^a = \tau^b$ can be characterized by a **system of RBSDEs**:

$$\begin{cases} -\mathrm{d}V_t^a = \tilde{G}_t^a(V_t^a, V_t^b)\mathrm{d}t - Z_t^a \mathrm{d}W_t + \mathrm{d}K_t^a, \quad V_T^a = 0, \\ -\mathrm{d}V_t^b = \tilde{G}_t^b(V_t^a, V_t^b)\mathrm{d}t - Z_t^b \mathrm{d}W_t - \mathrm{d}K_t^b, \quad V_T^b = 0, \\ V_t^a \geq V_t^b, \quad \forall t \in [0, T], \\ \int_0^T (V_t^a - V_t^b)\mathrm{d}K_t^a = 0, \quad \int_0^T (V_t^a - V_t^b)\mathrm{d}K_t^b = 0, \end{cases}$$

where K^a, K^b are continuous increasing processes starting at zero, and

$$\tilde{G}_t^a(V^a, V^b) = -\tilde{c}_t^a(V^a, V^b)V^a + \tilde{g}_t^a(V^a, V^b),$$
$$\tilde{G}_t^b(V^a, V^b) = -\tilde{c}_t^b(V^a, V^b)V^b + \tilde{g}_t^b(V^a, V^b),$$

with Lipschitz bounded functions $\tilde{c}^{a,b}$ and $\tilde{g}^{a,b}$.

Equivalent System

- Denote $K_t = K_t^a + K_t^b$. Then can write $dK_t^a = \alpha_t dK_t$, $dK_t^b = (1-\alpha_t)dK_t$, for some $\alpha_t \in [0,1]$.
- Assuming α is regular enough, we can change the variables, $Y_t^1 = V_t^a - V_t^b$, $Y_t^2 = (1-\alpha_t)V_t^a + \alpha_t V_t^b$ and derive a system of RBSDEs for (Y^1, Y^2):

$$\begin{cases} -dY_t^1 = \hat{G}_t^1(Y_t^1, Y_t^2)dt - Z_t^1 dW_t + dK_t, \quad Y_T^1 = 0, \\ -dY_t^2 = \hat{G}_t^2(Y_t^1, Y_t^2)dt - Z_t^2 dW_t, \quad Y_T^2 = 0, \\ Y_t^1 \geq 0, \\ \int_0^T Y_t^1 dK_t = 0, \end{cases}$$

where

$$\hat{G}_t^1(Y^1, Y^2) = -\hat{c}_t^{1,1}(Y^1, Y^2)Y^1 + \hat{c}_t^{1,2}(Y^1, Y^2)Y^2 + \hat{g}_t^1(Y^1, Y^2),$$
$$\hat{G}_t^2(Y^1, Y^2) = \hat{c}_t^{2,1}(Y^1, Y^2)Y^1 - \hat{c}_t^{2,2}(Y^1, Y^2)Y^2 + \hat{g}_t^2(Y^1, Y^2),$$

with Lipschitz bounded functions $\hat{c}^{i,j}$ and $\hat{g}^i$, s.t. $\hat{c}^{i,i} > 0$.

Dynamics between Market Orders: summary

- If the agents' beliefs are limited to the time of the first market order, it is possible to have a **non-degenerate equilibrium** without market-neutrality.
- Under additional assumptions, one can express the key elements of an equilibrium via a **two-dimensional system of RBSDEs**. This is one of the very few examples of tractable solutions to games with infinite number of players (e.g. mean field games).
- If Y is a relevant market factor, we can model the **compensator** of the jump measure of p^0, and the **demand size** $D_t(p + p_t^0)$, under every $\mathbb{P}^\alpha$, as functions of Y_t.
  - The driver of the system of RBSDEs, $\hat{G}_t(y^1, y^2)$, becomes a **function** of Y_t.
  - Then, the value functions and the **optimal strategies** of the agents can also be expressed as **functions** of Y.
  - Computing these functions, will allow us to see how **changes in relevant factors** affect agents' actions, and hence, the **liquidity**.

Existence

Theorem 2

- Assume that
  - $\mathbb{A}$ is a **singleton**,
  - $\log \sigma$ "does not oscillate too much",
  - the total demand never exceeds the total supply,
  - and p^0 is a **martingale**.
- Then, for all small enough Δt, the market model admits a **non-degenerate equilibrium**.

Zero-drift case

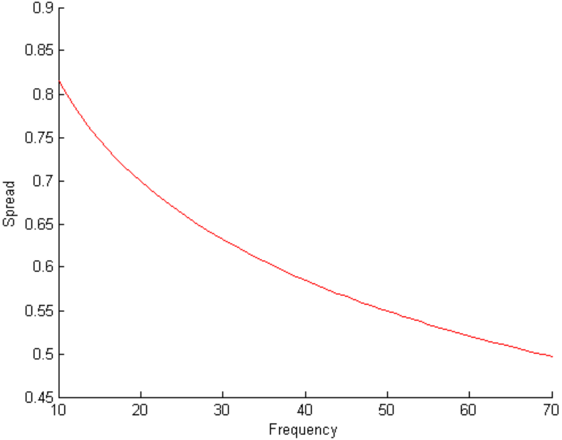

Figure: Time zero bid-ask spread as a function of trading frequency (measured in the number of steps).

Positive drift

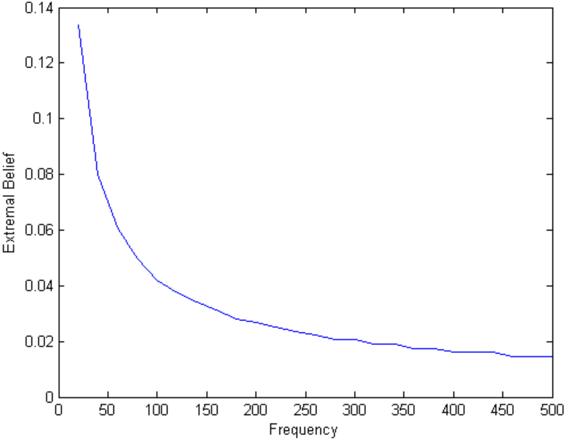

Figure: The **maximum** value of drift α that allows for a **non-degenerate equilibrium**, as a function of trading frequency (measured in the number of steps).

More extensions

- For the presented problem.
  - The continuum-player game is a limit of finite-player games.
  - A general existence result.
- How do the agents form their beliefs?
- Continuous time models.
- Test a class of such models against the market data.

Lemma

$$X_t = \int_0^t \mu_u du + \int_0^t \sigma_u dB_u, \quad t \geq 0, \quad X_0 = 0$$

- Assume that $|\mu| \leq C$, $1/C \leq \sigma \leq C$, and that there exists a deterministic $\varepsilon(\Delta t) \to 0$, as $\Delta t \to 0$, s.t.

$$\mathbb{E}\left((\sigma_{s\vee\tau} - \sigma_\tau)^2 \,|\, \mathcal{F}_\tau \right) \leq \varepsilon(\Delta t)$$

holds a.s. for all $0 \leq s \leq \Delta t$ and all stopping times $0 \leq \tau \leq s$.

- Then, $\exists C_1 > 0$ s.t., for all small enough $\Delta t > 0$,

$$\mathbb{P}\left(X_{\Delta t} > x + z \,|\, X_{\Delta t} > x \right) \leq C_1 e^{-z/\sqrt{\Delta t}}, \qquad \forall x, z \geq 0.$$

Lemma

$$X_t = \int_0^t \mu_u du + \int_0^t \sigma_u dB_u, \qquad t \in [0,1]$$

- Assume $\exists\, C > 1$, s. t. $|\sigma_\tau| \leq C$ and $|\mu_\tau| \leq C$ for any stopping time τ.
- Then the following holds.
  1. $\forall c > 0\ \exists C_1 > 0$, s.t.

$$\mathbb{P}\left(\sup_{t\in[0,1]} X_t > x + z\right) \leq C_1 e^{-cz}\, \mathbb{P}\left(\sup_{t\in[0,1]} X_t > x\right), \qquad \forall x, z \geq 0.$$

  2. $\forall c > 0\ \exists\, C_2, \varepsilon > 0$, s.t.

$$\mathbb{P}\left(\sup_{t\in[0,1]} X_t > x\right) \leq C_2 \mathbb{P}(X_1 > x), \qquad \forall x \geq 0,$$

provided $|\sigma_\tau| \geq c$, $\mu_\tau^2 \leq \varepsilon$ and $\mathbb{E}\left((\sigma_{s\vee\tau} - \sigma_\tau)^2 \,|\, \mathcal{F}_\tau\right) \leq \varepsilon$, for any $s \in [0,1]$ and any stopping time τ.

Sketch of the proof of Thm 1

- We need to show that, if the agents have a **signal** about future price movements, then, they will choose **not to post limit orders**.
- Consider the agents who are **long** the asset (i.e. they are trying to sell) and post limit orders around the **ask price**.
- There are two reasons why they may choose not to post limit orders:
  1. they are **bearish** – then, they submit a **market order**;
  2. they are **bullish** – then, they submit **nothing** and wait.
- If the long agents are **bearish**, eventually

$$\lambda_n^a = \lambda_{n+1}^a + \alpha \Delta t + \mathbb{E}_n^\alpha \left[\left(p_n^a - \lambda_{n+1}^a - \xi \right) \mathbf{1}_{\{\xi > p_n^a)\}} \right] \leq 0,$$

while

$$p_n^b \approx \lambda_n^b \approx 0$$

Hence,

$$\lambda_n^a < p_n^b,$$

and the agents submit a market order.

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while

$$p_n^b \approx \lambda_n^b \approx 0$$

Hence,

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Sketch of the proof: adverse selection

- If the long agents are **bullish**,

$$\lambda_n^a = \lambda_{n+1}^a + \alpha\Delta t + \mathbb{E}_n^\alpha \left[\left(p_n^a - \lambda_{n+1}^a - \xi \right) \mathbf{1}_{\{\xi > p_n^a)\}} \right] \geq 0,$$

  and, in turn, for any $p \in \mathbb{R}$,

$$\mathbb{E}_{n-1}^\alpha \left[(p - \lambda_n^a - \xi) \mathbf{1}_{\{\xi > p\}} \right] < 0,$$

  Hence, wherever the agents post their limit orders, they will **regret** doing it once the **orders are executed**, as, in that case, they would have been able to get more for their shares.

- This is precisely the **adverse selection** effect.

State process: implicit assumptions

$$S_m^{m,s,(p,q,r)} = s, \quad \Delta S_{n+1}^{m,s,(p,q,r)} = -q_n \mathbf{1}_{\{r_n=1\}}$$

$$-\mathbf{1}_{\{r_n=0\}} \left(q_n^+ \mathbf{1}_{\{D_{n+1}^+(p_n) > \nu_n^+((-\infty,p_n))\}} - q_n^- \mathbf{1}_{\{D_{n+1}^-(p_n) > \nu_n^-((p_n,\infty))\}} \right)$$

- In the above expression, we implicitly assume that each agent
  - is **small** so that her order is fully executed once the demand reaches it;
  - believes that her order will be **executed first among all orders with the same priority**.
- The latter assumption implies a possible **inconsistency** with the **market clearance condition**: i.e. the total executed demand may not coincide with the total change in the cumulative inventory.
- The above issue is resolved if $\nu_n(\cdot)$ is **continuous** and $r \equiv 0$.

Equilibrium: comments

- The equilibrium we chose is **sub-game perfect**.
- It uses the typical assumption of a game with **continuum players**: each player is too small to affect the LOB.
- It is very similar to a **Mean Field Game** with **purely common noise**.
- However, it only defines a **partial equilibrium**, as μ is given **exogenously**.
- To make it a **true equilibrium**, we need to require, in addition, that

$$\mu_n = \mu_0 \circ \left((s, \alpha) \mapsto \left(S_n^{0,s,(p,q,r)}, \alpha \right) \right)^{-1}$$

We call this an equilibrium with **endogenous** μ.

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Value function and DPP

$$V_n^{\nu}(s,\alpha) = \text{esssup}_{p,q,r} J_n^{\nu,(p,q,r)}(s,\alpha),$$

- $V_n^{\nu}(s,\alpha) = s^+ \lambda_n^a(\alpha) - s^- \lambda_n^b(\alpha)$, with $\lambda_N^a(\alpha) = p_N^b$ and $\lambda_N^b(\alpha) = p_N^a$;
- there always exists an optimal control $(\hat{p}, \hat{q}, \hat{r})$, s. t. $\hat{q}_n(s,\alpha) \in \{0, s\}$;
- for $s > 0$,
  - if $\hat{q}_n(s,\alpha) = s$ and $\hat{r}_n(s,\alpha) = 0$, then the **expected execution price** λ^a follows:

$$\lambda_n^a(\alpha) = \mathbb{E}_n^{\alpha} \lambda_{n+1}^a(\alpha) + \sup_{p \in \mathbb{R}} \mathbb{E}_n^{\alpha} \left[\left(p - \lambda_{n+1}^a(\alpha)\right) \mathbf{1}_{\left\{D_{n+1}^+(p) > \nu_n^+((-\infty,p))\right\}} \right],$$

    and $p = \hat{p}_n(s,\alpha)$ attains the above supremum,
  - if $\hat{q}_n(s,\alpha) = 0$ and $\hat{r}_n(s,\alpha) = 0$, then $\lambda_n^a(\alpha) = \mathbb{E}_n^{\alpha} \lambda_{n+1}^a(\alpha)$,
  - if $\hat{r}_n(s,\alpha) = 1$, then $\lambda_n^a(\alpha) = p_n^b$.

Terminal condition and LTC equilibrium

- To start the backward iteration, suggested by DPP, we need to resolve the last-time-step problem:

$$\begin{cases} p_{N-1}(1,\alpha) \in \arg\sup_{p\in\mathbb{R}} \mathbb{E}^{\alpha}_{N-1}\left[\left(p - p^b_N\right) \mathbf{1}_{\left\{D^+_N(p) > \nu^+_{N-1}((-\infty,p))\right\}}\right], \\ p_{N-1}(-1,\alpha) \in \arg\sup_{p\in\mathbb{R}} \mathbb{E}^{\alpha}_{N-1}\left[\left(p^a_N - p\right) \mathbf{1}_{\left\{D^-_N(p) > \nu^-_{N-1}((-\infty,p))\right\}}\right], \end{cases}$$

- The equilibrium condition

$$\nu^+_{N-1}((-\infty,x]) = \int_{(0,\infty)\times\mathbb{A}} \mathbf{1}_{\{p_{N-1}(1,\alpha)\leq x\}}\, s\mu_{N-1}(ds,d\alpha), \quad \forall\, x\in\mathbb{R},$$

  links ν_{N-1} and p_{N-1}, resulting in a **fixed-point problem** for ν_{N-1}.

- However, there is no fixed-point problem for ν_N: we can choose ν_N and, in turn, (p^a_N, p^b_N) arbitrarily, as the agents do not optimize their actions at time N.

- An equilibrium with LOB ν is **linear at terminal crossing (LTC)** if

$$\nu_N = \nu_{N-1} \circ \left(x \mapsto x + \Delta p^0_N\right)^{-1}, \quad \mathbb{P}\text{-a.s.}$$

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Degeneracy

$$\lambda^a_n(\alpha) = \mathbb{E}^\alpha_n \lambda^a_{n+1}(\alpha) + \sup_{p\in\mathbb{R}} \mathbb{E}^\alpha_n \left[\left(p - \lambda^a_{n+1}(\alpha)\right) \mathbf{1}_{\left\{D^+_{n+1}(p) > \nu^+_n((-\infty,p))\right\}} \right],$$

- If

$$\sup_{p\in\mathbb{R}} \mathbb{E}^\alpha_n \left[p - \lambda^a_{n+1}(\alpha) \,\middle|\, D^+_{n+1}(p) > \nu^+_n((-\infty,p)) \right] < 0,$$

then, the agents at (s,α) choose to **wait**, in which case $q_n(s,\alpha) = 0$ and

$$\lambda^a_n(\alpha) = \mathbb{E}^\alpha_n \lambda^a_{n+1}(\alpha)$$

- This may indeed occur in an equilibrium, and it can be attributed to the **adverse selection** effect.
- An equilibrium with LOB ν is **non-degenerate** if $\nu^+_n(\mathbb{R}) > 0$ and $\nu^-_n(\mathbb{R}) > 0$, for all n, $\mathbb{P}$-a.s..

Assumptions

- There exists deterministic $\varepsilon(\Delta t) \to 0$, as $\Delta t \to 0$, s.t., $\mathbb{P}$-a.s.,

$$\mathbb{P}_t^\alpha \left(\mathbb{E}^\alpha \left(\left(\sigma_{s \vee \tau} - \sigma_\tau \right)^2 \,|\, \mathcal{F}_\tau \right) \leq \varepsilon(\Delta t) \right) = 1, \qquad \mathbb{P} - a.s.,$$

  holds for all $t \leq s \leq t + \Delta t$, all stopping times $t \leq \tau \leq s$, and all $\alpha \in \mathbb{A}$.

- For any n, there exists a strictly decreasing random function $\kappa_{n-1}(\cdot)$, such that $\kappa_{n-1}(0) = 0$ and

$$\left| D_n \left(p + p_n^0 \right) \right| \geq \left| \kappa_{n-1}(p) \right|, \quad \forall p \in \mathbb{R}, \qquad \mathbb{P} - a.s.$$

- For any n, there exists a deterministic measure μ_n^0, s.t. $\mu_n \ll \mu_n^0$, $\mathbb{P}$-a.s..

Market neutrality as a necessary condition

- **Additional assumption**. For any $\alpha \in \mathbb{A}$ and any $t \in [0, T)$, there exists a deterministic $\varepsilon(\cdot) \geq 0$, s.t. $\varepsilon(\Delta t) \to 0$, as $\Delta t \to 0$, and, for any $t \leq t' \leq t'' \leq t + \Delta t$,

$$\mathbb{P}^{\alpha}_{t'}\left(\left|\mathbb{E}^{\alpha}_{t''}\int_t^T \mu^{\alpha}_s ds - \mathbb{E}^{\alpha}_{t'}\int_t^T \mu^{\alpha}_s ds\right| \geq \varepsilon(\Delta t)\right) \leq \varepsilon(\Delta t), \quad \mathbb{P}^{\alpha} - a.s.$$

- **Theorem 1**
  - Consider a family of $\{\Delta t > 0\}$, containing arbitrarily small Δt, and the associated market models satisfying the above assumptions.
  - Assume that every model admits a **non-degenerate LTC equilibrium**.
  - Then, for all $\alpha \in \mathbb{A}$, $\tilde{p}^0$ is a **martingale** under $\mathbb{P}^{\alpha}$.
  - Moreover, for all $\alpha \in \mathbb{A}$, $p^b_n < p^0_n < p^a_n$, $\mathbb{P}^{\alpha}$-a.s..

Existence: endogenous μ

- $\mathbb{A}$ is a **singleton**, σ_t is deterministic and non-increasing in $t \in [0, T]$, and p^0 is a **martingale**.
- For any n, there exists a strictly decreasing continuous (deterministic) function $\kappa_n(\cdot)$, with $\kappa_n(0) = 0$, s.t. $D_n\left(p^0_n + p\right) = \kappa_n(p)$.
- **Theorem 3**
  - Consider a market model and an initial empirical distribution μ_0.
  - Let the above assumptions hold, and assume that, in addition,

$$\mu_0^{1,c} > \sum_{n=1}^{N-1} \sup_{p\in\mathbb{R}} D_n^+(p), \qquad \mu_0^{2,c} > \sum_{n=1}^{N-1} \sup_{p\in\mathbb{R}} D_n^-(p)$$

  - Then, there exists an empirical distribution process μ, with the prescribed μ_0, s.t. the associated market model and μ admit a **non-degenerate LTC equilibrium**, in which the agents **do not post market orders**, the LOB is **continuous** (i.e. has no mass points in $\mathbb{R}$), and

$$\mu_n = \mu_0 \circ \left((s,\alpha) \mapsto \left(S_n^{0,s,(p,q,r)}, \alpha\right)\right)^{-1}$$