Default Contagion in Random Financial Block Models

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joint with:
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Introduction

Systemic risk: risk that in case of an adverse (local) shock substantial parts of the system default due to **contagion effects**.

Aim: **measurement and management of default contagion** in financial systems in terms of **statistical network characteristics**.

Tool: asymptotic analysis of default contagion in weighted, directed, inhomogeneous **random graphs**.

- *In the spirit of [Amini, Cont, Minca, 13], [Gai, Kapadia, 10], [Hurd, 16], [Detering, M.-B., Panagiotou, Ritter, 15,16]*

Random financial network

Stylized features of financial networks:

- strong heterogeneity (core/periphery structure)
- the graph is directed
- there are weights on the edges
- the graph is sparse: number of edges linear in network size
- The networks are large

Random financial network

Strong heterogeneity:

- in degrees (connectivity)
  - degree distribution might not have second moments (*[Boss et al., 2004], [Cont et al., 2013], [Craig and van Peter, 2014]*)
- in financial exposures
- in capital endowments
- assortative structure (*[Hurd, 16]*)
  - in connectivity
  - in financial exposures

Random financial network

- *[Amini, Cont, Minca, 13], [Gai, Kapadia, 10], [Hurd, 16]*: configuration model
  - degree distribution needs to have 2nd moment (no core/periphery)
  - exposures depend only on creditor
  - no flexible assortativity
- *[Detering, M.-B., Panagiotou, Ritter, 15,16]*: inhomogeneous random graph
  - degree distribution without 2nd moment possible
  - exposures depend only on creditor
  - no flexible assortativity

Random financial network

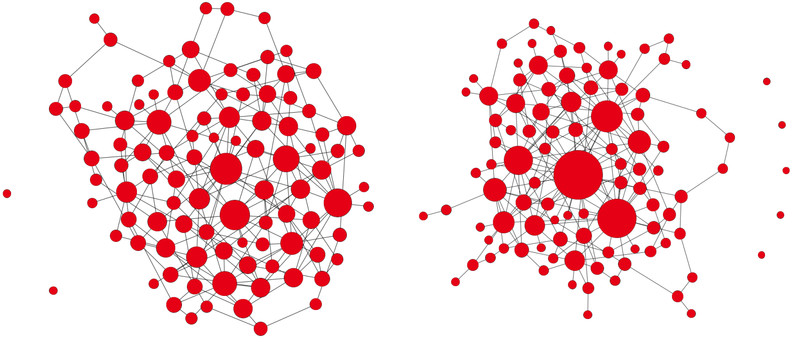

Figure: Inhomogeneous random graph with Pareto-distributed degrees with shape parameter (a) 3.5 (bounded second moment) respectively (b) 2.5 (unbounded second moment).
Node sizes scale with the corresponding degree.

Random financial network

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- We construct a random network with edge-weights (financial exposures) in $[R]$, $R \in \mathbb{N}$.
- Each bank $i \in [n]$ equipped with a set of non-negative vertex-weights: for all $r \in [R], \alpha \in [T]$
  - $w_i^{-,r,\alpha}$ describes tendency of bank i to develop incoming edges of weight r from institutions of type α
  - $w_i^{+,r,\alpha}$ describes tendency of bank i to develop outgoing edges of weight r to institutions of type α

Random financial network

- A directed edge from i to j of weight $r \in [R]$ appears with probability

$$p_{i,j}^r := \begin{cases} R^{-1} \wedge n^{-1} w_i^{+,r,\alpha_j} w_j^{-,r,\alpha_i}, & i \neq j, \\ 0, & i = j. \end{cases}$$

  - edge between banks appear independently
  - no multiple edges of different weights between banks
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- Each bank $i \in [n]$ is equipped with a capital level (net worth) $c_i \in \mathbb{N}_0 \cup \{\infty\}$.

Default contagion

Default cascades

- The set of initially defaulted banks:

$$\mathbb{D}_0 = \{i \in [n] \mid c_i \leq 0\}$$

- Default cascade $\mathbb{D}_0 \subseteq \mathbb{D}_1 \subseteq \ldots \subseteq \mathbb{D}_{n-1}$ triggered by $\mathbb{D}_0$:

$$\mathbb{D}_k = \left\{ i \in [n] \,:\, c_i \leq \sum_{r=1}^{R} r \sum_{j \in \mathcal{D}_{k-1}} X_{j,i}^r \right\}$$

  where $X_{j,i}^r$ is 1 if there is an edge of weight r going from j to i and 0 otherwise.

- $\mathbb{D}_n := \mathbb{D}_{n-1}$ is the final default cluster in the network generated by the fundamental defaults $\mathbb{D}_0$.

Systemic risk indicator

In the following we will focus on the final fraction of defaulted banks after contagion as systemic risk indicator:

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$\rightsquigarrow$ Asymptotic analysis for $n \to \infty$ of the default fraction via generalized bootstrap percolation in our weighted, directed block random graph.

Asymptotic default fraction

- For each network size n, the (random) financial network model is characterized by the empirical distribution of $(w_i^{-,r,\alpha}, w_i^{+,r,\alpha}, c_i, \alpha_i)_{i=1,\dots,n}$; i.e. by a corresponding random vector
$$(W_n^{-,r,\alpha}, W_n^{+,r,\alpha}, C_n, A_n)$$
- For $n \to \infty$ we assume
$$(W_n^{-,r,\alpha}, W_n^{+,r,\alpha}, C_n, A_n) \rightsquigarrow (W^{-,r,\alpha}, W^{+,r,\alpha}, C, A)$$
- Further we assume
$$\mathbb{E}[W_n^{-,r,\alpha}] \to \mathbb{E}[W^{-,r,\alpha}] < \infty \quad \text{and} \quad \mathbb{E}[W_n^{+,r,\alpha}] \to \mathbb{E}[W^{+,r,\alpha}] < \infty$$

Asymptotic default fraction

Define $f^{r,\alpha,\beta} : \mathbb{R}_{+,0}^{V} \to \mathbb{R}$, $(r,\alpha,\beta) \in V := R \times T^2$, and $g : \mathbb{R}_{+,0}^{V} \to \mathbb{R}_{+,0}$ by

$$f^{r,\alpha,\beta}(\mathbf{z}) = \mathbb{E}\left[W^{+,r,\alpha}\psi_C\left(\sum_{\gamma\in[T]} W^{-,1,\gamma}z^{1,\beta,\gamma},\ldots,\sum_{\gamma\in[T]} W^{-,R,\gamma}z^{R,\beta,\gamma}\right)\mathbf{1}\{A=\beta\}\right] - z^{r,\alpha,\beta},$$

$$g(\mathbf{z}) = \sum_{\beta\in[T]}\mathbb{E}\left[\psi_C\left(\sum_{\gamma\in[T]} W^{-,1,\gamma}z^{1,\beta,\gamma},\ldots,\sum_{\gamma\in[T]} W^{-,R,\gamma}z^{R,\beta,\gamma}\right)\mathbf{1}\{A=\beta\}\right]$$

where

$$\psi_l(x_1,\ldots,x_R) := \mathbb{P}\left(\sum_{s\in[R]} sX_s \geq l\right)$$

for independent Poisson random variables $X_s \sim \mathrm{Poi}(x_s)$.

Asymptotic default fraction

- Define

$$S := \bigcap_{(r,\alpha,\beta)\in V} \{\mathbf{z} \in \mathbb{R}^{V}_{+,0} \,:\, f^{r,\alpha,\beta}(\mathbf{z}) \geq 0\}$$

- Let S_0 denote the largest connected subset of S containing $\mathbf{0}$.
- Let $\mathbf{z}^*$ and $\hat{\mathbf{z}}$ be the largest and smallest joint root in S_0 of all the functions $f^{r,\alpha,\beta}$, $(r, \alpha, \beta) \in V$, respectively.

Asymptotic default fraction

Theorem: Let $\hat{\mathbf{z}}$ and $\mathbf{z}^*$ be the smallest respectively largest joint root in S_0 of the functions $\{f^{r,\alpha,\beta}\}_{(r,\alpha,\beta)\in V}$. Then

$$g(\hat{\mathbf{z}}) + o_p(1) \leq n^{-1}|\mathbb{D}_n| \leq g(\mathbf{z}^*) + o_p(1).$$

In particular, if $\hat{\mathbf{z}} = \mathbf{z}^*$, then

$$n^{-1}|\mathbb{D}_n| = g(\hat{\mathbf{z}}) + o_p(1).$$

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Identifying systemic risk for a given network with characteristics $(W^{-,r,\alpha}, W^{+,r,\alpha}, C, A)$**:**

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- Then shock system: some vertices default ex post
  - f.ex. banks default independently with probability $\epsilon > 0$
  - or only banks of certain types are affected by default
  - ...

$\rightsquigarrow$ new threshold sequence C_s with $\mathbb{P}(C_s = 0) > 0$

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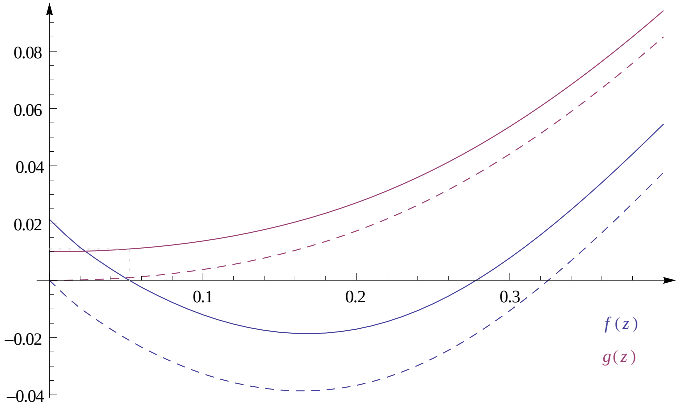
- Compute final default fraction $n^{-1}|\mathbb{D}_n^s|$ after contagion in shocked system
- Systemic risk: *small local shock spreads to substantial parts of the system*

Measure of systemic risk: resilience/non-resilience

Theorem (resilient system): Assume that $S_0 = \{\mathbf{0}\}$. Then for any $\epsilon > 0$ there exists $\delta > 0$ such that $n^{-1}|\mathbb{D}_n^s| \leq \epsilon$ w.h.p. for every ex post default shock satisfying $\mathbb{P}(C_s = 0) < \delta$.

- Small shocks remain local!
  Final default fraction goes to 0 when shock goes to 0!

Measure of systemic risk: resilience/non-resilience



Measure of systemic risk: resilience/non-resilience

Theorem (non-resilient system): Assume that $S_0 \neq \{\mathbf{0}\}$, i. e. $\mathbf{z}^* \neq \mathbf{0}$. Consider any ex post default shock which is independent of type A, vertex-weights $W^{\pm,r,\alpha}$, and capital C. Then w.h.p.

$$n^{-1}|\mathcal{D}_n^s| \geq g(\mathbf{z}^*) > 0$$

- Any small shock spreads to a substantial part (linear fraction)! Lower bound independent of shock size!

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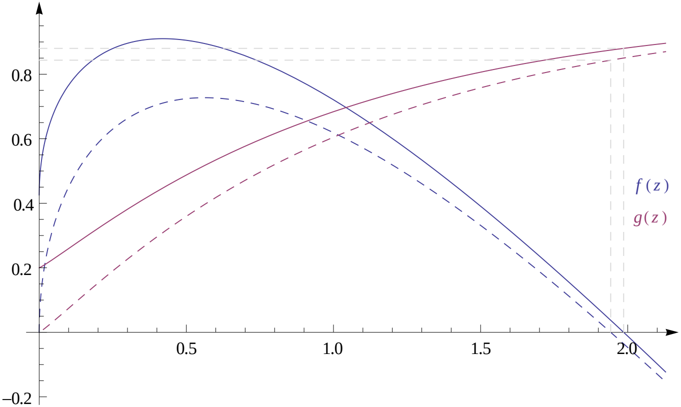
In the paper we provide more general analysis that allows only certain parts of system (certain types of banks) to be hit by ex post default.

Measure of systemic risk: resilience/non-resilience



Measure of systemic risk: resilience/non-resilience

Example: influence of non-resilient subsystem

- Global system consisting of two subsystems (R=1, T=2) given by
$$(W^{\pm,1}, W^{\pm,2}, C, A)$$
- Subsystem of banks of type 1 assumed to be non-resilient.
- Subsystem of banks of type 2 assumed to be resilient.

Applications

Now apply ex post shock to subsystem 1: $\mathbb{P}(C_s = 0, A = 1) > 0$

- Assume there are some banks in subsystem 1 lending to banks in the non-resilient sub-system 1:

$$W^{+,2}|_{A=1} > 0 \text{ and } \mathbb{P}(W^{-,1} > 0, C < \infty, A = 2) > 0$$

- Then every howsoever small shock to the non-resilient subsystem 1 spreads to a lower bounded fraction of finally defaulted banks in subsystem 2 as well.

Applications

Now apply ex post shock to subsystem 2: $\mathbb{P}(C_s = 0, A = 2) > 0$

- Assume there are some banks in non-resilient subsystem 1 lending to banks in the resilient subsystem 1:

$$W^{+,1}|_{A=2} > 0 \text{ and } \mathbb{P}(W^{-,2} > 0, C < \infty, A = 1) > 0$$

- Then every howsoever small shock to the resilient subsystem 2 spreads to a lower bounded fraction of finally defaulted banks in subsystem 2.
- That is, by connecting to the non-resilient subsystem 1 the a priori resilient subsystem 2 becomes non-resilient as well.

Applications

Even connecting a priori resilient subsystems may result in a global non-resilient system!

But:

Proposition: Consider a global financial network consisting of T resilient subnetworks (R=1 for simplicity) and assume that there exists a constant $K < \infty$ such that for all $\alpha \neq \beta \in [T]$

$$W^{\pm,\beta}|_{A=\alpha} \leq KW^{\pm,\alpha}|_{A=\alpha}$$

Then the global system is still resilient.

Applications

Example: counterparty dependent exposures

- Core/periphery network ($T = 2$) with 2 posiible exposure sizes ($R = 2$)
- $p = 1/3$ of all banks have type 1 (core), $1 - p = 2/3$ banks have type 2 (periphery)
- each possible edge appears with probability $4/n$
- all banks have capital $C = 2$

Applications

Counterparty-dependent exposures:

- Exposures between core banks are of size 2, all other exposures are of size 1

Creditor-dependent exposures:

- assign size 2 with probability p to any exposure of a core bank, all other exposures are of size 1.

Proposition: The network with counterparty-dependent exposures is non-resilient, while the network with creditor-dependent exposures is resilient.

Thank you!

References

- V. V. Acharya, L. H. Pedersen, T. Philippon, and M. Richardson. Measuring systemic risk. Working paper, March 2010.
- Adrian, T. and M. K. Brunnermeier (2011). Covar. Technical report, National Bureau of Economic Research.
- Amini, H., D. Filipovic & A. Minca (2013), Systemic risk with central counterparty clearing, Swiss Finance Institute Research Paper No. 13-34, Swiss Finance Institute
- Awiszus, K. & Weber, S. (2015): The Joint Impact of Bankruptcy Costs, Cross-Holdings and Fire Sales on Systemic Risk in Financial Networks, preprint
- Artzner, Delbaen, Eber, Heath: Coherent measures of risk, Math. Fin., 1999
- M. Boss, H. Elsinger, M. Summer, and S. Thurner. Network topology of the interbank market. Quantitative Finance, 4(6):677-684, 2004.
- Brunnermeier, M. K. and P. Cheridito (2013). Measuring and allocating systemic risk. Available at SSRN

References

- Cerreia-Vioglio, S., Maccheroni, F., Marinacci, M., and Montrucchio, L., Riskmeasures: rationality and diversication, Mathematical Finance, 21, 743 - 774 (2011)
- Chen, C., G. Iyengar, and C. C. Moallemi (2013). An axiomatic approach to systemic risk. Management Science 59(6), 13731388.
- Cifuentes, R., G. Ferrucci & H. S. Shin (2005), Liquidity risk and contagion, Journal of the European Economic Association 3 (2-3), 556566]
- R. Cont, A. Moussa, and E. Santos. Network structure and systemic risk in banking systems. In J.-P. Fouque and J. Langsam, editors, Handbook on Systemic Risk. Cambridge, 2013.
- B. Craig and G. von Peter. Interbank tiering and money center banks. Journal of Financial Intermediation, (0), 2014.
- Detlefsen, K. and G. Scandolo (2005). Conditional and dynamic convex risk measures. Finance and Stochastics 9(4), 539561.

References

- Föllmer, H., Spatial Risk Measures and their Local Specication: The Locally Law-Invariant Case, Statistics & Risk Modeling 31(1), 79103, 2014.
- Föllmer, H. & A. Schied (2011). Stochastic Finance: An introduction in discrete time (3rd ed.). De Gruyter.
- Föllmer, H., Klüppelberg, C., Spatial risk measures: local specification and boundary risk. In: Crisan, D., Hambly, B. and Zariphopoulou, T.: Stochastic Analysis and Applications 2014 - In Honour of Terry Lyons . Springer, 2014, 307-326
- Frittelli, M. & M. Maggis (2011). Conditional certainty equivalent. International Journal of Theoretical and Applied Finance 14 (01), 4159
- Gai, P. & Kapadia, S. (2010), Contagion in financial networks, Bank of England Working Papers 383, Bank of England

References

X. Huang, H. Zhou, and H. Zhu. A framework for assessing the systemic risk of major financial institutions. Journal of Banking & Finance, 33(11):20362049, 2009.

Kromer E., Overbeck, L., Zilch, K.. Systemic risk measures on general probability spaces, preprint, 2013.

A. Lehar. Measuring systemic risk: A risk management approach. Journal of Banking & Finance, 29(10): 25772603, 2005

Rogers, L. C. G. & L. A. M. Veraart (2013), Failure and rescue in an interbank network, Management Science 59(4), 882898.

Tutsch, S. (2007). Konsistente und konsequente dynamische Risikomae und das Problem der Aktualisierung. Ph. D. thesis, Humboldt-Universität zu Berlin, Mathematisch-Naturwissenschaftliche Fakultät II.