Optimal Investment, Derivative Demand & Arbitrage under Price Impact

Michail Anthropelos
University of Piraeus

Joint work with S. Robertson (BU) and K. Spiliopoulos (BU)

University of Southern California, Mathematical Finance Colloquium
November, 2019

Outline

1. Motivation & goals
2. Market model & initial steps
3. Connection to a constrained investment problem with no price impact
4. Derivative pricing under price impact
5. Conclusive remarks

Outline

1. Motivation & goals
2. Market model & initial steps
3. Connection to a constrained investment problem with no price impact
4. Derivative pricing under price impact
5. Conclusive remarks

Large Investor's Price Impact

- **Large** in the sense that their orders substantially change MMs' inventory and hence their quoted prices.
  $\rightarrow$ *causing price impact.*
- MMs are risk averse.
- Given MMs price-quoting, each investor places his flow of orders, aiming to increase his individual expected utility.

Goal No. 1

Find the continuous-time **optimal investment strategy** *under price-impact.*

- A key step: Optimal investment problem *upon market impact* can be written as a constrained optimal investment problem in a *fictitious market without market impact*.

$\rightarrow$ Impose conditions that make the **constraint set non-binding**.

$\rightarrow$ Exploit this representation to **solve the problem** (when possible).

Large Investor's Price Impact

- **Large** in the sense that their orders substantially change MMs' inventory and hence their quoted prices.
  $\rightarrow$ *causing price impact.*
- MMs are risk averse.
- Given MMs price-quoting, each investor places his flow of orders, aiming to increase his individual expected utility.

Goal No. 1

Find the continuous-time **optimal investment strategy** *under price-impact.*

- A key step: Optimal investment problem *upon market impact* can be written as a constrained optimal investment problem in a *fictitious market without market impact*.

$\rightarrow$ Impose conditions that make the **constraint set non-binding**.

$\rightarrow$ Exploit this representation to **solve the problem** (when possible).

Large Investor's Price Impact

- **Large** in the sense that their orders substantially change MMs' inventory and hence their quoted prices.
  $\rightarrow$ *causing price impact.*
- MMs are risk averse.
- Given MMs price-quoting, each investor places his flow of orders, aiming to increase his individual expected utility.

Goal No. 1

Find the continuous-time **optimal investment strategy** *under price-impact.*

- A key step: Optimal investment problem *upon market impact* can be written as a constrained optimal investment problem in a *fictitious market without market impact*.

$\rightarrow$ Impose conditions that make the **constraint set non-binding**.

$\rightarrow$ Exploit this representation to **solve the problem** (when possible).

Large Investor's Price Impact

- **Large** in the sense that their orders substantially change MMs' inventory and hence their quoted prices.
  $\to$ *causing price impact.*
- MMs are risk averse.
- Given MMs price-quoting, each investor places his flow of orders, aiming to increase his individual expected utility.

Goal No. 1

Find the continuous-time **optimal investment strategy** *under price-impact.*

- A key step: Optimal investment problem *upon market impact* can be written as a constrained optimal investment problem in a *fictitious market without market impact*.

$\to$ Impose conditions that make the **constraint set non-binding**.

$\to$ Exploit this representation to **solve the problem** (when possible).

Large Investor's Price Impact

Goal No. 2

What about **derivative pricing and demand** *under price impact?*

→ Hedging costs are **not** linear anymore.

→ Standard arbitrage-free arguments should be revisited.

→ Even if there is a derivative price that creates arbitrage, the induced gains are *limited*, due to price impact.

→ Since, investors are utility maximizers, they may **optimally ignore an arbitrage**!

Goal No. 3

Could these arbitrage prices arise **endogenously***?*

→ Indeed, through a partial equilibrium argument in segmented markets of the underlying assets.

Large Investor's Price Impact

Goal No. 2

What about **derivative pricing and demand** *under price impact?*

→ Hedging costs are **not** linear anymore.

→ Standard arbitrage-free arguments should be revisited.

→ Even if there is a derivative price that creates arbitrage, the induced gains are *limited*, due to price impact.

→ Since, investors are utility maximizers, they may **optimally ignore an arbitrage**!

Goal No. 3

Could these arbitrage prices arise **endogenously***?*

→ Indeed, through a partial equilibrium argument in segmented markets of the underlying assets.

Large Investor's Price Impact

Goal No. 2

What about **derivative pricing and demand** *under price impact?*

- → Hedging costs are **not** linear anymore.
- → Standard arbitrage-free arguments should be revisited.
- → Even if there is a derivative price that creates arbitrage, the induced gains are *limited*, due to price impact.
- → Since, investors are utility maximizers, they may **optimally ignore an arbitrage**!

Goal No. 3

Could these arbitrage prices arise **endogenously***?*

- → Indeed, through a partial equilibrium argument in segmented markets of the underlying assets.

Large Investor's Price Impact

Goal No. 2

What about **derivative pricing and demand** *under price impact?*

→ Hedging costs are **not** linear anymore.

→ Standard arbitrage-free arguments should be revisited.

→ Even if there is a derivative price that creates arbitrage, the induced gains are *limited*, due to price impact.

→ Since, investors are utility maximizers, they may **optimally ignore an arbitrage**!

Goal No. 3

Could these arbitrage prices arise **endogenously***?*

→ Indeed, through a partial equilibrium argument in segmented markets of the underlying assets.

Outline

1. Motivation & goals
2. Market model & initial steps
3. Connection to a constrained investment problem with no price impact
4. Derivative pricing under price impact
5. Conclusive remarks

Assets & Market Makers

- We begin with $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0 \leq t \leq T}, \mathbb{P})$, where $\{\mathcal{F}_t\}_{0 \leq t \leq T}$ is the natural filtration of a d-dimensional Brownian Motion and $T > 0$ the terminal time.
- A random vector $\Psi \in \mathbb{L}^0(\mathcal{F}_T, \mathbb{R}^d)$ denotes the payoff of the tradeable assets.
- There are M *risk averse* market makers (MMs) that quote prices for Ψ at any time $t \in [0, T]$.
- The utility function of kth MM for terminal wealth is denoted by U_k and his endowment by $\Sigma_0^k \in \mathbb{L}^0(\mathcal{F}_T, \mathbb{R})$, $k = 1, 2, ..., M$ and
- **Standing assumptions on utilities:** strict concavity, increasing, smooth on whole $\mathbb{R}$ with *bounded absolute risk aversion*.

Assets & Market Makers

- We begin with $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0\leq t\leq T}, \mathbb{P})$, where $\{\mathcal{F}_t\}_{0\leq t\leq T}$ is the natural filtration of a d-dimensional Brownian Motion and $T > 0$ the terminal time.
- A random vector $\Psi \in \mathbb{L}^0(\mathcal{F}_T, \mathbb{R}^d)$ denotes the payoff of the tradeable assets.
- There are M *risk averse* market makers (MMs) that quote prices for Ψ at any time $t \in [0, T]$.
- The utility function of kth MM for terminal wealth is denoted by U_k and his endowment by $\Sigma_0^k \in \mathbb{L}^0(\mathcal{F}_T, \mathbb{R})$, $k = 1, 2, ..., M$ and
- **Standing assumptions on utilities:** strict concavity, increasing, smooth on whole $\mathbb{R}$ with *bounded absolute risk aversion*.

Market Makers' Pricing Rule

- Let $\{Q_t\}_{t\in[0,T]}$ denote the aggregate order flow to MMs.
- Let $X_t(Q_t)_{t\in[0,T]}$ be the cash balance (price) asked by all the MMs at time t.
- The way $X_t(Q_t)$ is determined is the following:

MMs' Pricing Rule

1. At any time $t \in [0,T]$, the MMs' total endowment after the transaction:

$$\sum_{k=1}^{M} \Sigma_0^k + X_t(Q_t) - Q_t'\Psi$$

is redistributed among MMs in a **Pareto optimal way** and

2. each MM remains **at indifference**, i.e, there is no increase on the expected utility by entering into the trading of Ψ.

✓ When all MMs have exponential utility: $\rightarrow X_t(Q_t)$ is the *indifference pricing* of the *representative MM* with exponential utility and endowment $\Sigma_0 := \sum_{k=1}^{M} \Sigma_0^k$ and some risk aversion γ.

Market Makers' Pricing Rule

- Let $\{Q_t\}_{t\in[0,T]}$ denote the aggregate order flow to MMs.
- Let $X_t(Q_t)_{t\in[0,T]}$ be the cash balance (price) asked by all the MMs at time t.
- The way $X_t(Q_t)$ is determined is the following:

MMs' Pricing Rule

1. At any time $t \in [0, T]$, the MMs' total endowment after the transaction:

$$\sum_{k=1}^{M} \Sigma_0^k + X_t(Q_t) - Q_t'\Psi$$

   is redistributed among MMs in a **Pareto optimal way** and
2. each MM remains **at indifference**, i.e, there is no increase on the expected utility by entering into the trading of Ψ.

✓ When all MMs have exponential utility: $\rightarrow X_t(Q_t)$ is the *indifference pricing* of the *representative MM* with exponential utility and endowment $\Sigma_0 := \sum_{k=1}^{M} \Sigma_0^k$ and some risk aversion γ.

Market Makers' Pricing Rule

- Let $\{Q_t\}_{t\in[0,T]}$ denote the aggregate order flow to MMs.
- Let $X_t(Q_t)_{t\in[0,T]}$ be the cash balance (price) asked by all the MMs at time t.
- The way $X_t(Q_t)$ is determined is the following:

MMs' Pricing Rule

1. At any time $t \in [0, T]$, the MMs' total endowment after the transaction:

$$\sum_{k=1}^{M} \Sigma_0^k + X_t(Q_t) - Q_t'\Psi$$

   is redistributed among MMs in a **Pareto optimal way** and

2. each MM remains **at indifference**, i.e, there is no increase on the expected utility by entering into the trading of Ψ.

✓ When all MMs have exponential utility: $\rightarrow X_t(Q_t)$ is the *indifference pricing* of the *representative MM* with exponential utility and endowment $\Sigma_0 := \sum_{k=1}^{M} \Sigma_0^k$ and some risk aversion γ.

Some notation

Standing Assumption

For all $q \in \mathbb{R}$, $\quad \mathbb{E}[e^{-\gamma\Sigma_0 + q|\Psi|}] < \infty$.

Under the above assumption,

$$N_t(q) := \mathbb{E}\left[e^{-\gamma\Sigma_0 - \gamma q'\Psi} \middle| \mathcal{F}_t\right], \quad t \leq T,$$

is a strictly positive martingale, and by martingale representation we may write

$$\frac{N_t(q)}{N_0(q)} = \mathcal{E}\left(\int_0^{\cdot} H_s(q)' dB_s\right)_t, \quad t \leq T,$$

for some adapted process $H(q)$ such that $\int_0^T |H_t(q)|^2 dt < \infty$.
Then, define the class of processes

$$Q \in \mathcal{A}_{PI} := \left\{ Q \text{ adapted s.t. } \int_0^T |H_t(Q_t)|^2 dt < \infty \right\},$$

Some notation

Standing Assumption

For all $q \in \mathbb{R}$, $\quad \mathbb{E}[e^{-\gamma\Sigma_0 + q|\Psi|}] < \infty$.

Under the above assumption,

$$N_t(q) := \mathbb{E}\left[e^{-\gamma\Sigma_0 - \gamma q'\Psi} \middle| \mathcal{F}_t\right], \quad t \leq T,$$

is a strictly positive martingale, and by martingale representation we may write

$$\frac{N_t(q)}{N_0(q)} = \mathcal{E}\left(\int_0^{\cdot} H_s(q)' dB_s\right)_t, \quad t \leq T,$$

for some adapted process $H(q)$ such that $\int_0^T |H_t(q)|^2 dt < \infty$.

Then, define the class of processes

$$Q \in \mathcal{A}_{PI} := \left\{ Q \text{ adapted s.t. } \int_0^T |H_t(Q_t)|^2 dt < \infty \right\},$$

Some notation

Standing Assumption

For all $q \in \mathbb{R}$, $\quad \mathbb{E}[e^{-\gamma\Sigma_0 + q|\Psi|}] < \infty$.

Under the above assumption,

$$N_t(q) := \mathbb{E}\left[e^{-\gamma\Sigma_0 - \gamma q'\Psi}\middle|\mathcal{F}_t\right], \quad t \le T,$$

is a strictly positive martingale, and by martingale representation we may write

$$\frac{N_t(q)}{N_0(q)} = \mathcal{E}\left(\int_0^\cdot H_s(q)' dB_s\right)_t, \quad t \le T,$$

for some adapted process $H(q)$ such that $\int_0^T |H_t(q)|^2 dt < \infty$.
Then, define the class of processes

$$Q \in \mathcal{A}_{PI} := \left\{Q \text{ adapted s.t. } \int_0^T |H_t(Q_t)|^2 dt < \infty\right\},$$

A useful representation

- Consider a large investor who submits order flow $\{Q_t\}_{t\in[0,T]}$ to the MM(s).
- Let $\{V_t(Q_t)\}_{t\in[0,T]}$ be his **gains process**, i.e. $V_t(Q_t)$ represents the cash amount that he gets if he sells at time t his cumulative orders. For instance,

$$V_T(Q_T) = -X_T(Q_T) + Q_T'\Psi.$$

Based on the results of Bank & Kramkov ['15], we get the following:

Proposition

For $Q \in \mathcal{A}_{PI}$, and for all $t \in [0, T]$, the gains process takes the form

$$V_t(Q_t) = \frac{1}{\gamma}\int_0^t (H_s(Q_s) - H_s(0))'(dB_s - H_s(0)ds) - \frac{1}{2\gamma}\int_0^t |H_s(Q_s) - H_s(0)|^2\, ds$$

Investor's investment problem, under endowment Σ_1

$$u(x;\Sigma_1) := \sup_{Q\in\mathcal{A}_{PI}} \mathbb{E}[U(x + V_T(Q) + \Sigma_1)].$$

A useful representation

- Consider a large investor who submits order flow $\{Q_t\}_{t\in[0,T]}$ to the MM(s).
- Let $\{V_t(Q_t)\}_{t\in[0,T]}$ be his **gains process**, i.e. $V_t(Q_t)$ represents the cash amount that he gets if he sells at time t his cumulative orders. For instance,

$$V_T(Q_T) = -X_T(Q_T) + Q_T'\Psi.$$

Based on the results of Bank & Kramkov ['15], we get the following:

Proposition

For $Q \in \mathcal{A}_{PI}$, and for all $t \in [0, T]$, the gains process takes the form

$$V_t(Q_t) = \frac{1}{\gamma}\int_0^t (H_s(Q_s) - H_s(0))'(dB_s - H_s(0)ds) - \frac{1}{2\gamma}\int_0^t |H_s(Q_s) - H_s(0)|^2\, ds$$

Investor's investment problem, under endowment Σ_1

$$u(x;\Sigma_1) := \sup_{Q\in\mathcal{A}_{PI}} \mathbb{E}[U(x + V_T(Q) + \Sigma_1)].$$

A useful representation

- Consider a large investor who submits order flow $\{Q_t\}_{t\in[0,T]}$ to the MM(s).
- Let $\{V_t(Q_t)\}_{t\in[0,T]}$ be his **gains process**, i.e. $V_t(Q_t)$ represents the cash amount that he gets if he sells at time t his cumulative orders. For instance,

$$V_T(Q_T) = -X_T(Q_T) + Q_T'\Psi.$$

Based on the results of Bank & Kramkov ['15], we get the following:

Proposition

For $Q \in \mathcal{A}_{PI}$, and for all $t \in [0, T]$, the gains process takes the form

$$V_t(Q_t) = \frac{1}{\gamma}\int_0^t (H_s(Q_s) - H_s(0))'(dB_s - H_s(0)ds) - \frac{1}{2\gamma}\int_0^t |H_s(Q_s) - H_s(0)|^2\, ds$$

Investor's investment problem, under endowment Σ_1

$$u(x;\Sigma_1) := \sup_{Q\in\mathcal{A}_{PI}} \mathbb{E}[U\left(x + V_T(Q) + \Sigma_1\right)].$$

Outline

1 Motivation & goals

2 Market model & initial steps

3 Connection to a constrained investment problem with no price impact

4 Derivative pricing under price impact

5 Conclusive remarks

A fictitious related market

Define

$$\frac{dS_t}{S_t} = \lambda_t dt + dB_t, \quad t \in [0, T],$$

for an adapted d-dimensional process λ, such that $\int_0^T |\lambda_t|^2 dt < \infty$.
By construction, there is a unique measure $\mathbb{Q}_0$ on $\mathcal{F}_T$ under which S is a martingale. $\mathbb{Q}_0$ has density

$$\left.\frac{d\mathbb{Q}_0}{d\mathbb{P}}\right|_{\mathcal{F}_T} = \mathcal{E}\left(-\int_0^T \lambda_t' dB_t\right).$$

Self-financing trading strategies are denoted by π (proportions of wealth) and the induced wealth process' dynamics

$$\frac{dX_t(\pi)}{X_t(\pi)} = \pi_t' \left(\lambda_t dt + dB_t\right), \quad t \in [0, T].$$

With initial wealth $X_0 = e^{\gamma x}$ the terminal wealth is

$$X_T(\pi) = \exp\left(\gamma x + \int_0^T \pi_t'(dB_t + \lambda_t dt) - \frac{1}{2}\int_0^T |\pi_t|^2 dt\right).$$

Some simple observations

Recall that large investor's gain process is

$$V_t(Q_t) = \frac{1}{\gamma}\int_0^t (H_s(Q_s) - H_s(0))'(dB_s - H_s(0)ds) - \frac{1}{2\gamma}\int_0^t |H_s(Q_s) - H_s(0)|^2\, ds$$

A key observation

Set $\lambda_t = -H_t(0)$, assume that $\pi_t = H_t(Q_t) - H_t(0)$ and compare $X_T(\pi)$ with $V_T(Q)$. We get that

$$X_T(\pi) = e^{\gamma x + \gamma V_T(Q)} \quad \Longrightarrow \quad x + V_T(Q) = \frac{1}{\gamma}\log(X_T(\pi)).$$

- For $Q \in \mathcal{A}_{PI}$ we can construct π. For the reverse we need π_t to belong in the random constraint set $\mathcal{K}_t^o$, where

$$\mathcal{K}_t := \left\{H_t(q) : q \in \mathbb{R}^d\right\}, \quad \mathcal{K}_t^o := \left\{H_t(q) - H_t(0) \mid q \in \mathbb{R}^d\right\}.$$

- Therefore, we define the acceptable strategies

$$\mathcal{A} := \left\{\pi \text{ adapted} : \int_0^T |\pi_t|^2 dt < \infty\right\}, \mathcal{A}_C := \left\{\pi \in \mathcal{A} : \pi_t \in \mathcal{K}_t^0, t \leq T\right\}.$$

Some simple observations

Recall that large investor's gain process is

$$V_t(Q_t) = \frac{1}{\gamma}\int_0^t (H_s(Q_s) - H_s(0))'(dB_s - H_s(0)ds) - \frac{1}{2\gamma}\int_0^t |H_s(Q_s) - H_s(0)|^2\, ds$$

A key observation

Set $\lambda_t = -H_t(0)$, assume that $\pi_t = H_t(Q_t) - H_t(0)$ and compare $X_T(\pi)$ with $V_T(Q)$. We get that

$$X_T(\pi) = e^{\gamma x + \gamma V_T(Q)} \quad \Longrightarrow \quad x + V_T(Q) = \frac{1}{\gamma}\log(X_T(\pi)).$$

- For $Q \in \mathcal{A}_{PI}$ we can construct π. For the reverse we need π_t to belong in the random constraint set $\mathcal{K}_t^o$, where

$$\mathcal{K}_t := \left\{H_t(q) : q \in \mathbb{R}^d\right\}, \quad \mathcal{K}_t^o := \left\{H_t(q) - H_t(0) \mid q \in \mathbb{R}^d\right\}.$$

- Therefore, we define the acceptable strategies

$$\mathcal{A} := \left\{\pi \text{ adapted} : \int_0^T |\pi_t|^2 dt < \infty\right\}, \mathcal{A}_C := \left\{\pi \in \mathcal{A} : \pi_t \in \mathcal{K}_t^0, t \le T\right\}.$$

Some simple observations

Recall that large investor's gain process is

$$V_t(Q_t) = \frac{1}{\gamma}\int_0^t (H_s(Q_s) - H_s(0))'(dB_s - H_s(0)ds) - \frac{1}{2\gamma}\int_0^t |H_s(Q_s) - H_s(0)|^2\, ds$$

A key observation

Set $\lambda_t = -H_t(0)$, assume that $\pi_t = H_t(Q_t) - H_t(0)$ and compare $X_T(\pi)$ with $V_T(Q)$. We get that

$$X_T(\pi) = e^{\gamma x + \gamma V_T(Q)} \quad \Longrightarrow \quad x + V_T(Q) = \frac{1}{\gamma}\log(X_T(\pi)).$$

- For $Q \in \mathcal{A}_{PI}$ we can construct π. For the reverse we need π_t to belong in the random constraint set $\mathcal{K}_t^o$, where
$$\mathcal{K}_t := \left\{H_t(q) : q \in \mathbb{R}^d\right\}, \quad \mathcal{K}_t^o := \left\{H_t(q) - H_t(0) \mid q \in \mathbb{R}^d\right\}.$$

- Therefore, we define the acceptable strategies

$$\mathcal{A} := \left\{\pi \text{ adapted} : \int_0^T |\pi_t|^2 dt < \infty\right\}, \mathcal{A}_C := \left\{\pi \in \mathcal{A} : \pi_t \in \mathcal{K}_t^0, t \leq T\right\}.$$

Some simple observations

Recall that large investor's gain process is

$$V_t(Q_t) = \frac{1}{\gamma}\int_0^t (H_s(Q_s) - H_s(0))'(dB_s - H_s(0)ds) - \frac{1}{2\gamma}\int_0^t |H_s(Q_s) - H_s(0)|^2\, ds$$

A key observation

Set $\lambda_t = -H_t(0)$, assume that $\pi_t = H_t(Q_t) - H_t(0)$ and compare $X_T(\pi)$ with $V_T(Q)$. We get that

$$X_T(\pi) = e^{\gamma x + \gamma V_T(Q)} \quad \Longrightarrow \quad x + V_T(Q) = \frac{1}{\gamma}\log(X_T(\pi)).$$

- For $Q \in \mathcal{A}_{PI}$ we can construct π. For the reverse we need π_t to belong in the random constraint set $\mathcal{K}_t^o$, where
  $$\mathcal{K}_t := \left\{H_t(q) : q \in \mathbb{R}^d\right\}, \quad \mathcal{K}_t^o := \left\{H_t(q) - H_t(0) \mid q \in \mathbb{R}^d\right\}.$$
- Therefore, we define the acceptable strategies

$$\mathcal{A} := \left\{\pi \text{ adapted} : \int_0^T |\pi_t|^2 dt < \infty\right\}, \mathcal{A}_C := \left\{\pi \in \mathcal{A} : \pi_t \in \mathcal{K}_t^0,\, t \le T\right\}.$$

A representation of optimal investment problem

Define the utility field $\tilde{U}(w,\omega) : (0,\infty) \times \Omega$ by

$$\tilde{U}(w,\omega) \; := \; U\left(\frac{1}{\gamma}\log(w) + \Sigma_1(\omega)\right),$$

and the value functions

$$\tilde{u}_C(x;\Sigma_1) \; := \; \sup_{\pi \in \mathcal{A}_C} \mathbb{E}[\tilde{U}\left(X_T(\pi),\Sigma_1\right) | X_0 = e^{\gamma x}],$$

and

$$\tilde{u}(x;\Sigma_1) \; := \; \sup_{\pi \in \mathcal{A}} \mathbb{E}[\tilde{U}\left(X_T(\pi),\Sigma_1\right) | X_0 = e^{\gamma x}].$$

Theorem

With the above notation

$$u(x;\Sigma_1) = \tilde{u}_C(x;\Sigma_1) \leq \tilde{u}(x;\Sigma_1).$$

A representation of optimal investment problem

Define the utility field $\tilde{U}(w, \omega) : (0, \infty) \times \Omega$ by

$$\tilde{U}(w, \omega) := U\left(\frac{1}{\gamma}\log(w) + \Sigma_1(\omega)\right),$$

and the value functions

$$\tilde{u}_C(x; \Sigma_1) := \sup_{\pi \in \mathcal{A}_C} \mathbb{E}[\tilde{U}(X_T(\pi), \Sigma_1) \,|X_0 = e^{\gamma x}],$$

and

$$\tilde{u}(x; \Sigma_1) := \sup_{\pi \in \mathcal{A}} \mathbb{E}[\tilde{U}(X_T(\pi), \Sigma_1) \,|X_0 = e^{\gamma x}].$$

Theorem

With the above notation

$$u(x; \Sigma_1) = \tilde{u}_C(x; \Sigma_1) \leq \tilde{u}(x; \Sigma_1).$$

An indicative example

- Let $U(x) = -e^{-\alpha x}$ and define

$$\left.\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}}\right|_{\mathcal{F}_T} = \frac{e^{-\alpha\Sigma_1}}{\mathbb{E}[e^{-\alpha\Sigma_1}]}.$$

Proposition

$$u(0;\Sigma_1) = \tilde{u}_C(0;\Sigma_1) = \frac{\alpha}{\gamma}\mathbb{E}[e^{-\alpha\Sigma_1}]\left(\sup_{\pi\in\mathcal{A}_C}\tilde{\mathbb{E}}\left[\frac{1}{p}\left(X_T(\pi)\right)^p|X_0 = 1\right]\right)$$

$$\tilde{u}(0;\Sigma_1) = \frac{\alpha}{\gamma}\mathbb{E}[e^{-\alpha\Sigma_1}]\left(\sup_{\pi\in\mathcal{A}}\tilde{\mathbb{E}}\left[\frac{1}{p}\left(X_T(\pi)\right)^p|X_0 = 1\right]\right)$$

where $p := -\alpha/\gamma$.

✓ *From exponential and price impact to power with no price impact (but with constrains).*

An indicative example

- Let $U(x) = -e^{-\alpha x}$ and define

$$\left.\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}}\right|_{\mathcal{F}_T} = \frac{e^{-\alpha\Sigma_1}}{\mathbb{E}[e^{-\alpha\Sigma_1}]}.$$

Proposition

$$u(0;\Sigma_1) = \tilde{u}_C(0;\Sigma_1) = \frac{\alpha}{\gamma}\mathbb{E}[e^{-\alpha\Sigma_1}]\left(\sup_{\pi\in\mathcal{A}_C}\tilde{\mathbb{E}}\left[\frac{1}{p}\left(X_T(\pi)\right)^p|X_0 = 1\right]\right)$$

$$\tilde{u}(0;\Sigma_1) = \frac{\alpha}{\gamma}\mathbb{E}[e^{-\alpha\Sigma_1}]\left(\sup_{\pi\in\mathcal{A}}\tilde{\mathbb{E}}\left[\frac{1}{p}\left(X_T(\pi)\right)^p|X_0 = 1\right]\right)$$

where $p := -\alpha/\gamma$.

✓ *From exponential and price impact to power with no price impact (but with constrains).*

On constrained problem

- In the related literature on utility maximization under random constrains the standard assumption is that constrained set is convex and closed.
- However, here $\mathcal{K}_t^o$ is typically neither convex nor closed!

Bachelier model

Let

$$\Sigma_0 = \int_0^T f_t' dB_t \quad \text{and} \quad \Psi = \int_0^T \psi_t' dB_t,$$

where $f \in \mathbb{L}^2([0,T];\mathbb{R}^d)$ and $\psi \in \mathbb{L}^2([0,T];\mathbb{R}^{d\times d})$. Then,

$$N_t(q) = e^{\frac{1}{2}\gamma^2 \int_0^t |f_s + \psi_s q|^2 ds} \mathcal{E}\left(-\gamma \int_0^t (f_s + \psi_s q)' dB_s\right) \quad t \in [0,T].$$

Thus,

$$H_t(q) = -\gamma(f_t + \psi_t q) \quad \text{and} \quad H_t(q) - H_t(0) = -\gamma \psi_t q.$$

Hence, if ψ_t is invertible, $\mathcal{K}_t = \mathcal{K}_t^o = \mathbb{R}^d$ with $\pi_t = H_t(Q_t) - H_t(0)$ implying $Q_t = -(\gamma\psi_t)^{-1}\pi_t$.

On constrained problem

- In the related literature on utility maximization under random constrains the standard assumption is that constrained set is convex and closed.
- However, here $\mathcal{K}_t^o$ is typically neither convex nor closed!

Bachelier model

Let

$$\Sigma_0 = \int_0^T f_t' dB_t \quad \text{and} \quad \Psi = \int_0^T \psi_t' dB_t,$$

where $f \in \mathbb{L}^2([0, T]; \mathbb{R}^d)$ and $\psi \in \mathbb{L}^2([0, T]; \mathbb{R}^{d \times d})$. Then,

$$N_t(q) = e^{\frac{1}{2}\gamma^2 \int_0^t |f_s + \psi_s q|^2 ds} \mathcal{E}\left(-\gamma \int_0^t (f_s + \psi_s q)' dB_s\right) \quad t \in [0, T].$$

Thus,

$$H_t(q) = -\gamma(f_t + \psi_t q) \quad \text{and} \quad H_t(q) - H_t(0) = -\gamma \psi_t q.$$

Hence, if ψ_t is invertible, $\mathcal{K}_t = \mathcal{K}_t^o = \mathbb{R}^d$ with $\pi_t = H_t(Q_t) - H_t(0)$ implying $Q_t = -(\gamma \psi_t)^{-1} \pi_t$.

Solving the optimal investment problem

Impose the standing assumption and that large investor has exponential utility. Then,

$$\frac{e^{-\frac{\alpha\gamma}{\alpha+\gamma}(\Sigma_1+\Sigma_0)}}{\mathbb{E}\left[e^{-\frac{\alpha\gamma}{\alpha+\gamma}(\Sigma_1+\Sigma_0)}\right]} = \mathcal{E}\left(\int_0^T M_t' dB_t\right),$$

for some adapted process M with $\int_0^T |M_t|^2 dt < \infty$.

A key assumption

$$M_t \in \mathcal{K}_t, \qquad \forall t \in [0, T].$$

Proposition

Under the standing and key assumptions, the **constrain set is non-binding** and in fact,

$$\begin{aligned} u(0;\Sigma_1) &= \tilde{u}_C(0;\Sigma_1) = \tilde{u}(0;\Sigma_1) \\ &= -\mathbb{E}\left[e^{-\gamma\Sigma_0}\right]^{-\frac{\alpha}{\gamma}} \times \mathbb{E}\left[e^{-\frac{\alpha\gamma}{\alpha+\gamma}(\Sigma_0+\Sigma_1)}\right]^{\frac{\alpha+\gamma}{\gamma}}. \end{aligned}$$

Also, $\hat{\pi}_t = M_t - H_t(0)$, for all $t \le T$.

Solving the optimal investment problem

Impose the standing assumption and that large investor has exponential utility. Then,

$$\frac{e^{-\frac{\alpha\gamma}{\alpha+\gamma}(\Sigma_1+\Sigma_0)}}{\mathbb{E}\left[e^{-\frac{\alpha\gamma}{\alpha+\gamma}(\Sigma_1+\Sigma_0)}\right]} = \mathcal{E}\left(\int_0^T M_t' dB_t\right),$$

for some adapted process M with $\int_0^T |M_t|^2 dt < \infty$.

A key assumption

$$M_t \in \mathcal{K}_t, \qquad \forall t \in [0, T].$$

Proposition

Under the standing and key assumptions, the **constrain set is non-binding** and in fact,

$$u(0;\Sigma_1) = \tilde{u}_C(0;\Sigma_1) = \tilde{u}(0;\Sigma_1)$$
$$= -\mathbb{E}\left[e^{-\gamma\Sigma_0}\right]^{-\frac{\alpha}{\gamma}} \times \mathbb{E}\left[e^{-\frac{\alpha\gamma}{\alpha+\gamma}(\Sigma_0+\Sigma_1)}\right]^{\frac{\alpha+\gamma}{\gamma}}.$$

Also, $\hat{\pi}_t = M_t - H_t(0)$, for all $t \leq T$.

Solving the optimal investment problem

Impose the standing assumption and that large investor has exponential utility. Then,

$$\frac{e^{-\frac{\alpha\gamma}{\alpha+\gamma}(\Sigma_1+\Sigma_0)}}{\mathbb{E}\left[e^{-\frac{\alpha\gamma}{\alpha+\gamma}(\Sigma_1+\Sigma_0)}\right]} = \mathcal{E}\left(\int_0^T M_t' dB_t\right),$$

for some adapted process M with $\int_0^T |M_t|^2 dt < \infty$.

A key assumption

$$M_t \in \mathcal{K}_t, \qquad \forall t \in [0, T].$$

Proposition

Under the standing and key assumptions, the **constrain set is non-binding** and in fact,

$$\begin{aligned} u(0;\Sigma_1) &= \tilde{u}_C(0;\Sigma_1) = \tilde{u}(0;\Sigma_1) \\ &= -\mathbb{E}\left[e^{-\gamma\Sigma_0}\right]^{-\frac{\alpha}{\gamma}} \times \mathbb{E}\left[e^{-\frac{\alpha\gamma}{\alpha+\gamma}(\Sigma_0+\Sigma_1)}\right]^{\frac{\alpha+\gamma}{\gamma}}. \end{aligned}$$

Also, $\hat{\pi}_t = M_t - H_t(0)$, for all $t \leq T$.

Example I

Endowments as portfolios of Ψ

Let $\Sigma_0 = k_0'\Psi$ and $\Sigma_1 = k_1'\Psi$ for some $k_0, k_1 \in \mathbb{R}^d$. Recall that

$$N_t(q) := \mathbb{E}\left[e^{-\gamma\Sigma_0 - \gamma q'\Psi}\middle|\mathcal{F}_t\right], \quad \text{and} \quad \frac{N_t(q)}{N_0(q)} = \mathbb{E}\left(\int_0^{\cdot} H_s(q)'dB_s\right)_t$$

We immediately get that

$$M_t = H_t\left(\frac{\alpha k_1 - \gamma k_0}{\alpha + \gamma}\right), \quad \text{i.e.,} \quad M_t \in \mathcal{K}_t, \forall t \in [0, T]$$

and since $\hat{\pi}_t = M_t - H_t(0)$, we also get that

$$\hat{Q}_t \equiv \hat{q} = \frac{\alpha k_1 - \gamma k_0}{\alpha + \gamma}, \quad \forall t \in [0, T].$$

✓ *We have a similar situation when $\Sigma_0 = k_0'\Psi + Y_0$ and $\Sigma_1 = k_1'\Psi + +Y_1$, where (Y_0, Y_1) and Ψ are independent.*

Example I

Endowments as portfolios of Ψ

Let $\Sigma_0 = k_0'\Psi$ and $\Sigma_1 = k_1'\Psi$ for some $k_0, k_1 \in \mathbb{R}^d$. Recall that

$$N_t(q) := \mathbb{E}\left[e^{-\gamma\Sigma_0 - \gamma q'\Psi} \middle| \mathcal{F}_t\right], \quad \text{and} \quad \frac{N_t(q)}{N_0(q)} = \mathbb{E}\left(\int_0^\cdot H_s(q)'dB_s\right)_t$$

We immediately get that

$$M_t = H_t\left(\frac{\alpha k_1 - \gamma k_0}{\alpha + \gamma}\right), \quad \text{i.e.,} \quad M_t \in \mathcal{K}_t, \forall t \in [0, T]$$

and since $\hat{\pi}_t = M_t - H_t(0)$, we also get that

$$\hat{Q}_t \equiv \hat{q} = \frac{\alpha k_1 - \gamma k_0}{\alpha + \gamma}, \quad \forall t \in [0, T].$$

✓ *We have a similar situation when $\Sigma_0 = k_0'\Psi + Y_0$ and $\Sigma_1 = k_1'\Psi + +Y_1$, where (Y_0, Y_1) and Ψ are independent.*

Example II

Bachelier Model

- Recall that $\Sigma_0 = \int_0^T f_t' dB_t$ and $\Psi = \int_0^T \psi_t' dB_t$.
- We have seen that $\mathcal{K}_t^o = \mathbb{R}^d$, so the crucial assumption holds.
- Assume also that $\Sigma_1 = \int_0^T g_t' dB_t$.
- *What is the optimal demand?*
- We have seen that $H_t(q) - H_t(0) = -\gamma\psi_t q$, and we readily have that $\forall t \in [0, T]$

$$M_t = -\frac{\alpha\gamma}{\alpha+\gamma}(f_t + g_t) \quad \text{and} \quad \hat{\pi}_t = M_t - H_t(0) = \frac{\gamma}{\alpha+\gamma}(\gamma f_t - \alpha g_t).$$

Thus, the equality $-\gamma\psi_t\hat{Q}_t = H_t(\hat{Q}_t) - H_t(0) = \hat{\pi}_t$ gives

$$\hat{Q}_t = \frac{1}{\alpha+\gamma}\psi_t^{-1}(\alpha g_t - \gamma f_t).$$

Example II

Bachelier Model

- Recall that $\Sigma_0 = \int_0^T f_t' dB_t$ and $\Psi = \int_0^T \psi_t' dB_t$.
- We have seen that $\mathcal{K}_t^o = \mathbb{R}^d$, so the crucial assumption holds.
- Assume also that $\Sigma_1 = \int_0^T g_t' dB_t$.
- *What is the optimal demand?*
- We have seen that $H_t(q) - H_t(0) = -\gamma\psi_t q$, and we readily have that $\forall t \in [0, T]$

$$M_t = -\frac{\alpha\gamma}{\alpha+\gamma}\left(f_t + g_t\right) \quad \text{and} \quad \hat{\pi}_t = M_t - H_t(0) = \frac{\gamma}{\alpha+\gamma}\left(\gamma f_t - \alpha g_t\right).$$

Thus, the equality $-\gamma\psi_t\hat{Q}_t = H_t(\hat{Q}_t) - H_t(0) = \hat{\pi}_t$ gives

$$\hat{Q}_t = \frac{1}{\alpha+\gamma}\psi_t^{-1}\left(\alpha g_t - \gamma f_t\right).$$

Outline

1. Motivation & goals
2. Market model & initial steps
3. Connection to a constrained investment problem with no price impact
4. Derivative pricing under price impact
5. Conclusive remarks

Introducing a derivative contract

- Consider a single contingent claim with $\mathcal{F}_T$ measurable payoff h.
- MMs do *not* make the market of h.
- However, investor could hedge his positions on h by trading the underlying market Ψ through MMs.
- Note that if $\mathcal{K}_t^o = \mathbb{R}^d$, $\forall t \in [0, T]$, large investor can fully hedge.
- Indeed, for every $u \neq 0$ units of h, there is an order flow $Q \in \mathcal{A}_{PI}$ and a (per unit) initial capital $\overline{h}(u)$ such that

$$u\overline{h}(u) + V_T(Q) = uh.$$

- In fact, $\overline{h}(u)$ is the MM's indifference value of selling u units of h, given by

$$\overline{h}(u) := \frac{1}{\gamma u} \log\left(\mathbb{E}^0\left[e^{\gamma u h}\right]\right).$$

- Note that the value $\overline{h}(u)$ is increasing for $u > 0$, but not linear.

Introducing a derivative contract

- Consider a single contingent claim with $\mathcal{F}_T$ measurable payoff h.
- MMs do *not* make the market of h.
- However, investor could hedge his positions on h by trading the underlying market Ψ through MMs.
- Note that if $\mathcal{K}_t^o = \mathbb{R}^d$, $\forall t \in [0, T]$, large investor can fully hedge.
- Indeed, for every $\mathfrak{u} \neq 0$ units of h, there is an order flow $Q \in \mathcal{A}_{PI}$ and a (per unit) initial capital $\overline{h}(\mathfrak{u})$ such that

$$\mathfrak{u}\overline{h}(\mathfrak{u}) + V_T(Q) = \mathfrak{u}h.$$

- In fact, $\overline{h}(\mathfrak{u})$ is the MM's indifference value of selling $\mathfrak{u}$ units of h, given by

$$\overline{h}(\mathfrak{u}) := \frac{1}{\gamma\mathfrak{u}} \log\left(\mathbb{E}^0\left[e^{\gamma\mathfrak{u}h}\right]\right).$$

- Note that the value $\overline{h}(\mathfrak{u})$ is increasing for $\mathfrak{u} > 0$, but not linear.

Introducing a derivative contract

- Consider a single contingent claim with $\mathcal{F}_T$ measurable payoff h.
- MMs do *not* make the market of h.
- However, investor could hedge his positions on h by trading the underlying market Ψ through MMs.
- Note that if $\mathcal{K}_t^o = \mathbb{R}^d$, $\forall t \in [0, T]$, large investor can fully hedge.
- Indeed, for every $\mathfrak{u} \neq 0$ units of h, there is an order flow $Q \in \mathcal{A}_{PI}$ and a (per unit) initial capital $\overline{h}(\mathfrak{u})$ such that

$$\mathfrak{u}\overline{h}(\mathfrak{u}) + V_T(Q) = \mathfrak{u}h.$$

- In fact, $\overline{h}(\mathfrak{u})$ is the MM's indifference value of selling $\mathfrak{u}$ units of h, given by

$$\overline{h}(\mathfrak{u}) := \frac{1}{\gamma\mathfrak{u}} \log\left(\mathbb{E}^0\left[e^{\gamma\mathfrak{u}h}\right]\right).$$

- Note that the value $\overline{h}(\mathfrak{u})$ is increasing for $\mathfrak{u} > 0$, but not linear.

Introducing a derivative contract

- Consider a single contingent claim with $\mathcal{F}_T$ measurable payoff h.
- MMs do *not* make the market of h.
- However, investor could hedge his positions on h by trading the underlying market Ψ through MMs.
- Note that if $\mathcal{K}_t^o = \mathbb{R}^d$, $\forall t \in [0, T]$, large investor can fully hedge.
- Indeed, for every $\mathfrak{u} \neq 0$ units of h, there is an order flow $Q \in \mathcal{A}_{PI}$ and a (per unit) initial capital $\overline{h}(\mathfrak{u})$ such that

$$\mathfrak{u}\overline{h}(\mathfrak{u}) + V_T(Q) = \mathfrak{u}h.$$

- In fact, $\overline{h}(\mathfrak{u})$ is the MM's indifference value of selling $\mathfrak{u}$ units of h, given by

$$\overline{h}(\mathfrak{u}) := \frac{1}{\gamma\mathfrak{u}} \log\left(\mathbb{E}^0\left[e^{\gamma\mathfrak{u}h}\right]\right).$$

- Note that the value $\overline{h}(\mathfrak{u})$ is increasing for $\mathfrak{u} > 0$, but not linear.

Derivative pricing, price impact and *arbitrage*

Arbitrage-free price for all positions

A price $p \in \mathbb{R}$ is an arbitrage-free price for all position in h, when:
For all $Q \in \mathcal{A}_{PI}$ and $\mathfrak{u} \in \mathbb{R}$, if $\mathfrak{u}p + V_T(Q) - \mathfrak{u}h \geq 0$ a.s., then $\mathfrak{u}p + V_T(Q) = \mathfrak{u}h$ a.s.

Arbitrage-free price for position $\mathfrak{u}$

A price $p \in \mathbb{R}$ is an arbitrage-free price for a position $\mathfrak{u} > 0$ in h, when:
For all $Q \in \mathcal{A}_{PI}$, if $\mathfrak{u}p + V_T(Q) - \mathfrak{u}h \geq 0$ a.s., then $\mathfrak{u}p + V_T(Q) = \mathfrak{u}h$ a.s.

Proposition

- The range of arbitrage-free prices for h is the singleton $\mathbb{E}^0[h]$.
- For any fixed $\mathfrak{u} > 0$, the range of arbitrage-free prices for h at position $\mathfrak{u}$ is the closed interval $[-\bar{h}(\mathfrak{u}); \bar{h}(\mathfrak{u})]$.
- If p is an arbitrage-free price for position $\mathfrak{u} > 0$, then p is an arbitrage-free price at all positions $\mathfrak{u}' \geq \mathfrak{u}$.

Derivative pricing, price impact and *arbitrage*

Arbitrage-free price for all positions

A price $p \in \mathbb{R}$ is an arbitrage-free price for all position in h, when:
For all $Q \in \mathcal{A}_{PI}$ and $\mathfrak{u} \in \mathbb{R}$, if $\mathfrak{u}p + V_T(Q) - \mathfrak{u}h \geq 0$ a.s., then $\mathfrak{u}p + V_T(Q) = \mathfrak{u}h$ a.s.

Arbitrage-free price for position $\mathfrak{u}$

A price $p \in \mathbb{R}$ is an arbitrage-free price for a position $\mathfrak{u} > 0$ in h, when:
For all $Q \in \mathcal{A}_{PI}$, if $\mathfrak{u}p + V_T(Q) - \mathfrak{u}h \geq 0$ a.s., then $\mathfrak{u}p + V_T(Q) = \mathfrak{u}h$ a.s.

Proposition

- The range of arbitrage-free prices for h is the singleton $\mathbb{E}^0[h]$.
- For any fixed $\mathfrak{u} > 0$, the range of arbitrage-free prices for h at position $\mathfrak{u}$ is the closed interval $[-\overline{h}(\mathfrak{u}); \overline{h}(\mathfrak{u})]$.
- If p is an arbitrage-free price for position $\mathfrak{u} > 0$, then p is an arbitrage-free price at all positions $\mathfrak{u}' \geq \mathfrak{u}$.

Derivative pricing, price impact and *arbitrage*

Arbitrage-free price for all positions

A price $p \in \mathbb{R}$ is an arbitrage-free price for all position in h, when:
For all $Q \in \mathcal{A}_{PI}$ and $\mathfrak{u} \in \mathbb{R}$, if $\mathfrak{u}p + V_T(Q) - \mathfrak{u}h \geq 0$ a.s., then $\mathfrak{u}p + V_T(Q) = \mathfrak{u}h$ a.s.

Arbitrage-free price for position $\mathfrak{u}$

A price $p \in \mathbb{R}$ is an arbitrage-free price for a position $\mathfrak{u} > 0$ in h, when:
For all $Q \in \mathcal{A}_{PI}$, if $\mathfrak{u}p + V_T(Q) - \mathfrak{u}h \geq 0$ a.s., then $\mathfrak{u}p + V_T(Q) = \mathfrak{u}h$ a.s.

Proposition

- The range of arbitrage-free prices for h is the singleton $\mathbb{E}^0[h]$.
- For any fixed $\mathfrak{u} > 0$, the range of arbitrage-free prices for h at position $\mathfrak{u}$ is the closed interval $[-\overline{h}(\mathfrak{u}); \overline{h}(\mathfrak{u})]$.
- If p is an arbitrage-free price for position $\mathfrak{u} > 0$, then p is an arbitrage-free price at all positions $\mathfrak{u}' \geq \mathfrak{u}$.

Derivative pricing, price impact and *arbitrage*

Arbitrage-free price for all positions

A price $p \in \mathbb{R}$ is an arbitrage-free price for all position in h, when:
For all $Q \in \mathcal{A}_{PI}$ and $\mathfrak{u} \in \mathbb{R}$, if $\mathfrak{u}p + V_T(Q) - \mathfrak{u}h \geq 0$ a.s., then $\mathfrak{u}p + V_T(Q) = \mathfrak{u}h$ a.s.

Arbitrage-free price for position $\mathfrak{u}$

A price $p \in \mathbb{R}$ is an arbitrage-free price for a position $\mathfrak{u} > 0$ in h, when:
For all $Q \in \mathcal{A}_{PI}$, if $\mathfrak{u}p + V_T(Q) - \mathfrak{u}h \geq 0$ a.s., then $\mathfrak{u}p + V_T(Q) = \mathfrak{u}h$ a.s.

Proposition

- The range of arbitrage-free prices for h is the singleton $\mathbb{E}^0[h]$.
- For any fixed $\mathfrak{u} > 0$, the range of arbitrage-free prices for h at position $\mathfrak{u}$ is the closed interval $[-\overline{h}(\mathfrak{u}); \overline{h}(\mathfrak{u})]$.
- If p is an arbitrage-free price for position $\mathfrak{u} > 0$, then p is an arbitrage-free price at all positions $\mathfrak{u}' \geq \mathfrak{u}$.

Limited arbitrage

- If the large investor gets a price for h different than $\mathbb{E}^0[h]$, an arbitrage opportunity arises thanks to his price impact.
- However, because of not linearity, the arbitrage cannot be exploited for arbitrarily large units of h.
- In other words, the gains from the arbitrage are limited up to a certain position u^*.
- Note however that large investors is a utility maximizer with hedging needs. Hence, exploiting the limited arbitrage may be less preferable than reducing the risk exposure.
- Investor may ignore certain cash in favor of a higher expected utility.

✓ *But who is going to ask/bid an arbitrage price?*

Limited arbitrage

- If the large investor gets a price for h different than $\mathbb{E}^0[h]$, an arbitrage opportunity arises thanks to his price impact.
- However, because of not linearity, the arbitrage cannot be exploited for arbitrarily large units of h.
- In other words, the gains from the arbitrage are limited up to a certain position $\mathfrak{u}^*$.
- Note however that large investors is a utility maximizer with hedging needs. Hence, exploiting the limited arbitrage may be less preferable than reducing the risk exposure.
- Investor may ignore certain cash in favor of a higher expected utility.

✓ *But who is going to ask/bid an arbitrage price?*

Limited arbitrage

- If the large investor gets a price for h different than $\mathbb{E}^0[h]$, an arbitrage opportunity arises thanks to his price impact.
- However, because of not linearity, the arbitrage cannot be exploited for arbitrarily large units of h.
- In other words, the gains from the arbitrage are limited up to a certain position $\mathfrak{u}^*$.
- Note however that large investors is a utility maximizer with hedging needs. Hence, exploiting the limited arbitrage may be less preferable than reducing the risk exposure.
- Investor may ignore certain cash in favor of a higher expected utility.

✓ *But who is going to ask/bid an arbitrage price?*

Partial equilibrium in segmented markets

- Suppose that there are two large investors, labeled A and B.
- They trade with different MMs in **segmented** markets (possibly with different securities too).
- The large investors trade to each other the derivative h at a partial equilibrium price & quantity (PEPQ), as introduced by A. and Žitković ['10].
- A PEPQ of h is a pair $(p^*, \mathfrak{u}^*) \in \mathbb{R}^2$ if

$$\mathfrak{u}^* \in \underset{\mathfrak{u}\in\mathbb{R}}{\operatorname{argmax}}\{u_A(x_A - p^*\mathfrak{u}, \Sigma_A + \mathfrak{u}h)\} \bigcap \underset{\mathfrak{u}\in\mathbb{R}}{\operatorname{argmax}}\{u_B(x_B + p^*\mathfrak{u}, \Sigma_B - \mathfrak{u}h)\}.$$

Proposition

Let both large investors have exponential utility and assume that $\gamma_A\Sigma_0^A - \gamma_B\Sigma_0^B$ and h are not constants. Then,

i. There is a unique PEPQ.

ii. If the key assumption holds for both markets, PEPQ creates arbitrage opportunity for at least one of the investors.

✓ *However, the arbitrage cannot be arbitrarily large, or even exploited.*

Partial equilibrium in segmented markets

- Suppose that there are two large investors, labeled A and B.
- They trade with different MMs in **segmented** markets (possibly with different securities too).
- The large investors trade to each other the derivative h at a partial equilibrium price & quantity (PEPQ), as introduced by A. and Žitković ['10].
- A PEPQ of h is a pair $(p^*, \mathfrak{u}^*) \in \mathbb{R}^2$ if

$$\mathfrak{u}^* \in \underset{\mathfrak{u}\in\mathbb{R}}{\operatorname{argmax}}\{u_A(x_A - p^*\mathfrak{u}, \Sigma_A + \mathfrak{u}h)\} \bigcap \underset{\mathfrak{u}\in\mathbb{R}}{\operatorname{argmax}}\{u_B(x_B + p^*\mathfrak{u}, \Sigma_B - \mathfrak{u}h)\}.$$

Proposition

Let both large investors have exponential utility and assume that $\gamma_A\Sigma_0^A - \gamma_B\Sigma_0^B$ and h are not constants. Then,

i. There is a unique PEPQ.

ii. If the key assumption holds for both markets, PEPQ creates arbitrage opportunity for at least one of the investors.

✓ *However, the arbitrage cannot be arbitrarily large, or even exploited.*

Partial equilibrium in segmented markets

- Suppose that there are two large investors, labeled A and B.
- They trade with different MMs in **segmented** markets (possibly with different securities too).
- The large investors trade to each other the derivative h at a partial equilibrium price & quantity (PEPQ), as introduced by A. and Žitković ['10].
- A PEPQ of h is a pair $(p^*, \mathfrak{u}^*) \in \mathbb{R}^2$ if

$$\mathfrak{u}^* \in \operatorname*{argmax}_{\mathfrak{u}\in\mathbb{R}}\{u_A(x_A - p^*\mathfrak{u}, \Sigma_A + \mathfrak{u}h)\} \bigcap \operatorname*{argmax}_{\mathfrak{u}\in\mathbb{R}}\{u_B(x_B + p^*\mathfrak{u}, \Sigma_B - \mathfrak{u}h)\}.$$

Proposition

Let both large investors have exponential utility and assume that $\gamma_A\Sigma_0^A - \gamma_B\Sigma_0^B$ and h are not constants. Then,

i. There is a unique PEPQ.

ii. If the key assumption holds for both markets, PEPQ creates arbitrage opportunity for at least one of the investors.

✓ *However, the arbitrage cannot be arbitrarily large, or even exploited.*

Partial equilibrium in segmented markets

- Suppose that there are two large investors, labeled A and B.
- They trade with different MMs in **segmented** markets (possibly with different securities too).
- The large investors trade to each other the derivative h at a partial equilibrium price & quantity (PEPQ), as introduced by A. and Žitković ['10].
- A PEPQ of h is a pair $(p^*, \mathfrak{u}^*) \in \mathbb{R}^2$ if

$$\mathfrak{u}^* \in \operatorname*{argmax}_{\mathfrak{u}\in\mathbb{R}}\{u_A(x_A - p^*\mathfrak{u}, \Sigma_A + \mathfrak{u}h)\} \bigcap \operatorname*{argmax}_{\mathfrak{u}\in\mathbb{R}}\{u_B(x_B + p^*\mathfrak{u}, \Sigma_B - \mathfrak{u}h)\}.$$

Proposition

Let both large investors have exponential utility and assume that $\gamma_A \Sigma_0^A - \gamma_B \Sigma_0^B$ and h are not constants. Then,

i. There is a unique PEPQ.

ii. If the key assumption holds for both markets, PEPQ creates arbitrage opportunity for at least one of the investors.

✓ *However, the arbitrage cannot be arbitrarily large, or even exploited.*

Outline

1 Motivation & goals

2 Market model & initial steps

3 Connection to a constrained investment problem with no price impact

4 Derivative pricing under price impact

5 Conclusive remarks

Highlights...

- The optimal investment problem under price impact can be written as an optimal investment constrained problem without market impact.
- There is a specific condition that guarantees that constrain set is non-binding and the problem can be solved.
- Derivative pricing upon price impact on the underlying market differs from the standard arbitrage-free pricing.
- Arbitrage is limited $\longrightarrow$ Investors may optimally ignore it!
- In segmented markets, arbitrage-price may arise as the equilibrium price!

The End

Thank you for your attention!